Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **29** (2003), 167-176

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

DUALITY IN CONSTRAINED DC-OPTIMIZATION VIA TOLAND'S DUALITY APPROACH

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Communicated by R. Lucchetti

Abstract. In this paper we reconsider a nonconvex duality theory established by B. Lemaire and M. Volle (see [4]), related to a primal problem of minimizing the difference of two convex functions subject to a DC-constraint. The purpose of this note is to present a new method based on Toland-Singer duality principle. Applications to the case when the constraints are vector-valued are provided.

1. Introduction. It is well known that the theory of DC-optimization is now very well developed because of its theoretical aspects as well as its wide range of applications. The developments of this theory has been stimulated by the diversity of applications in optimization, economics, operations research, optimal control, mechanics and others(see [2], and references therein).

In a recent work, a duality theory associated with an important large class of DC-programming problems, was developed by B. Lemaire and M. Volle [4], in

2000 *Mathematics Subject Classification:* 90C48, 49N15, 90C25.
Key words: Conjugate, difference convex, Toland-Singer duality, DC-constraint.

the setting of locally convex real vector space. That is minimizing a difference of two extended real-valued convex functions subject to a DC-constraint i.e. it concerns the primal problem

$$(\mathcal{P}) \qquad \inf\{g_1(x) - g_2(x) : \quad h_1(x) - h_2(x) < 0 \}$$

where g_1, g_2, h_1 and h_2 are extended real-valued convex functions on the Hausdorff locally convex real vector space X. This problem covers various situations in nonconvex and convex optimization.

This duality extends a duality theory initially examined in [3] by B. Lemaire for the case where $g_2 \equiv 0$ and $h_1 \equiv 0$. This case is usually called reverse convex programming problem. The technique used in [4] for stating the duality theory associated with problem $(\mathcal{P})$ makes an intensive use of convex analysis theory and essentially the" inf sup" theorem of J. J. Moreau [6].

In the presents work, we address a main question, that is: how to obtain the duality theory in constrained DC-programming via Toland-Singer duality approach? In fact, the answer of this question presents a new method completely different from that of [4] for establishing the duality result associated with problem $(\mathcal{P})$.

The outline of this paper is as follows. In section 2 we recall some definitions and some preliminary results, section 3 is devoted to the main result. Finally in section 4 we give two illustrations of our main result which consists in both cases to minimize a DC-objective function subject in the first case to a vector DC-constraint and in the second case to a mixed constraint composed by vector convex constraint and a vector reverse constraint. The mappings introduced in the vector constraints take together their values in a partially order topological vector space.

2. Preliminaries. Let us begin by recalling some definitions, which can be found in [1]. We suppose throughout this paper that X and Y are two locally convex topological real linear spaces and X^* , Y^* respectively their dual spaces. In both cases, we denote the separating duality by $\langle \ , \ \rangle$ understanding in each case that we consider the duality (X^*, X) or (Y^*, Y). We suppose these spaces are supplied with topologies compatible with this duality. In the sequel, we assume that the space Y is endowed with a partial order (*symbol* $:\leq_Y$) induced by a convex cone Y_+ and we denote by Y_+^* the dual positive cone of Y^* defined by

$$Y_+^* := \{y^* \in Y^* : \langle y^*, y\rangle \geq 0, \quad \forall y \in Y_+\}.$$

The cone Y_+ is called the set of nonnegative element of Y. If we adjoint an abstract maximal element $+\infty$ to Y, a mapping $h : X \longrightarrow Y \cup \{+\infty\}$ is said to be Y_+-convex if the following inequality

$$h(\alpha x_1 + \beta x_2) \leq_Y \alpha h(x_1) + \beta h(x_2)$$

holds whenever $x_1,\ x_2 \in X$ and α, β are positive numbers with $\alpha + \beta = 1$. A function $g : Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ is said to be Y_+-nondecreasing on Y if for each $y_1,\ y_2 \in Y$ satisfying $y_1 \leq_Y y_2$ we have $g(y_1) \leq g(y_2)$. For a given function $f : X \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$, one usually denotes by $domf := \{x \in X : f(x) < +\infty\}$ its effective domain and by $f^* : X^* \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$, $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x), x \in X\}$ its Legendre-Fenchel conjugate function. We recall that f coincides with its biconjugate $f^{**} = (f^*)^*$ whenever f is convex , lower semicontinous and proper (proper means that f does not take the value $-\infty$ and it is non identically equal to $+\infty$). By $\Gamma(X)$ (resp. $\Gamma_0(X)$) we denote the set of convex lower semicontinuous proper functions plus the constant $+\infty$ and $-\infty$ (resp. the set of convex lower semicontinuous proper functions). For a subset $C \subset X$, we denote by δ_C the indicator function defined by $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ otherwise.

