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AN APPROACH TO WEALTH MODELLING

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Abstract. The change in the wealth of a market agent (an investor, a company, a bank etc.) in an economy is a popular topic in finance. In this paper, we propose a general stochastic model describing the wealth process and give some of its properties and special cases. A result regarding the probability of default within the framework of the model is also offered.

1. Introduction.

Different models describing the wealth changes of an entity in an economy (an investor, a company, a bank etc.) have been used over the past decades in mathematical finance.

The foundations of modern financial investigations were laid in 1959 by Markowitz [26] and Tobin [34] who created the Capital Asset Pricing Model (CAPM), further developed by Sharpe [31]. The model is based on the so called *mean-deviation approach* which is used now in banks and for which Markowitz won a Nobel prize.

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In 1973, Black and Scholes [2] proposed a model which today is considered a fundamental financial model. The key insight of this model is that, under some conditions, the absence of arbitrage suffices to derive unique prices for European options on stocks.

The Black-Scholes model is reformulated in terms of the semimartingale theory by Harrison and Pliska [10], who apply the general theory of stochastic processes to financial modelling.

Many papers are devoted to derivative pricing and hedging in incomplete markets. Some of them are based on a general equilibrium framework [6].

Different kinds of probabilistic models are used to describe the underlyings in the markets, for example jump diffusions, bivariate diffusions, discrete ARCH time series etc.

Since the redistribution of capital is an essential action of the investor, transaction costs can not be ignored. This problem is considered by Magill and Constantinides [25], Davis and Norman [4], Soner and Shreve [32].

It is clear from the preceding description that there is a need to systematize and generalize the existing models in the form of a single, simple easy to modify model.

Here we consider a class of models which is an attempt to satisfy this need. Most of the modern financial models outline the aggregate of stochastic processes used presently for investigation in finance. This is the area of intersection of three quite general classes of stochastic processes - Markov processes, martingales and processes with stationary increments. Specifically, some special kinds of random walks with properties analogous to the properties of the martingales belong to this area.These processes are the semimartingales, with some of their generalizations and special cases. In this paper we consider and explore a class of processes called *generalized Lévy processes* which coincide very closely with the mentioned group processes. So, we can also call them *financial processes.* By using the generalized Lévy processes we introduce a class of models describing wealth changes called *Wealth Motion Models.*

In the second part of the paper we consider the credit risk in the framework of the proposed model. A key point here is the determination of the default probabilities. Presently, there are two approaches to this problem: the structural approach and the intensity-based approach.

In the first approach, pioneered by Merton [28], the default time is a stopping time in the filtration of the prices. Therefore the evaluation of the wealth with possible default is reduced to the problem of the pricing of a defaultable claim which is measurable with respect to the filtration of the prices. This is a standard, though difficult, problem which is reduced – in a complete market case – to the computation of the expectation of the discounted payoff under the risk neutral probability measure. Main papers to be quoted here are [2, 3, 8, 35, 5, 27].

In the second method, the aim is also to compute the value of a defaultable claim; however it could be that this claim is not measurable with respect to the σ-algebra generated by prices. In this case, it is generally assumed that the market is complete for a larger filtration, which means that the defaultable claim is hedgeable. In order to compute the expectation of wealth under risk-neutral probability, it is convenient to introduce the notion of the intensity of the default. Then, under some assumptions, the intensity of the default time acts as a change of the spot interest rate in the pricing formula. This more recent approach than the structural one is also known as the *reduced-form approach*, and has been introduced in [15, 19, 20, 7]. New contributions are [11, 12, 1, 30, 22, 23].

In this paper we follow the structural approach. Within the framework of the *Homogeneous General Wealth Motion Model* we consider a special kind of wealth process called *Brownian motion with returns to zero* for which we formulate and prove a result regarding the above boundary for the probability of default.

2. Wealth motion models. Here we will introduce a class of models denoted "Wealth Motion Models". We consider an entity in an economy (a company, a bank, an investor etc.). This entity we will call *an investor*. He starts his life with an initial capital u_0 and, from that moment on, his wealth changes depending on the income and costs.

