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ON A STRATIFICATION DEFINED BY REAL ROOTS OF POLYNOMIALS

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Abstract. We consider the family of polynomials $P(x,a) = x^n + a_1x^{n-1} + \ldots + a_n$, $x, a_i \in \mathbf{R}$, and the stratification of $\mathbf{R}^n \cong \{(a_1,\ldots,a_n)|a_i \in \mathbf{R}\}$ defined by the multiplicity vector of the real roots of P. We prove smoothness of the strata and a transversality property of their tangent spaces.

1. Formulation of the result. For $n \in \mathbf{N}^*$ fixed consider the family of polynomials $P(x,a) = x^n + a_1x^{n-1} + \ldots + a_n$, $x, a_i \in \mathbf{R}$. A *multiplicity vector (MV)* is a vector whose components are the multiplicities of the real roots of P (for a fixed) listed in increasing order. E.g. for $n = 9$ the MV $[3,1,2,1]$ means that for the real roots x_i one has $x_1 = x_2 = x_3 < x_4 < x_5 = x_6 < x_7$ and there is a complex conjugate couple about whose real and imaginary part the MV gives no information.

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Define the *length* (resp. the *multiplicity surplus*) of a MV $[r_1, r_2, \ldots, r_q]$ as the integer $l = r_1 + \ldots + r_q$ (resp. $r = \sum_{j=1}^{q}(r_j - 1) = l - q$). The number of complex conjugate couples of roots of P (counted with the multiplicities) equals $(n - l)/2$.

Stratify the space $\mathbf{R}^n \cong \{(a_1, \ldots, a_n) | a_i \in \mathbf{R}\}$ – each MV with multiplicity surplus r defines a stratum of codimension r.

Remarks 1. 1) It is clear that if S_i, $i = 1, 2$, are two strata defined by the MVs V_i and if S_1 belongs to the closure of S_2, then for the lengths l_i of the MVs V_i one has $l_1 \geq l_2$. Indeed, to pass from S_1 to S_2 one or several real roots must bifurcate. If all new roots are real, then $l_1 = l_2$. If there is at least one complex conjugate couple, then $l_1 > l_2$.

2) If the MVs V_i are of the same length, then S_1 belongs to the closure of S_2 if and only if V_1 is obtained from V_2 by replacing groups of consecutive multiplicities by their sums (call this an *operation of type A*; it corresponds to confluence of real roots).

3) An *operation of type B* consists either in adding a component equal to 2 at an arbitrary place of the MV (a complex conjugate couple becomes a double real root different from the other real roots of P) or in increasing of one of the components of the MV by 2 (a complex conjugate couple becomes a double real root which coincides with one of the other real roots). If for the lengths l_i of S_i one has $l_1 = l_2 + 2k$, $k \in \mathbf{N}$, then S_1 belongs to the closure of S_2 if and only if V_1 is obtained from V_2 either by an operation of type A followed by k operations of type B or just by k operations of type B.

4) Given a stratum T of dimension $d < n$, with MV $\vec{v}$, one can obtain all MVs of strata of dimension $d+1$ adjacent to T either by replacing some component $m > 1$ of $\vec{v}$ by two consecutive components $m' \geq 1, m'' \geq 1$, $m' + m'' = m$ or by deleting from $\vec{v}$ a component equal to 2 (which means that a double real root becomes a complex conjugate couple). Indeed, to pass to a stratum of next dimension adjacent to the given one a root must bifurcate into two roots. If these two roots are complex conjugate, then they cannot be multiple because the stratification does not take into account the multiplicities of the complex roots.

The first aim of the present paper is to prove the following

Theorem 2. *A stratum of codimension r is a smooth real contractible algebraic variety of dimension $n - r$. It is the graph of a smooth r-dimensional vector-function defined on the projection of the stratum in $Oa_1 \ldots a_{n-r}$. The field of tangent spaces to the stratum is continuously extended to the strata belonging to its closure. The extension is everywhere transversal to the space $Oa_{n-r+1} \cdots a_n$ and contains the tangent space to the stratum on which the extension is done.*

The theorem is proved in Section 4. It is illustrated by an example in Section 2. We discuss in Section 3 (in view of the theorem and of previous results from [4]) the mutual disposition of adjacent strata.

Remarks 3. 1) The theorem generalizes Theorem 1.8 from [5]. The latter treats the case when P is *hyperbolic*, i.e. all roots are real.

2) The above stratification (or a similar one) has been considered (at least in some aspects) by other authors as well, see for instance [1], [2] and their bibliographies, [3], [7] and [8]; the list is anything but exhaustive.