In order to state our main result, we shall need the following results due respectively to C. Combari, M. Laghdir and L. Thibault [1], J. F. Toland [9] and I. Singer [8]. The first result concerns the composition of conjugate function of the composition of a nondecreasing convex function with a convex mapping taking values in a partially ordered topological vector space. The second result established the dual problem related to an unconstrained DC-mathematical programming problem.

Proposition 2.1 ([1]). *Let X and Y be two Hausdorff locally convex real vector spaces, $F : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a convex function, $G : Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a convex and Y_+-nondecreasing function and $H : X \longrightarrow Y \cup \{+\infty\}$ is a Y_+-convex mapping. If there exists some $\overline{x} \in dom\ F \cap dom\ H$ such that G is finite and continuous at $H(\overline{x}) \in Y$, then we have for any $x^* \in X^*$.*

$$(F + G \circ H)^*(x^*) = \min_{y^* \in Y_+^*} \{G^*(y^*) + (F + y^* \circ H)^*(x^*)\}$$

Proposition 2.2 ([8],[9]). *Let $f_1 : X \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ be any function and $f_2 : X \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ be a convex and lower semicontinuous function. Then we have*

$$\inf_{x \in X} \{f_1(x) - f_2(x)\} = \inf_{x^* \in X^*} \{f_2^*(x^*) - f_1^*(x^*)\}.$$

Throughout, we adopt the following convention

$$x \longrightarrow (G \circ H)(x) := \begin{cases} G(H(x)) & \text{if } H(x) \in Y \\ \\ \sup\limits_{y \in Y} G(y), & \text{otherwise,} \end{cases} \tag{2.1}$$

and the following extensions of the addition and the product in $\mathbb{R} \cup \{-\infty, +\infty\}$

$$(+\infty) + (-\infty) = (-\infty) + (+\infty) = +\infty, \quad 0 \times (-\infty) = 0, \quad 0 \times (+\infty) = +\infty.$$

3. The main result. Before embarking on the development of duality theory related to the problem $(\mathcal{P})$, we start with a lemma that will be needed in the sequel.

Lemma 3.1. *Let X be a Hausdorff locally convex real vector space and let $g_1, g_2, h_1, h_2 : X \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ be convex functions.*
i) If g_2 is lower semicontinuous and $\{g_2 > -\infty\} \cap dom\ g_1 \cap dom\ h_1 \neq \emptyset$, then we have:

$$\inf_{h_1(x)<0} \{g_1(x) - g_2(x)\} = \inf_{h_1(x) \leq 0} \{g_1(x) - g_2(x)\}$$

ii) If we set for any $x^* \in X^* : x \longrightarrow k_{x^*}(x) := h_1(x) - \langle x^*, x\rangle + h_2^*(x^*)$ then we have:

$$dom\ g_1 \cap \{x \in X : k_{x^*}(x) < 0\} \neq \emptyset \Longleftrightarrow h_2^*(x^*) - (\delta_{dom g_1} + h_1)^*(x^*) < 0.$$

P r o o f. i) This statement is proved in [4]
ii)($\Longrightarrow$) We have

$$dom\ g_1 \cap \{x \in X : k_{x^*}(x) < 0\} \neq \emptyset \iff \exists x \in dom\ g_1 : \ k_{x^*}(x) < 0,$$

which implies

$$h_2^*(x^*) < \langle x^*, x\rangle - h_1(x) - \delta_{dom g_1}(x) \leq (\delta_{dom g_1} + h_1)^*(x^*)$$

i.e.