The processes considered below are all defined in a filtered probability space (Ω, A, F, P) and are adapted to the filtration F. Here we give some definitions from the theory of stochastic processes.

Definition 2.1. *A predictable sigma-algebra A^{Pr} in the space $\Omega\times[0,\infty)$ is a sigma-algebra with respect to which any càdlàg process is measurable.*

Definition 2.2. *For any R^d-valued càdlàg stochastic process $X(t)$ the random measure of jumps μ^X is defined by*

$$\mu^X(\omega; dt, dx) = \sum_s 1_{\{R^d\setminus\{0\}\}}(\Delta X(\omega, s))\epsilon_{(s,\Delta X(\omega,s))}(dt, dx), \tag{1}$$

where $\Delta X(s,\omega) = X(s) - X(s-)$ is the jump of $X(t)$ at time s and ϵ_P is the Dirac measure in point P.

The following model represents the changes in the capital held by an economic entity.

$$U(t;u_0) = u_0 + R(t) - C(t). \tag{2}$$

Here $U(t;u_0)$ is the capital held by the entity at time t; $u_0 = U(0;u_0)$ is the initial capital; $R(t) \geq 0$ is the stochastic income ($R(0) = 0$); $C(t) \geq 0$ are the stochastic costs ($C(0) = 0$).

For $R(t)$ and $C(t)$ we suppose that they are adapted to F. It follows that $U(t;u_0)$ is adapted too.

The time t may be considered to be either a continuous variable ($t \in R^+$) or a discrete variable ($t \in N$), where R^+ is the set of non-negative real numbers ($R^+ = [0,+\infty)$) and N is the set of non-negative integers ($N \in \{0,1,\ldots\}$). So, the model may be considered as either a continuous or discrete model.

So far, we have not made any assumptions about the capital structure, i. e. about the way the assets on the financial instruments are distributed. Since, we will call the model (2) *Homogeneous General Wealth Motion Model – HGWMM.*

Now, we consider the financial instruments $I_0, I_1, \ldots, I_m$ traded in a financial market M.

Definition 2.3. *The vector process $P_M(t) = (P_0(t), P_1(t), \ldots, P_m(t))$ where $P_k(t)$ ($k = 0,1,\ldots,m$) is the price of the instrument I_k at time t, is called vector of the prices.*

Definition 2.4. A process of dividends *is a vector process $D_M(t) = (D_0(t), D_1(t), \ldots, D_m(t))$ where $D_k(0) = 0$ ($k = 0,1,\ldots,m$).*

The dividends process models income (expenditures) which is the result of *special events*, dividends from financial instruments, and transaction costs. The special events are the kind of events which occur relatively rarely in a system but have considerable influence on the system's behaviour. The special events are related to jumps, cataclysms and sharp changes in the system under consideration.

Definition 2.5. A vector of generalized prices *is a vector $G_M(t) = (G_0(t), G_1(t), \ldots, G_m(t))$ where $G_M(t) = P_M(t) + D_M(t)$.*

Definition 2.6. *The vector $A_M(t) = (A_0(t), A_1(t), \ldots, A_m(t))$, where $A_k(t)$ ($k = 0,1,\ldots,m$) is the quantity of the instrument I_k held by the investor at time t, is called absolute portfolio in quantities. $A_k(t) = 0$ in the case where there is not any quantity of the instrument I_k held in the portfolio.*

Definition 2.7. *The process $V_M(t) = \sum_{k=0}^{m} A_k(t)P_k(t)$ is called* value of the portfolio at time t *for the market M.*

Below we omit the subscript M in the notations.

On the market, the capital is distributed between the traded financial instruments. For obtaining a structured wealth motion model, we make the following assumptions:

1) In the beginning all the wealth is allocated between the financial instruments traded on the market. So, $u_0 = V(0) = P(0)A(0)$.

2) At any given time in the future we have

$$U(t; u_0) = A(t)G(t) = A(t)(P(t) + D(t)), \tag{3}$$

where $G(t) = P(t) + D(t)$ is the generalized market price for the capital.