2. An example. On Fig. 1. we show for $n = 4$, $a_1 = 0$, the well-known picture of the swallowtail, i.e. the surface $\Sigma = \{(a_2, a_3, a_4) \in \mathbf{R}^3 | \mathrm{Res}(P, P') = 0\}$. The three strata of dimension 3 and their respective MVs are the open subset of $\mathbf{R}^3$ "above" Σ, with empty MV (no real roots), its open subset "below" Σ, with MV $[1, 1]$, and the interior of the curvilinear pyramid $\Pi = OABC$, with MV $[1, 1, 1, 1]$.

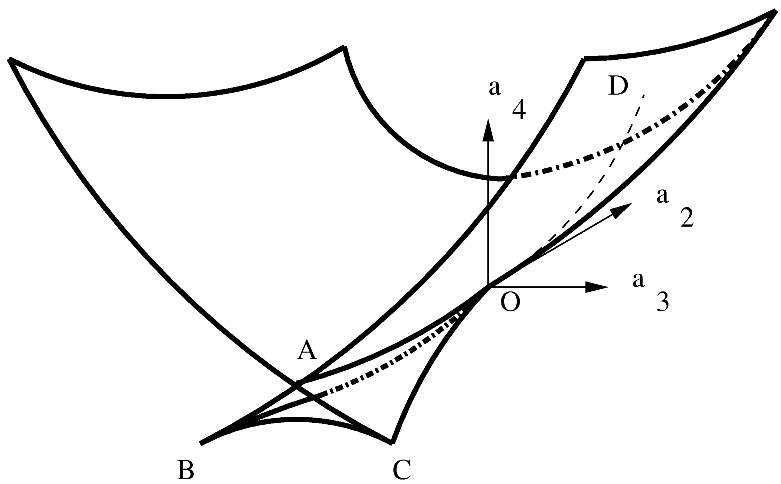

Fig. 1

The two-dimensional stratum S defined by the MV [2] is the swallowtail without the boundary of Π. The limits of the tangent spaces to S are different on the different sides of the self-intersection curve AO (excepting O where the limit is unique), see Remark 4. The strata ABO and ACO (without the boundaries) are defined respectively by the MVs $[2, 1, 1]$ and $[1, 1, 2]$; the stratum BCO (without the boundary) is defined by the MV $[1, 2, 1]$.

The one-dimensional strata BO and CO (without the point O) are defined respectively by the MVs $[3,1]$ and $[1,3]$ while AO (without the point O) is defined by the MV $[2,2]$. Finally, the point O is defined by the MV $[4]$.

The self-intersection curve AO has an analytic continuation OD (represented by a dashed line on Fig. 1) which consists of polynomials of the form $(x^2+b^2)^2$, $b \in \mathbf{R}$, i.e. having a conjugate complex couple of roots of multiplicity 2. This curve is not a stratum of the stratification (although it belongs to Σ) because the multiplicities of the complex roots are not taken into account, see part 4) of Remarks 1.

Remark 4. The following proposition is proved in [6], p. 52-53.

Proposition 5. *If P, P_1, ..., P_μ are monic polynomials where $P = P_1P_2\ldots P_\mu$ and P_i have two by two no root in common, then there exist neighbourhoods U, U_1, ..., U_μ of P, P_1, ..., P_μ such that the product map $U_1\times\ldots\times U_\mu \to U$, $(Q_1,\ldots,Q_\mu)\mapsto Q$, is a diffeomorphism.*

This result implies that at any point of the open arc AO the swallowtail is locally diffeomorphic to two intersecting planes in $\mathbf{R}^3$. Indeed, on AO the polynomial P has two double real roots, hence, is of the form $(x-a)^2(x+a)^2$, $a \in \mathbf{R}$. One can set $P_1 = (x-a)^2$, $P_2 = (x+a)^2$ and $Q_1 = (x-a)^2+\alpha(x-a)+\beta$, $Q_2 = (x+a)^2-\alpha(x+a)+\gamma$. The two planes are defined respectively by the variables (α,β) and (α,γ).

Proposition 5 can be applied in the general case as well (i.e. when P has any MV) to understand what the set $\{(a_1,\ldots,a_n)\in\mathbf{R}^n|\mathrm{Res}(P,P')=0\}$ is locally like up to a diffeomorphism.

3. On the mutual disposition of strata. Consider a point A of a stratum U of dimension $s \leq n-2$. Intersect U by the affine space $\mathcal{F}$ of dimension 2 containing A and parallel to $Oa_{s+1}a_{s+2}$. By Theorem 2 the intersection is the point A. The intersections with $\mathcal{F}$ of the strata of dimension $s+1$ are curves containing A in their closures and having non-vertical limits at A of their tangent lines (Theorem 2). Each such curve (considered locally, at A) projects only "to the left" or "to the right" of A on Oa_{s+1}. The intersections with $\mathcal{F}$ of the strata of dimension $s+2$ are sectors delimited by these curves.