$$h_2^*(x^*) < (\delta_{dom g_1} + h_1)^*(x^*). \tag{3.1}$$

($\Longleftarrow$) Conversely, let us note that, according to the definition of $(\delta_{dom g_1} + h_1)^*(x^*)$, the above strict inequality (3.1) becomes equivalent to

$$\inf_{x \in X} \{h_2^*(x^*) - \langle x^*, x\rangle + h_1(x) + \delta_{dom\ g_1}(x)\} < 0,$$

which yields that there exists some $x \in X$ such that

$$h_2^*(x^*) - \langle x^*, x\rangle + h_1(x) + \delta_{dom g_1}(x) < 0$$

and then, according to convention $(+\infty) + (-\infty) = +\infty$, it follows that $x \in$ *dom* g_1 and $h_2^*(x^*) - \langle x^*, x\rangle + h_1(x) < 0$, i.e.

$$dom\ g_1 \cap \{x \in X : k_{x^*}(x) < 0\} \neq \emptyset,$$

which completes the proof. □

By setting

$$\alpha := \inf\{g_1(x) - g_2(x) : \quad h_1(x) - h_2(x) < 0\}$$

we can state, now, the associated dual problem given by

Theorem 3.1. *Let X be a Hausdorff locally convex real vector space and let $g_1, g_2, h_2 : X \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ and $h_1 : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be convex functions with $g_2 \in \Gamma(X)$ and $h_2 \in \Gamma_0(X)$ or $h_2 \equiv -\infty$. Then we have*

$$\alpha = \inf_{x^*, p^* \in X^*} \max_{\lambda \geq 0} \{g_2^*(p^*) + \lambda h_2^*(x^*) - (g_1 + \lambda h_1)^*(p^* + \lambda x^*) : h_2^*(x^*) - (\delta_{dom g_1} + h_1)^*(x^*) < 0\}.$$

P r o o f. First of all, let us note that the following equality

$$\{x \in X, h_1(x) - h_2(x) < 0\} = \bigsqcup_{x^* \in X^*} \{x \in X : k_{x^*}(x) < 0\}, \tag{3.2}$$

follows merely from the fact that

$$h_2(x) = h_2^{**}(x) = \sup_{x^* \in X^*} \{\langle x^*, x\rangle - h_2^*(x^*)\}, \quad \forall x \in X.$$

Therefore, by virtue of the above equality (3.2), we may write

$$\alpha = \inf_{x^* \in X^*} \inf_{x \in X} \{g_1(x) - g_2(x) : k_{x^*}(x) < 0\},$$

which yields according to Lemma 3.1

$$\begin{aligned}
\alpha &= \inf_{x^* \in X^*} \inf_{x \in X} \{g_1(x) - g_2(x) : k_{x^*}(x) \leq 0,\ dom\ g_1 \cap \{x \in X, k_{x^*}(x) < 0\} \neq \emptyset\} \\
&= \inf_{x^* \in X^*} \inf_{x \in X} \{(g_1 + \delta_{-\mathbb{R}_+} \circ k_{x^*})(x) - g_2(x) : h_2^*(x^*) - (\delta_{dom g_1} + h_1)^*(x^*) < 0\}.
\end{aligned}$$

Since $g_2 \in \Gamma(X)$, we obtain by Proposition 2.2
(3.3)
$$\alpha = \inf_{x^* \in X^*} \inf_{p^* \in X^*} \{g_2^*(p^*) - (g_1 + \delta_{-\mathbb{R}_+} \circ k_{x^*})^*(p^*) : h_2^*(x^*) - (\delta_{dom g_1} + h_1)^*(x^*) < 0\}.$$

Let us note that the indicator function $y \longrightarrow \delta_{-\mathbb{R}_+}(y)$ is convex and nondecreasing on the whole space $\mathbb{R}$ and by means of the convexity of the function $x \longrightarrow k_{x^*}(x)$ we check easily that the composite function $\delta_{-\mathbb{R}_+} \circ k_{x^*}$ is so convex. The condition *dom* $g_1 \cap \{x \in X : k_{x^*}(x) < 0\} \neq \emptyset$ asserts that the indicator function $\delta_{-\mathbb{R}_+}$ is finite and continuous at some point of the nonempty subset $k_{x^*}(dom\ g_1 \cap dom\ k_{x^*})$ and hence by assuming g_1 proper, it follows from Proposition 2.1 that for any $p^* \in X^*$:

$$(g_1 + \delta_{-\mathbb{R}_+} \circ k_{x^*})^*(p^*) = \min_{\lambda \geq 0} \{\delta^*_{-\mathbb{R}_+}(\lambda) + (g_1 + \lambda k_{x^*})^*(p^*)\}.$$