For the price we suppose

$$P(t) = P^c(t) + P^d(t) + P^f(t). \tag{4}$$

Here $P^c(t)$ is a continuous local martingale which typically models changes in the capital, related to market fluctuations of real prices. The process $P^d(t)$ is a discontinued local martingale which typically models changes in the capital, related to sharpe changes in the real prices. The process $P^f(t)$ is a process with finite variation which typically models changes of the capital, related to market trends of the prices. At time 0 we have $P^c(t) = P^d(t) = 0$.

For the dividends we suppose

$$D(t) = D^c(t) + D^d(t) + D^f(t). \tag{5}$$

Here $D^c(t)$ is a continuous local martingale which typically models changes in the capital, caused by dividends and taxes, proportional to the prices of the financial instruments. The process $D^d(t)$ is a discontinued local martingale which typically models changes of the capital, related to special events, extraordinary income or loss. The process $D^f(t)$ is a process with finite variation which typically models changes of the capital, related to transaction costs. At time 0 we have $D^c(0) = D^d(0) = D^f(0) = 0$.

As a result, for the process of generalized prices we have

$$G(t) = G^c(t) + G^d(t) + G^f(t), \tag{6}$$

where $G^c(t) = P^c(t) + D^c(t)$ is the continuous part of the generalized prices, $G^d(t) = P^d(t) + D^d(t)$ is the discontinued part, and $G^f(t) = P^f(t) + D^f(t)$ is the part with finite variation.

Finally, we obtain

$$U(t; u_0) = A(t)G(t) = A(t)(G^c(t) + G^d(t) + G^f(t)). \tag{7}$$

The model defined by (7) we will call *Structured General Wealth Motion Model – SGWMM*. It may be continuous or discrete. The models introduced by us (2, 7) we will call generally *Wealth Motion Models – WMM.* Their discrete versions we will call *Discrete Wealth Motion Models – DWMM.* Their continuous versions we will call *Continuous Wealth Motion Models – CWMM.* The concept of WMM is originally introduced in [33]. Below we will talk about a Wealth Motion Model having in mind the entire family of related models.

The next considerations will be made in the context of WMM. WMM is a maximally simple, feasible and flexible model. It has many special cases with specific fields of application. On the other hand, it is a generalization of many common enough models in finance.

It is clear that the process of generalized prices is a semimartingale as a sum of semimartingales – see for example [29].

Below we will narrow the class of the semimartingales which may be used for modelling the process of the generalized prices. We will consider the following type of processes.

Definition 2.8. A generalized Lévy process (a financial process) *is a process $X(t)$ with the following properties*

1) *$X(t)$ may be represented in the form*

$$X(t) = X^c(t) + X^d(t) + X^f(t),$$

where $X^c(t)$ is a continuous local martingale, $X^d(t)$ is a discontinued local martingale, and $X^f(t)$ is a process with finite variation. For the process $X^f(t)$ we have

$$X^f(t) = \int_0^t x_0^f(s)ds + \int_0^t x_1^f(s)dN^{\Theta}(s) + \int_0^t x_2^f(s)dN(s), \tag{8}$$

where

1.1.) *The processes $\{x_i^f(t);\ t \geq 0\}$ are processes with values in the d-dimentional Euclidian space R^d ($i = 1, 2, 3$); for x_2^f we assume that it is stationary with distribution $q^{x_2^f}$;*

1.2.) *The process $N^{\Theta}(t)$ where $\Theta \subset R^+$ is a deterministic process with values $N^{\Theta}(t) = 1$ for $t \in \Theta$ and $N^{\Theta}(t) = 0$ outside of Θ where Θ is a discrete set. For the second integral in* (8) *taken on that process we have*

$$\int_0^t x_1^f(s)dN^{\Theta}(s) = \sum_{s\in\Theta \bigcap [0,t]} x_1^f(s);$$

1.3.) *The process $N(s)$ is a standard Poisson process with parameter λ.*

For the third integral in (8) *taken on that process we have*

$$\int_0^t x_2^f(s)dN(s) = \sum_{T_i<t} x_2^f(T_i),$$

where T_i are the moments of jumps of the process $N(t)$.

