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ON THE EXTINCTION PROBABILITY FOR BISEXUAL BRANCHING PROCESSES IN VARYING ENVIRONMENTS*

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Abstract. In this paper, the bisexual branching process in varying environments introduced in [9] is considered and some sufficient conditions for the existence of positive probability of non-extinction are established.

1. Introduction. Recently, from the bisexual branching process (BP) introduced in [2], new bisexual branching models have been developed (see [5], [6], [7], [8] and [9]). In particular, in [9] a bisexual process which allows that

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the offspring probability distribution to be different in each generation has been defined and, for such a model, some necessary and sufficient conditions for its almost sure extinction have been established. In this paper, we continue the research about this bisexual branching process. In Section 2, we provide its mathematical description and some auxiliary definitions. Section 3 is devoted to investigate sufficient conditions for the existence of positive probability of non-extinction.

2. The probabilistic model. The bisexual branching process in varying environments (BPVE) is a two-type stochastic model $\{(F_n, M_n)\}_{n\geq 1}$ defined in the form:

$$(1) \qquad (F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n} (f_{ni}, m_{ni}), \; Z_{n+1} = L(F_{n+1}, M_{n+1}), \; n = 0, 1, \ldots$$

where the empty sum is considered to be $(0,0)$, $Z_0 = N \in \mathbb{Z}^+$, for every $n = 0, 1, \ldots$, $\{(f_{ni}, m_{ni})\}_{i\geq 1}$, is a sequence of i.i.d. non negative, integer valued random variables, and the mating function $L : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is assumed to be monotonic non-decreasing in each argument, integer-valued for integer-valued arguments and such that $L(x, y) \leq xy$. Intuitively, (f_{ni}, m_{ni}), represents the number of females and males produced by the i-th mating unit in the n-th generation, being $\{p_{jk}^{(n)}\}_{j,k\geq 0}$, the corresponding offspring probability distribution, namely $p_{jk}^{(n)} := P(f_{n1} = j, m_{n1} = k)$, $n = 0, 1, \ldots$. Thus, from (1), (F_{n+1}, M_{n+1}) will be the total number of females and males in the $(n+1)$-th generation. These females and males form $Z_{n+1} = L(F_{n+1}, M_{n+1})$ mating units, which reproduce independently.

Remark 1. This branching model describes reasonably the probabilistic evolution of two-sex dynamics population with sexual reproduction in which, for several kind of reasons (environmental, social or other), it is possible that the probability distribution associated to the reproduction changes in each generation. From (1), it can be easily proved that $\{(F_n, M_n)\}_{n\geq 1}$ and its associated sequence of mating units $\{Z_n\}_{n\geq 0}$ are Markov chains not necessarily homogeneous. This lack of homogeneity establishes an important difference with the standard BP introduced in [2] and it will play an crucial role in our study.

Definition 1. *A BPVE is said to be* superadditive *when its mating function L is superadditive, i.e. if, for $k = 2, 3, \dots,$ it verifies:*

$$L\left(\sum_{i=1}^{k}(x_i, y_i)\right) \geq \sum_{i=1}^{k} L(x_i, y_i), \quad x_i, y_i \in \mathbb{R}^+, \quad i = 1, \dots, k$$

Remark 2. The superadditivity is an intuitive and logic condition. Moreover, it is not a serious restriction, most of the mating functions considered in bisexual branching processes theory are superadditive.

Definition 2. *For a BPVE we introduce the* mean growth rates per mating unit *as:*

$$r_{nj} := j^{-1}E[Z_{n+1}|Z_n = j], \quad n = 0, 1, \dots; \ j = 1, 2, \dots$$

Remark 3. For a superadditve BPVE it is verified that $r_{n1} = \inf_{j\geq 1} r_{nj}$, $n = 0, 1, \dots$. In fact,

$$r_{nj} = j^{-1}E\left[L\left(\sum_{i=1}^{j}(f_{ni}, m_{ni})\right)\right] \geq j^{-1}\sum_{i=1}^{j} E[L(f_{ni}, m_{ni})] = r_{n1}, \quad j = 1, 2, \dots$$

Definition 3. *Given a BPVE, we define its associated asexual process in varying environments (APVE), denoted as $\{\widetilde{Z}_n\}_{n=0}^{\infty}$, in the form:*

$$\widetilde{Z}_0 = Z_0 = N \ , \quad \widetilde{Z}_{n+1} = \sum_{i=1}^{\widetilde{Z}_n} X_{ni} \ , \quad n = 0, 1, \dots$$

where $X_{ni} := L(f_{ni}, m_{ni})$.