It is explained in [4], Subsection 1.2, how the above curves are situated near A in the case when U is a stratum of hyperbolic polynomials. We generalize here these results in the case of an arbitrary stratum U.

Denote by $\vec{v} = [r_1,\ldots,r_q]$ the MV of the stratum U. Denote by $U_{i,j}$ the stratum with MV obtained from $\vec{v}$ by replacing the component r_i by two components – j, $r_i - j$ – where $j = 1, \ldots, r_i - 1$. If $r_i = 2$, denote by V_i the stratum whose MV is obtained from $\vec{v}$ by deleting the component r_i. By part 4) of Remarks 1, these are all strata of dimension $s + 1$ adjacent to U. We use the notation $U_{i,j}$, V_i also for the intersections of these strata with $\mathcal{F}$. In what follows we assume that the equations of the limits at A of the tangent lines to the curves $U_{i,j}$ are given in the form $a_{s+2} = k_i a_{s+1} + \theta_i$. Here k_i is the slope of the limit of the tangent line.

Lemma 6. *The slopes of the limits at A of the tangent lines to the curves U_{i_1,j_1}, U_{i_2,j_2}, are the same for $i_1 = i_2$ and different for $i_1 \neq i_2$.*

The lemma is proved by full analogy with Lemma 16 from [4] (using Proposition 5).

Lemma 7. *If $r_i = 2$, then the tangent lines to $U_{i,1}$ and V_i are the same. The curves $U_{i,1}$, V_i and the point A are parts of one and the same curve smooth at A.*

Proof. In the particular case when $P = x^2 + \lambda$ the strata $U_{i,1}$ and V_i are the half-lines $\{\lambda < 0\}$ and $\{\lambda > 0\}$. In the general case the lemma is proved by analogy with Lemma 16 from [4] (using Proposition 5). □

Lemma 8. *For the slopes k_i of the limits at A of the tangent lines to the curves $U_{i,j}$ one has $k_1 > \ldots > k_q$.*

Proof. 1^0. In the case when the stratum U consists of hyperbolic polynomials the lemma is proved in [4], see Lemma 22 there. Suppose that U does not consist of hyperbolic polynomials. Denote by U' the stratum whose MV is obtained from $\vec{v}$ by adding to the right $n - l$ components equal to 1. Hence, U' consists of hyperbolic polynomials.

2^0. Choose a point $B \in U'$. Connect it with A by a continuous curve (parametrized by $\sigma \in [0, 1]$) passing only through strata with MVs whose first q components are the same as the ones of $\vec{v}$ and such that for $(n - l)/2$ distinct values σ_ν of σ the greatest two of the real roots become equal after which they form a complex conjugate couple. The slopes k_i can be defined for any $\sigma \in [0, 1]$. For different i they remain different throughout the deformation (even for $\sigma = \sigma_\nu$ which can be proved like Lemma 6). For all σ they are finite. Hence, their order is the same at A and at B. □

Lemma 9. *If i is even, then the projection on Oa_{s+1} of $U_{i,j}$ is "to the right" of the one of A; if i is odd, then "it is on its left".*

Proof. In the case when U consists of hyperbolic polynomials this is Lemma 18 from [4]. In the general case the lemma is proved like the previous one

– being "to the left or to the right" does not change throughout the deformation, because it depends continuously on σ, i.e. in fact it does not depend on σ. $\square$

Remark 10. It follows from Lemmas 7 and 9 that if $r_i = 2$ and if the projection of $U_{i,1}$ on Oa_{s+1} is "to the right" (resp. "to the left") of the one of A, then the projection of V_i on Oa_{s+1} is "to the left" (resp. "to the right") of the one of A.

Lemma 11. *For i fixed the curve U_{i,j_1} is "above" the curve U_{i,j_2} if and only if either i is odd and $j_1 > j_2$ or i is even and $j_1 < j_2$.*

Proof. If the stratum U consists of hyperbolic polynomials, then this is Lemma 20 from [4]. If not, then use the same deformation as in the proof of Lemma 8. For any value of σ and for any i fixed the curves $U_{i,j}$ have the same mutual disposition. Indeed, one can apply Proposition 5 – the curves $U_{i,j}$ with one and the same i correspond to one and the same neighbourhood U_i and to the respective curves constructed after the polynomial $(x-a)^{r_i}$. The latter's mutual disposition does not depend on σ or on a. $\square$

Remark 12. On Fig. 2 we show the curves $U_{i,j}$ and V_i in the case when $l = 10$, $n = 10 + 2h$, $h \in \mathbf{N}$. The reader can check the above lemmas on this example.