The assumptions made about the process $X^f(t)$ mean that this process contains an absolutely continuous part and jumps. Some of the jumps are at deterministic times, defined by the set Θ, and others are at the random times of jumps of the standard Poisson process.

2) *For the continuous process $C(t)$ with values in $R^d \times R^d$, given by $C_{ij}(t) =< X_i^c, X_j^c >$ for any $i, j \in \{1, 2, \ldots, d\}$ holds*

$$C(t) = \int_0^t c(s)ds,$$

where $c(t)$ is a predictable process with values in $R^d \times R^d$ which are symmetric, non-negative definite matrices.

3) *There exists a specific transition kernel Z from $(\Omega \times R^+, A^{Pr})$ into (R^d, B^d) such that*

$$\nu([0,t] \times G) = \int_0^t Z(\omega, s, G)ds + \sum_{s\in\Theta\bigcap[0,t]} \nu(\{s\} \times G) \tag{9}$$

for any $G \in B^d$. Here ν is the compensator of the random measure μ^X of the jumps of X.

4) *There exists a transition kernel K from $(\Omega\times[0,+\infty), A^{Pr})$ into (R^d, B^d) such that*

$$K(\omega, t, G) = \nu(\{t\} \times G) + \epsilon_0(G)(1 - \nu(\{t\} \times R^d));\ t \in \Theta \tag{10}$$

and $K(\omega, t, G) = 0$ else. For almost all $\omega \in \Omega$ with respect to P and for any $t \in \Theta$ we have that $K(\omega, t, G)$ is a probability measure. For any $t \in \Theta$ we have that $x_1^f(t) = \int xK(\omega, t, dx)$ almost certainly with respect to P. For any $t \in \Theta$ and $G \in B^d$ we have

$$q^{(x_1^f(t)|F_{t-})} = K(\omega, t, G)$$

almost certainly with respect to P. Here $q^{(x_1^f(t)|F_{t-})}$ is the conditional distribution of the random variable $x_1^f(t)$ with respect to F_{t-}.

Definition 2.9. *A* characteristic *of the generalized Lévy process, defined by the above definition is an ordered nineplet*

$$(\Theta, \lambda, q^0, x_0^f, x_1^f, x_2^f, c, Z, K),$$

where Θ, λ, x_0^f, x_1^f, x_2^f, c, Z *and* K *are as in* Definition 2.8., *and* q^0 *is a probabilistic measure in* R^d *giving the initial distribution of* $X(0)$.

Here we will consider some simple examples for the generalized Lévy process to make their nature clearer.

1. *A Poisson process with parameter* λ.

The Poisson process $N(t)$ is a very special case of the generalized Lévy process.

Its representation according to the definition is:

$$N(t) = N^c(t) + N^d(t) + N^f(t) = 0 + (N(t) - \lambda t) + \lambda t.$$

It is clear that the continuous local martingale $N^c(t) = 0$, the discontinued local martingale is $N^d(t) = N(t) - \lambda t$, and the process with finite variation is λt. Here we use the fact that λt is the compensator of $N(t)$, i. e. $N(t) - \lambda t$ is a (discontinued) local martingale.

If we consider the part with finite variation, $N^f(t)$, we have

$$N^f(t) = \int_0^t x_0^f(s)ds + \int_0^t x_1^f(s)dN^\Theta(s) + \int_0^t x_2^f(s)dN(s) = \lambda t.$$

It is clear that $x_1^f(t) = x_2^f(t) = 0 = \text{const}$ and $x_0^f(t) = \lambda$.

Another representation of the Poisson process according to its definition is:

$$N(t) = N^c(t) + N^d(t) + N^f(t) = 0 + 0 + N(t).$$

Here $N^c(t) = N^d(t) = 0$ and $N^f(t) = N(t)$, i. e. the process with finite variation is the very Poisson process.