If we assume now there exists some $x \in X$ such that $g_1(x) = -\infty$, then according to the previous conventions we obtain obviously

$$(g_1 + \delta_{-\mathbb{R}_+} \circ k_{x^*})^*(p^*) = \min_{\lambda \geq 0} \{\delta^*_{-\mathbb{R}_+}(\lambda) + (g_1 + \lambda k_{x^*})^*(p^*)\} = +\infty.$$

As $\delta^*_{-\mathbb{R}_+} = \delta_{\mathbb{R}_+}$, we obtain
(3.4)
$$\begin{aligned}(g_1 + \delta_{-\mathbb{R}_+} \circ k_{x^*})^*(p^*) &= \min_{\lambda \geq 0} (g_1 + \lambda k_{x^*})^*(p^*) \\ &= \min_{\lambda \geq 0} \sup_{x \in X} \{\langle p^*, x\rangle - g_1(x) - \lambda k_{x^*}(x)\} \\ &= \min_{\lambda \geq 0} \sup_{x \in X} \{\langle p^* + \lambda x^*, x\rangle - g_1(x) - \lambda h_1(x) - \lambda h_2^*(x^*)\} \\ &= \min_{\lambda \geq 0} \{(g_1 + \lambda h_1)^*(p^* + \lambda x^*) - \lambda h_2^*(x^*)\}.\end{aligned}$$

By replacing the expression (3.4) in (3.3), we get our desired result. □

4. Applications.

Let us consider the following abstract DC-mathematical programming problem

$$(\mathcal{Q}) \qquad \beta := \inf\{g_1(x) - g_2(x) : \ h_1(x) - h_2(x) \notin Y_+\},$$

where $g_1, g_2 : X \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ are two convex functions and $h_1, h_2 : X \longrightarrow Y \cup \{+\infty\}$ are two convex vector valued mappings taking their values in a real partially ordered topological vector space Y. The partial order is induced by a convex cone $Y_+ \subset Y$. Let us note that the problem $(\mathcal{P})$ is a particular case of $(\mathcal{Q})$ by taking $Y = \mathbb{R}$ and $Y_+ = \mathbb{R}_+$.

In what follows, the aim is to derive from Theorem 3.1 the corresponding dual problem related to problem$(\mathcal{Q})$.

Before stating our duality result, let us recall a concept of lower semicontinuity adapted to vector valued mappings. For this, let X be a topological space. Following [7] and [5], one say that a mapping $h : X \longrightarrow Y \cup \{+\infty\}$ is lower semicontinuous (l.s.c) at $\overline{x} \in h^{-1}(Y)$ if for any neighbourhood V of $h(\overline{x}) \in Y$ there exists a neighbourhood U of $\overline{x}$ such that

$$h(U) \subset (V + Y_+) \cup \{+\infty\}. \tag{4.1}$$

h is said to be l.s.c at $\overline{x} \in h^{-1}(+\infty)$ if for any $y \in Y$, any neighborhood V of y there exists a neighborhood U of $\overline{x}$ such that(4.1) holds.

In the case when $Y = \mathbb{R}$ and $Y_+ = \mathbb{R}_+$, we recover the usual notion of lower semicontinuity.

Concerning the lower semicontinuity of a composite function we have

Proposition 4.1. ([5]). *Let $h : X \longrightarrow Y \cup \{+\infty\}$ be l.s.c on X and assume that $g : Y \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is Y_+-nondecreasing. If dom $h = X$ and g is l.s.c on $h(X)$, then $g \circ h$ is l.s.c on X. If dom $h \neq X$ and g is l.s.c on Y then $g \circ h$ is l.s.c on X.*

Now, we are ready to derive from Theorem 3.1 the dual problem associated with $(\mathcal{Q})$.