Remark 4. Intuitively, the variable X_{ni} represents the number of mating units originated by the offspring of the i-th mating unit in the generation n.

3. The extinction probability. In this Section, we will consider a superadditive BPVE and applying some classical results from the asexual branching processes in varying environments theory (see [1], [3] and [4]) to its APVE we will determine sufficient conditions for the existence of a positive probability of non-extinction. We interpret that a BPVE becomes extinct when, from certain generation on, there are not mating units in the population. Let us denote by $q_N := P(Z_n \to 0 \mid Z_0 = N)$, the extinction probability when the process starts with N mating units, $N = 1, 2, \ldots$. Let us also consider the functions

$$h_n(s) := E[s^{Z_n}], \quad g_n(s) := E[s^{X_{n1}}], \quad 0 \le s \le 1, \quad n = 0, 1, \ldots$$

i.e. the probability generating function (p.g.f.) of Z_n and X_{n1}, respectively. Since $Z_0 = N$, it is clear that $h_0(s) = s^N$, $0 \le s \le 1$.

Remark 5. For a superadditive BPVE, it has been proved in [9] the inequality:

$$h_n(s) \le ((g_0 \circ \cdots \circ g_{n-1})(s))^N, \quad 0 \le s \le 1 \quad n = 1, 2, \ldots \tag{2}$$

Theorem 1. *If it is verified that:*

(*i*) $g''_n(1) < \infty$, $n = 0, 1, \ldots$ *and* $\inf\limits_{n \ge n_0} r_{n1}^{-1} g''_n(1) > 0$ *for some* $n_0 \ge 0$.

(*ii*) $\sum_{n=0}^{\infty} E[\widetilde{Z}_n]^{-1} < \infty$.

then $q_N < 1$, $N = 1, 2, \ldots$

P r o o f. It will be sufficient to prove the result for $N = 1$. Note that $q_1 = \lim\limits_{n \to \infty} h_n(0)$.

Now, in [1] it was proved that for any p.g.f. g_n such that $g''_n(1) < \infty$, it is verified that

$$g_n(s) \le \phi_n(s), \quad 0 \le s \le 1, \quad n = 0, 1, \ldots \tag{3}$$

where ϕ_n is a fractional linear generating function(1) such that $\phi'_n(1) = g'_n(1)$ and $\phi''_n(1) = 2g''_n(1)$.

(1) A fractional linear generating function is a function which can be written in the following form $\phi(s) = 1 - (1 - c)^{-1} b + (1 - cs)^{-1} bs$, where b and c are non negative constants such that $b + c \le 1$.

By (3), and taking into account that g_k and ϕ_k are monotone increasing functions, we obtain that

$$(g_0 \circ \cdots \circ g_{n-1})(s) \leq (\phi_0 \circ \cdots \circ \phi_{n-1})(s) \tag{4}$$

Now, it is verified that

$$(\phi_0 \circ \cdots \circ \phi_{n-1})(s) = 1 - \left[\frac{1}{(1-s)E[\widetilde{Z}_n]} + \sum_{j=0}^{n-1} \frac{g_j''(1)}{r_{j1}E[\widetilde{Z}_{j+1}]}\right]^{-1} \tag{5}$$

and consequently, from (2), (4) and (5), we have that

$$h_n(0) \leq 1 - \left[\frac{1}{E[\widetilde{Z}_n]} + \sum_{j=0}^{n-1} \frac{g_j''(1)}{r_{j1}E[\widetilde{Z}_{j+1}]}\right]^{-1} \tag{6}$$

Finally, taking limit as $n \to \infty$ and considering conditions (i) and (ii) we deduce that $q_1 = \lim_{n\to\infty} h_n(0) < 1$ and this complete the proof. □

Theorem 2. *Suppose that*

(i) $r_{n1} \geq b\,\mathrm{Var}[X_{n1}],\ \ n = 1, 2, \ldots$ *for some* $b > 0$.