For the part with finite variation $N^f(t)$ in this case we have

$$N^f(t) = \int_0^t x_0^f(s)ds + \int_0^t x_1^f(s)dN^\Theta(s) + \int_0^t x_2^f(s)dN(s) =$$

$$= \int_0^t x_2^f(s)dN(s) = \sum_{T_i<t} 1.$$

Here $x_0^f(t) = x_1^f(t) = 0 = \text{const}$ and $x_2^f(t) = 1 = \text{const}$.

It is clear from this example that the representation as well as the characteristic of the Lévy process is not uniquely defined.

2. *A Poisson process as a sum of two independent Poisson processes.*

If we present $N(t)$ as a sum of two Poisson processes, i. e.

$$N(t) = N_1(t) + N_2(t),$$

where $N_1(t)$ is with parameter λ_1, and $N_2(t)$ is with parameter λ_2 ($\lambda_i > 0$ for $i = 1, 2$ and $\lambda_1 + \lambda_2 = \lambda$), we obtain the representation:

$$N(t) = N^c(t) + N^d(t) + N^f(t) = 0 + (N_1(t) - \lambda_1 t) + (\lambda_1 t + N_2(t)).$$

Again $N^c(t) = 0$, the discontinued local martingale is $N^d(t) = N_1(t) - \lambda_1 t$, and the process with finite variation is $\lambda_1 t + N_2(t)$. Here we use $\lambda_1 t$ as the compensator of $N_1(t)$, i. e. $N_1(t) - \lambda_1 t$ is a (discontinued) local martingale.

Then, for the part with finite variation $N^f(t)$ we have

$$N^f(t) = \int_0^t x_0^f(s)ds + \int_0^t x_1^f(s)dN^\Theta(s) + \int_0^t x_2^f(s)dN(s) =$$

$$= \int_0^t \lambda_1 ds + 0 + \int_0^t 1 dN_2(s) = \lambda_1 t + N_2(t).$$

It is clear that $x_0^f(t) = \lambda_1 = \text{const}$, $x_1^f(t) = 0 = \text{const}$, and $x_2^f(t) = 1 = \text{const}$.

The non-uniqueness found in the characteristic of the generalized Lévy process can be resumed in the next:

Theorem 2.1. *Let the generalized Lévy process have characteristic:*

$$(\Theta, \lambda, q^0, x_0^f, x_1^f, x_2^f = 1, c, Z, K).$$

Then the same process admits the characteristic:

$$(\Theta, \lambda - \lambda_0, q^0, x_0^f + \lambda_0, x_1^f, x_2^f = 1, c, Z, K)$$

for $0 < \lambda_0 < \lambda$ *and in the special case when* $\lambda_0 = \lambda$ – *characteristic*

$$(\Theta, 0, q^0, x_0^f + \lambda, x_1^f, 0, c, Z, K).$$

Proof. For the part with finite variation we have

$$X^f(t) = \int_0^t x_0^f(s)ds + \int_0^t x_1^f(s)dN^\Theta(s) + \int_0^t x_2^f(s)dN(s) = \tag{11}$$

$$= \int_0^t x_0^f(s)ds + \int_0^t x_1^f(s)dN^\Theta(s) + N(t) =$$

$$= \int_0^t x_0^f(s)ds + \int_0^t x_1^f(s)dN^\Theta(s) + N_0(t) - \lambda_0 t + \lambda_0 t + N_1(t),$$

where $N_0(t)$ is a Poisson process with parameter λ_0, and $N_1(t)$ – the Poisson process with parameter $\lambda_1 = \lambda - \lambda_0$. Then $N_0(t) - \lambda_0 t$ belongs to the discontinued local-martingale part of the process, and the new process, giving the jumps of the part with finite variation, is the process $N_1(t)$ with parameter $\lambda_1 = \lambda - \lambda_0$. At $\lambda = \lambda_0$ it is clear that $N_1(t)$ completely disappears. $\square$

In the case $\lambda = \lambda_0$, when the part with finite variation has no jumps at random times, the characteristic of the generalized Lévy process is equivalent to the characteristic

$$(\Theta, q^0, x_0^f, c, Z, K),$$

given by Kallsen [16] for the so called generalized Grigellionis processes, a concept which is very close to the concept of generalized Lévy processes.