Proposition 4.2. *Let X and Y be two Hausdorff localy convex vector spaces, $g_1, g_2 : X \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ two convex functions, and $h_1, h_2 : X \longrightarrow Y \cup \{+\infty\}$ two Y_+-convex mappings. We assume that $g_2 \in \Gamma(X)$, dom $h_1 = X$ and h_2 is proper and l.s.c., then*

$$\beta = \inf_{\substack{(x^*,p^*)\in X^*\times X^* \\ y^*\in Y_+^*\setminus\{0\}}} \max_{\lambda\geq 0}\{g_2^*(p^*) + \lambda(y^*\circ h_2)^*(x^*) - (g_1 + \lambda y^*\circ h_1)^*(p^* + \lambda x^*) :$$
$$(y^*\circ h_2)^*(x^*) - (\delta_{domg_1} + y^*\circ h_1)^*(x^*) < 0\}.$$

P r o o f. It is not difficult to check that:

$$\{x \in X : h_1(x) - h_2(x) \notin Y_+\} = \bigsqcup_{y^*\in Y_+^*\setminus\{0\}} \{x \in X :\ (y^*\circ h_1)(x) - (y^*\circ h_2)(x) < 0\}$$

and hence problem $(\mathcal{Q})$ becomes

$$\beta = \inf_{y^* \in Y_+^* \setminus \{0\}} \inf_{x \in X} \{g_1(x) - g_2(x) : \ (y^* \circ h_1)(x) - (y^* \circ h_2)(x) < 0\}.$$

In order to derive our desired duality result from Theorem 3.1, it suffices to prove that $y^* \circ h_2$ is proper, lower semicontinuous and convex for each $y^* \in Y_+^* \setminus \{0\}$. For this, let us observe that for any $y^* \in Y_+^*$, y^* is Y_+-nondecreasing on the whole space Y and according to convention (2.1) we have $(y^* \circ h_2)(x) = +\infty$ for $y^* \neq 0$ and $h_2(x) = +\infty$. By continuity of y^* and the fact that h_2 is proper and lower semicontinuous we easily obtain by Proposition 4.1 that $y^* \circ h_2$ is proper and lower semicontinuous. The convexity of $y^* \circ h_2$ follows from the monotonicity of $y^* \in Y_+^*$ combined with the convexity of h_2. To conclude the proof, it follows from Theorem 3.1 that

$$\beta = \inf_{\substack{(x^*,p^*) \in X^* \times X^* \\ y^* \in Y_+^* \setminus \{0\}}} \max_{\lambda \geq 0} \{g_2^*(p^*) + \lambda (y^* \circ h_2)^*(x^*) - (g_1 + \lambda y^* \circ h_1)^*(p^* + \lambda x^*) :$$
$$(y^* \circ h_2)^*(x^*) - (\delta_{dom g_1} + y^* \circ h_1)^*(x^*) < 0\}.$$

This finishes the proof. □

Let us consider now the case when the D.C objective function $g_1 - g_2$ is subject simultaneously to a vector convex constraint and a vector reverse constraint i.e.

$$(\mathcal{L}) \qquad \gamma := \inf\{g_1(x) - g_2(x) : h_1(x) \in -Y_+ \ \text{ and } \ h_2(x) \notin -Y_+\}$$

where X, Y, g_1, g_2, h_1 and h_2 are as in the above Proposition 4.2. This problem may be rewritten equivalently as

$$(4.2) \qquad \gamma = \inf\{g_1(x) - g_2(x) + (\delta_{-Y_+} \circ h_1)(x) : \ \ (\delta_{-Y_+} \circ h_2)(x) > 0\}.$$

Now, we are in position to state the duality result associated with primal problem $(\mathcal{L})$.

Proposition 4.3. *Let X and Y be two Hausdorff locally convex vector spaces, g_1, $g_2 : X \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ two convex functions and $h_1, h_2 : X \longrightarrow Y \cup \{+\infty\}$ two Y_+-convex mappings. We assume that $g_2 \in \Gamma(X)$, h_2 is proper and l.s.c, Y_+ is closed and there exist some $\overline{x} \in dom\ g_1 \cap dom\ h_1 \cap h_1^{-1}(-\operatorname{int} Y_+)$ and $\overline{u} \in h_2^{-1}(-\operatorname{int} Y_+)$. Then*

$$\gamma = \inf_{(x^*,p^*) \in X^* \times X^*} \min_{y^* \in Y_+^* \setminus \{0\}} \max_{(\lambda, z^*) \in \mathbb{R}_+ \times Y_+^*} \{g_2^*(p^*) + \lambda (y^* \circ h_2)^*(x^*) -$$
$$(g_1 + z^* \circ h_1)^*(p^* + \lambda x^*) : (y^* \circ h_2)^*(x^*) - \delta^*_{h_1^{-1}(-Y_+) \cap dom g_1}(x^*) < 0\}.$$