Remark 12. If for some i one has $r_i = r_{i+1} = 2$, then the MVs of the strata V_i and V_{i+1} are the same. Hence, this is one and the same stratum but it gives rise to two (or more) different curves V_i at A, with different slopes of their tangent lines at A, see Lemma 6). The simplest example of such a situation is the one of the previous section (the stratum defined by the MV [2] admitting two different limits of the tangent space along the curve AO).

4. Proof of Theorem 2. 1^0. Contractibility follows from the contractibility of the parameter space (the roots play the role of parameters and define the coefficients a_i via the Vieta formulas; these formulas define a homeomorphism).

Further we prove some statements of the theorem not for the stratification of $\mathbf{R}^n$, the space of the coefficients $a_i = (-1)^i\sigma_i$ (where σ_i is the i-th symmetric function of the roots of P counted with the multiplicities), but for the space of the Newton functions b_i which are the sums of the i-th powers of the roots. The statements formulated for the two spaces (of the quantities a_i or b_i) are equivalent because there exist polynomials q_j, q_j^* such that

$$ja_j = -nb_j + q_j(b_1, \ldots, b_{j-1}), \quad nb_j = -ja_j + q_j^*(a_1, \ldots, a_{j-1}).$$

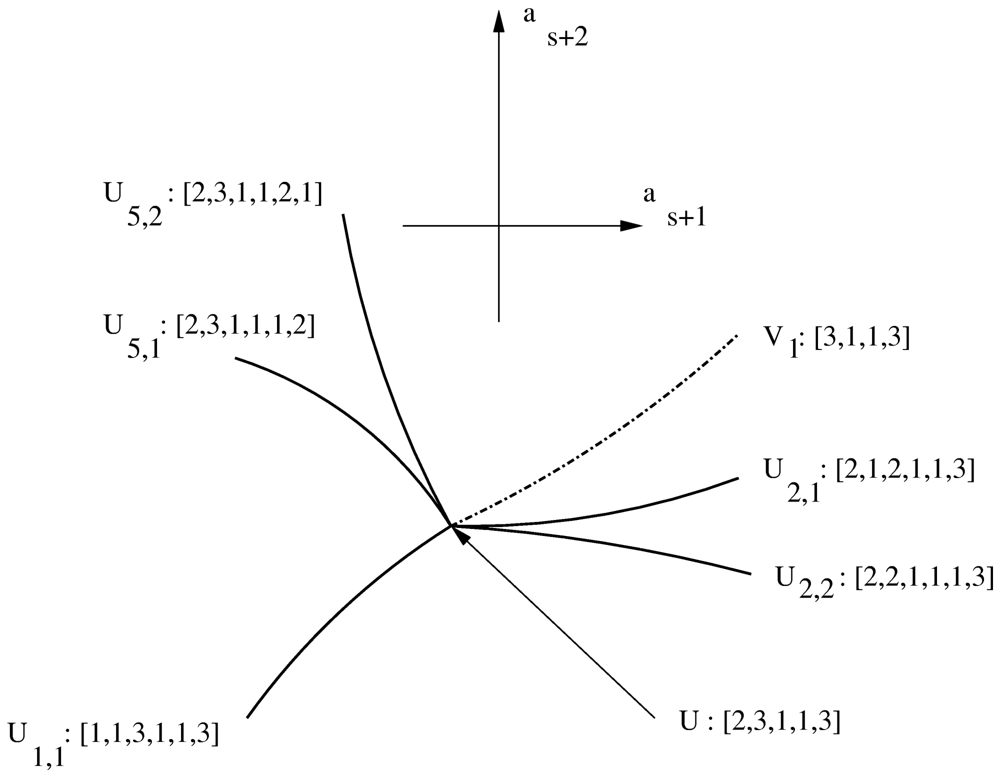

Fig. 2

2^0. Denote by $(x_1, \ldots, x_{n-r})$ the roots of P where the real roots are distinct while the complex ones might not be all distinct. The Jacobian matrix corresponding to the mapping $(x_1, \ldots, x_{n-r}) \mapsto (b_1, \ldots, b_{n-r})$ is obtained from the Vandermonde matrix $W(x_1, \ldots, x_{n-r})$ by multiplying the columns of the real roots by their respective multiplicities. Hence, if all complex roots are distinct, then the determinant J of the matrix W is nonzero and at such a point the mapping is a local diffeomorphism which means that the stratum is locally of dimension $n-r$. At such a point one can express the roots x_i as smooth functions of $(b_1, \ldots, b_{n-r})$, and then express $(b_{n-r+1}, \ldots, b_n)$ as smooth functions of x_i; hence, as smooth functions of $(b_1, \ldots, b_{n-r})$.