The parametrization given here is more modifiable because it permits the separation of a term of the Poisson's part of the process in order to play a more significant role, i.e. the connection to other processes, and more generally the fluctuations explained or generated by the environment.

The parameter λ (the second parameter in the characteristic of the generalized Lévy process) is called *parameter of adaptation to the environment* or *intensity of exterior influence.*

Definition 2.10. A Special Structured General Wealth Motion Model (SSGWMM) *is an ordered pair of financial processes (generalized Lévy processes)* $(P(t), D(t))$ *with characteristic*

$$C_P = (\Theta_P, \lambda, q_P^0, x_{0P}^f, x_{1P}^f, x_{2P}^f, c_P, Z_P, K_P)$$

(for $P(t)$) and

$$C_D = (\Theta_D, \lambda, q_D^0, x_{0D}^f, x_{1D}^f, x_{2D}^f, c_D, Z_D, K_D)$$

(for $D(t)$). Here for $P(t)$ and $D(t)$ we have one and the same standard Poisson process $N^\lambda(t)$ defining the random times of the jumps, and q_P^0 and q_D^0 are independent distributions. Also, we suppose that for the continuous local-martingale parts of $D(t)$ and $P(t)$, i. e. $P^c(t)$ and $D^c(t)$, holds

$$D^c(t) = bP^c(t)$$

where b is a constant, i. e. $D_i^c(t) = bP_i^c(t)$ for any $i = 0, \dots, d$.

Theorem 2.2. *The process of generalized prices $G(t)$ in SSGWMM is a generalized Lévy process with characteristic*

$$(\Theta_P \bigcup \Theta_D, \lambda, q_P^0 * q_D^0, x_{0P}^f + x_{0D}^f,$$

$$x_{1P}^f 1_{\{s\in\Theta_P\}} + x_{1D}^f 1_{\{s\in\Theta_D\}}, x_{2P}^f + x_{2D}^f, (1+b)^2 c_P, Z_G, K_G)$$

where for K_G we have

$$K_G(\omega, t, G) = 1_{\{t\in\Theta_P \bigcap \overline{\Theta_D}\}} K_P(\omega, t, G) + 1_{\{t\in\Theta_D \bigcap \overline{\Theta_P}\}} K_D(\omega, t, G) +$$

$$+ 1_{\{t\in\Theta_P \bigcap \Theta_D\}} K_P * K_D + 1_{\{t\in\overline{\Theta_P \bigcup \Theta_D}\}} 0,$$

for Z_G we have

$$Z_G = Z_P + Z_D,$$

and $Q * R$ is the convolution of the probability measures (distributions) Q and R.

P r o o f. For the process of the generalized prices we have:

$$G(t) = (P^c(t) + D^c(t)) + (P^d(t) + D^d(t)) + (P^f(t) + D^f(t)).$$

For $G^f(t) = P^f(t) + D^f(t)$ we have

$$P^f(t) = \int_0^t x^f_{0P}(s)ds + \int_0^t x^f_{1P}(s)dN^{\Theta_P}(s) + \int_0^t x^f_{2P}(s)dN^{\lambda}(s),$$

$$D^f(t) = \int_0^t x^f_{0D}(s)ds + \int_0^t x^f_{1D}(s)dN^{\Theta_D}(s) + \int_0^t x^f_{2D}(s)dN^{\lambda}(s),$$

and

$$G^f(t) = \int_0^t x^f_{0G}(s)ds +$$

$$+ \int_0^t x^f_{1G}(s)dN^{\Theta_G}(s) +$$

$$+ \int_0^t x^f_{2G}(s)dN^{\lambda}(s).$$

Here

$$x^f_{0G}(s) = x^f_{0P}(s) + x^f_{0D}(s)$$

and

$$x^f_{1G}(s) = x^f_{1P}(s)1_{\{s\in\Theta_P\}} + x^f_{1D}(s)1_{\{s\in\Theta_D\}}.$$

The process $N^{\Theta_G}(s)$ is different from zero at the points $\Theta_P \bigcup \Theta_D$, i. e. $\Theta_G = \Theta_P \bigcup \Theta_D$. We also have

$$x^f_{2G}(s) = x^f_{2P}(s) + x^f_{2D}(s). \tag{12}$$

The equation (12) holds, because the jumps follow the one and the same standard Poisson process $N^{\lambda}(t)$ for the prices $P(t)$ and $D(t)$.