(ii) $r_1 := \lim_{n\to\infty} r_{n1}$ *exists and* $r_1 > 1$.

then it is verified that $q_N < 1, \quad N = 1, 2, \ldots.$

Proof. Consider again $N = 1$. From (6), we have that

$$1 - h_n(0) \geq \left[\frac{1}{E[\widetilde{Z}_n]} + \sum_{j=0}^{n-1} \frac{g_j''(1)}{r_{j1}E[\widetilde{Z}_{j+1}]}\right]^{-1} = \left[1 + \sum_{j=0}^{n-1} \frac{\mathrm{Var}[X_{j1}]}{r_{j1}E[\widetilde{Z}_{j+1}]}\right]^{-1}$$

and by using that $r_{n1} \geq b\,\mathrm{Var}[X_{n1}],\ \ n = 1, 2, \ldots$, we obtain that

$$1 - h_n(0) \geq \left[1 + \frac{1}{b}\sum_{j=0}^{n-1} \frac{1}{E[\widetilde{Z}_{j+1}]}\right]^{-1} \tag{7}$$

Now, since $E[\widetilde{Z}_{n+1}] = E[\widetilde{Z}_n]E[X_{n1}]$, we deduce that

$$\lim_{n\to\infty} E[\widetilde{Z}_n]E[\widetilde{Z}_{n+1}]^{-1} = \lim_{n\to\infty} E[X_{n1}]^{-1} = \lim_{n\to\infty} r_{n1}^{-1} = r_1^{-1} < 1$$

and therefore $\sum_{n=1}^{\infty} E[\widetilde{Z}_n]^{-1} < \infty$. Hence, taking limit as $n \to \infty$ in (7), we derive that $1 - q_1 > 0$. $\square$

Remark 6. In the previous theorems it have been necessary to consider that the variables X_{n1} have finite variance. In the next result this assumption is not required.

Theorem 3. *Suppose that*

(*i*) $\prod_{j=0}^{n-1} r_{j1} \geq Ac^n$, $\quad n = 0, 1, \ldots$, *for some constants* $A > 0$ *and* $c > 1$.

(*ii*) *The sequence* $\{r_{n1}^{-1} X_{n1}\}_{n\geq 0}$ *is stochastically smaller*[(2)] *than* X, *where* X *is a r.v. with* $E[X] < \infty$.

then it is verified that $q_N < 1$, $\quad N = 1, 2, \ldots$.

Proof. We proved in previous theorem that

$$1 - h_n(0) \geq \left[1 + \sum_{j=0}^{n-1} \frac{\mathrm{Var}[X_{j1}]}{r_{j1} E[\widetilde{Z}_{j+1}]}\right]^{-1} \tag{8}$$

The idea is to use this inequality to show that $1 - h_n(0) = P(Z_n > 0)$ is bounded below. However to cope with the possibility that $\mathrm{Var}[X_{j1}]$ may be infinite we must use a truncation procedure.

Let $d \in (1, c)$ and b such that $E[X 1_{\{X \geq b\}}] \leq \varepsilon$, where $\varepsilon = (c - d)c^{-1}$ and 1_S denotes the indicator function of the set S.

Let $B \geq b$ and we define the truncated variables

$$X_{n1}^{(B)} = X_{n1} 1_{\{X_{n1} < r_{n1} B\}}, \quad \widetilde{Z}_n^{(B)} = \widetilde{Z}_n 1_{\{X_{n1} < r_{n1} B\}}, \quad n = 0, 1, \ldots$$

Taking into account the conditions (*i*) and (*ii*) in theorem, it is matter of some straightforward calculation to obtain that

$$E[X_{n1}^{(B)}] \geq r_{n1}(1 - \varepsilon), \quad \mathrm{Var}[X_{n1}^{(B)}] \leq r_{n1}^2 B^2$$

and

$$E[\widetilde{Z}_n^{(B)}] \geq E[\widetilde{Z}_n](1 - \varepsilon)^n \geq Ac^n (1 - \varepsilon)^n = Ad^n$$

(2) Given the random variables Y and X, we say that Y is stochastically smaller than X if for all $u \in \mathbb{R}$, $P(X \leq u) \leq P(Y \leq u)$.

So, from (8), we derive that

$$1 - q_1 \geq \left[1 + \sum_{j=0}^{\infty} \frac{r_{j1}^2 B^2}{r_{j1}^2 (1-\varepsilon)^2 A d^j}\right]^{-1} = \left[1 + \frac{B^2}{A(1-\varepsilon)^2} \frac{d}{d-1}\right]^{-1} > 0$$

and this conclude the proof. □

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