3^0. Prove the smoothness of the stratum regardless of whether the complex roots are all distinct or not. (We use the same ideas here as in [6], p. 52-53.)

To this end set $P=QR$, $Q=x^{2k}+c_1x^{2k-1}+\ldots+c_{2k}$, $R=x^{n-2k}+d_1x^{n-2k-1}+\ldots+d_{n-2k}$ where all roots of Q are complex and all roots of R are real. The mapping

$$(c_1,\ldots,c_{2k},d_1,\ldots,d_{n-2k})\mapsto(a_1,\ldots,a_n)$$

(where c_i, d_j are regarded as free parameters) is a local diffeomorphism. Indeed, its Jacobian matrix is the Sylvester matrix of Q and R; the latter's determinant equals $\mathrm{Res}(Q,R)$ which is nonzero because Q and R have no root in common.

4^0. *The field of tangent spaces to the given stratum is continuously extended to the strata of lower dimension belonging to its closure and to the points where some complex roots coincide. The extension is everywhere transversal to the space $Oa_{n-r+1}\ldots a_n$.*

The rest of the theorem follows from the statement and from Remark 14. The statement implies in particular that even in a neighbourhood of a point of the stratum where some complex roots coincide, the stratum is locally the graph of a smooth r-dimensional function defined on the projection of the stratum in $Oa_1\ldots a_{n-r}$, see 2^0-3^0.

5^0. To prove the statement from 4^0 compute the partial derivatives $\partial b_k/\partial b_u$, $k\geq n-r+1$, $u\leq n-r$ bearing in mind that b_j is the j-th Newton function of the roots x_i. (We follow here the same ideas as the ones used in the proof of Theorem 1.8 from [5].) Denote by m_i the quantity equal to the multiplicity of x_i if x_i is a real root and to 1 if it is a complex one. One has

$$\begin{aligned}\partial b_k/\partial b_u&=\sum_{i=1}^{n-r}(\partial b_k/\partial x_i)(\partial x_i/\partial b_u)\\&=k\sum_{i=1}^{n-r}(m_ix_i^{k-1})(\partial x_i/\partial b_u)=k\sum_{i=1}^{n-r}(m_ix_i^{k-1}A_{u,i})/w\end{aligned}\tag{1}$$

where $w=\det\|\partial b_j/\partial x_\nu\|=g\prod_{q<v}(x_q-x_v)$, $g\neq0$, and $A_{u,i}$ is the cofactor of the element $\partial b_u/\partial x_i$ in the matrix $\|\partial b_j/\partial x_\nu\|$.

6^0. Put $x_\mu=x_\nu$ in (1). Then for $i\neq\mu,\nu$ one has $A_{u,i}=0$ (two proportional columns). Suppose first that the roots x_μ, x_ν are complex. Then one has $m_\mu=m_\nu=1$ and $m_\mu A_{u,\mu}+m_\nu A_{u,\nu}=A_{u,\mu}+A_{u,\nu}=0$ (to be checked directly). This means that the numerator of the right hand-side of (1) is 0 when $x_\mu=x_\nu$, i.e. it is representable in the form $wh(x_1,\ldots,x_{n-r})$ for some polynomial h. Hence,

$$\partial b_k/\partial b_u=h\tag{2}$$

If the roots x_μ, x_ν are real, then one again checks directly that $m_\mu A_{u,\mu} + m_\nu A_{u,\nu} = 0$ and again one has (2).

7^0. The closure of the stratum can be defined by a continuous parametrization of the roots x_i by some parameters z (one can choose as such parameters part of the variables b_j; in general, these variables are more than the parameters needed). By (2), the partial derivatives $\partial b_k/\partial b_u$ are bounded continuous functions of the parameters z. Hence, the limits of these partial derivatives exist on the closure of the stratum. This proves the statement.

Remark 14. The limit of the tangent space of a stratum S when approaching a stratum T from its closure contains the tangent space of T. Indeed, it is sufficient to prove this only when $\nu :=\dim S =\dim T + 1$. In this case the couple (T, S) is locally diffeomorphic to $((\mathbf{R}^{\nu-1}, 0), \{(a, b)|\ a \in (\mathbf{R}^{\nu-1}, 0), b > 0\})$ and the claim is evident.

The theorem is proved. □

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