For the initial distribution (the distribution of $G(0)$) we have that it is a distribution of the sum of two independent random variables, i. e. the convolution $q_P^0 * q_D^0$ of q_P^0 and q_D^0.

Let $0 = t_0^n \le t_1^n \le \ldots \le t_i^n \le \ldots \le t_{k_n}^n$ be a sequence σ_n of stopping times. Let this sequence tend as $n \to \infty$ to the identity, i.e.

$$\lim_{n\to\infty} \sup_k t_k^n = \infty$$

almost certainly and

$$\| \sigma_n \| = \sup_k \mid t_{k+1}^n - t_k^n \mid \to 0$$

almost certainly. Then for the quadratic covariation of the components $G_i^c(t)$ and $G_j^c(t)$ of the process of the generalized prices $G(t)$ we have

$$\langle G_i^c(t), G_j^c(t)\rangle = \langle P_i^c(t) + D_i^c(t), P_j^c(t) + D_j^c(t)\rangle =$$

$$= (P_i^c(0) + D_i^c(0))(P_j^c(0) + D_j^c(0)) +$$

$$+ \lim_{n\to\infty} \sum_k ((P_i^c + D_i^c)(\min(t, t_{k+1}^n)) - (P_i^c + D_i^c)(\min(t, t_k^n)))\times$$

$$\times((P_j^c + D_j^c)(\min(t, t_{k+1}^n)) - (P_j^c + D_j^c)(\min(t, t_k^n))) =$$

$$= P_i^c(0)P_j^c(0) + \lim_{n\to\infty} \sum_k ((P_i^c(\min(t, t_{k+1}^n)) - P_i^c(\min(t, t_k^n)))\times$$

$$\times(P_j^c(\min(t, t_{k+1}^n)) - P_j^c(\min(t, t_k^n))))+$$

$$+D_i^c(0)D_j^c(0) + \lim_{n\to\infty} \sum_k ((D_i^c(\min(t, t_{k+1}^n)) - D_i^c(\min(t, t_k^n)))\times$$

$$\times(D_j^c(\min(t, t_{k+1}^n)) - D_j^c(\min(t, t_k^n))))+$$

$$+P_i^c(0)D_j^c(0) + \lim_{n\to\infty} \sum_k ((P_i^c(\min(t, t_{k+1}^n)) - P_i^c(\min(t, t_k^n)))\times$$

$$\times(D_j^c(\min(t, t_{k+1}^n)) - D_j^c(\min(t, t_k^n))))+$$

$$+D_i^c(0)P_j^c(0) + \lim_{n\to\infty} \sum_k ((D_i^c(\min(t, t_{k+1}^n)) - D_i^c(\min(t, t_k^n)))\times$$

$$\times(P_j^c(\min(t,t_{k+1}^n)) - P_j^c(\min(t,t_k^n)))) =$$

$$= \langle P_i^c(t), P_j^c(t)\rangle + \langle D_i^c(t), D_j^c(t)\rangle + \rangle P_i^c(t), D_j^c(t)\rangle +$$

$$+\langle P_j^c(t), D_i^c(t)\rangle.$$

Since $D_i^c(t) = bP_i^c(t)$ $i = 0, \dots, d$, we have

$$\langle G_i^c(t), G_j^c(t)\rangle = (1 + 2b + b^2)\langle P_i^c(t), P_j^c(t)\rangle.$$

We obtain that $c_G(t) = (1+b)^2 c_P(t)$.