P r o o f. Let us observe that, according to (4.2), the problem $(\mathcal{L})$ is a particular case of the problem $(\mathcal{P})$. In order to apply Theorem 3.1 for deriving our duality result, it suffices to prove that $\delta_{-Y_+} \circ h_2 \in \Gamma_0(X)$ and $\delta_{-Y_+} \circ h_1$ is convex. For this, let us note that the indicator function $y \longrightarrow \delta_{-Y_+}(y)$ defined on Y is convex, l.s.c, proper and Y_+−nondecreasing on the whole space Y (see [1]) and by adding the convexity of the mappings h_1 and h_2, it is straightforward to see that the composite functions $\delta_{-Y_+} \circ h_1$ and $\delta_{-Y_+} \circ h_2$ are so convex. The semicontinuity of $\delta_{-Y_+} \circ h_2$ follows from Proposition 4.1. By applying Theorem 3.1 we obtain

$$\gamma = \inf_{(x^*,p^*)\in X^*\times X^*} \max_{\lambda\in\mathbb{R}_+} \{g_2^*(p^*) + \lambda(\delta_{-Y_+} \circ h_2)^*(x^*) - (g_1 + \delta_{-Y_+} \circ h_1)^*(p^* + \lambda x^*) : \\ (\delta_{-Y_+} \circ h_2)^*(x^*) - \delta^*_{dom(g_1+\delta_{-Y_+}\circ h_1)}(x^*) < 0\}.$$

Since there exist some $\overline{x} \in dom\ g_1 \cap dom\ h_1 \cap h_1^{-1}(-\text{int}Y_+)$ and $\overline{u} \in h_2^{-1}(-\text{int}Y_+)$, it follows from Proposition 2.1 that

$$(g_1 + \delta_{-Y_+} \circ h_1)^*(p^* + \lambda x^*) = \min_{z^*\in Y_+^*} \{\delta^*_{-Y_+}(z^*) + (g_1 + z^* \circ h_1)^*(p^* + \lambda x^*)\}$$

$$(\delta_{-Y_+} \circ h_2)^*(x^*) = \min_{y^*\in Y_+^*} \{\delta^*_{-Y_+}(y^*) + (y^* \circ h_2)^*(x^*)\}$$

and as $\delta^*_{-Y_+} = \delta_{Y_+^*}$ and $dom(g_1 + \delta_{-Y_+} \circ h_1) = h_1^{-1}(-Y_+) \cap dom\ g_1$ we obtain

$$\gamma = \inf_{(x^*,p^*)\in X^*\times X^*} \min_{y^*\in Y_+^*} \max_{(\lambda,z^*)\in\mathbb{R}_+\times Y_+^*} \{g_2^*(p^*) + \lambda(y^* \circ h_2)^*(x^*) - \\ (g_1 + z^* \circ h_1)^*(p^* + \lambda x^*) : (y^* \circ h_2)^*(x^*) - \delta^*_{h_1^{-1}(-Y_+)\cap dom g_1}(x^*) < 0\}.$$

Now, it remains to claim that the minimum over $y^* \in Y_+^*$ is, indeed, taken over $Y_+^*\backslash\{0\}$ i.e. the following strict inequality

$$(y^* \circ h_2)^*(x^*) - \delta^*_{h_1^{-1}(-Y_+)\cap dom g_1}(x^*) < 0 \tag{4.3}$$

does not hold for $y^* \equiv 0$. Suppose the contrary and by taking into account of the convention (2.1) the strict inequality (4.3) becomes

$$\delta_{\{0\}}(x^*) - \delta^*_{h_1^{-1}(-Y_+)\cap dom g_1}(x^*) < 0,$$

i.e.

$$\delta^*_{h_1^{-1}(-Y_+)\cap dom g_1}(0) = \sup_{x\in X}\{-\delta_{h_1^{-1}(-Y_+)\cap dom g_1}(x)\} > 0.$$

This contradicts the fact that $\delta^*_{h_1^{-1}(-Y_+)\cap dom g_1}(0) \leq 0$ since $\delta_{h_1^{-1}(-Y_+)\cap dom g_1}(x) \geq 0$ for any $x \in X$, which completes the proof. $\square$

Acknowledgment. The author would like to thank an anonymous referee for his valuable remarks and suggestions.

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Received December 9, 2002