For the compensator of the random measure of jumps of $G(t)$ we have $\nu_G = \nu_P + \nu_D$. Since

$$\nu_P([0,t] \times V) = \int_0^t Z_P(\omega, s, V)ds + \sum_{s\in\Theta_P \bigcap [0,t]} \nu(\{s\} \times V)$$

and

$$\nu_D([0,t] \times V) = \int_0^t Z_D(\omega, s, V)ds + \sum_{s\in\Theta_D \bigcap [0,t]} \nu(\{s\} \times V),$$

we have $Z_G(\omega, t, V) = Z_P + Z_D$.

We have $x_{1G}^f(t) = \int xK_G(\omega, t, dx)$ and

$$x_{1G}^f(s) = x_{1P}^f(s)1_{\{s\in\Theta_P\}} + x_{1D}^f(s)1_{\{s\in\Theta_D\}}.$$

We see that if $t \in \overline{\Theta_P \bigcup \Theta_D}$ then K_G must be zero, if $t \in \Theta_P \bigcap \overline{\Theta_D}$ then $K_G = K_P$, if $t \in \Theta_D \bigcap \overline{\Theta_P}$ then $K_G = K_D$, and if $t \in \Theta_P \bigcap \Theta_D$ then $K_G = K_P * K_D$. $\square$

3. Examples of SSGWMM. Here we will consider some important special cases of SSGWMM. Firstly, we will consider some discrete models.

1. *A two-period model.*

As the simplest example of SSGWMM we could consider a model where the vector of generalized prices $G(t) = P(t) + D(t)$ is two-dimensional. The vector $P(t) = (P_0(t), P_1(t))$ where $t \in \{0,1,2\}$ is such that $P_0(t) = 1 = \text{const}$ and for $P_1(t)$ holds that $P_1(0) = 100$ and $P_1(t) = P_1(t-1) + \frac{1}{10}P_1(t-1)X$ where X takes values 0, 1 and (-1) with probabilities $p_0 = 0.4$, $p_1 = 0.4$ and $p_{-1} = 0.2$ respectively. Here $P(t)$ is a process in the probability space (Ω, A, P) with filtration F_t, generated by $P(t)$. The vector $D(t) = (D_0(t), D_1(t))$ where $t \in \{0,1,2\}$ is such that $D_0(t) = a = \text{const} < 1$ for $t > 0$ and $D_0(0) = 0$ and for $D_1(t)$ holds that $D_1(0) = 0$ and $D_1(t) = bP_1(t)$, i.e. the dividends of the second financial instrument are proportional to its price. Here $0 < b = \text{const} < 1$. We have $D_1(t) = D_1(t-1) + \frac{1}{10}D_1(t-1)X$ where X is the same as above.

The characteristics of that process are

$$C_P = (\{1,2\}, (0,0), \epsilon_{(1,100)}, 0, 0, 0, 0, 0, K_P(t))$$

and

$$C_D = (\{1,2\}, (0,0), \epsilon_{(0,0)}, 0, 0, 0, 0, 0, K_D(t))$$

where ϵ_A is the Dirac measure in point A, and $K_P(t)$ is the transition kernel from (Ω, R^+) into (R^2, B^2), for which holds

$$K_P(t,G) = \epsilon_0 \times (0.4\epsilon_{(0.1P_1(t-1))} + 0.4\epsilon_0 + 0.2\epsilon_{(-0.1P_1(t-1))})$$

for $t \in \{1,2\}$ and $K_P(t,G) = 0$ else. Also, $K_D(t)$ is the transition kernel from (Ω, R^+) into (R^2, B^2), for which holds

$$K_D(t,G) = \epsilon_0 \times (0.4\epsilon_{(0.1bP_1(t-1))} + 0.4\epsilon_0 + 0.2\epsilon_{(-0.1bP_1(t-1))})$$

for $t \in \{1,2\}$ and $K_D(t,G) = 0$ else. In accordance with *Theorem 2.3.* we have for the characteristic of $G(t)$ the following nineplet

$$C_G = (\{1,2\}, (0,0), \epsilon_{(1,100)}, 0, 0, 0, 0, 0, K_G(t)),$$

where

$$K_G(\omega, t, V) = \epsilon_0 \times (0.4\epsilon_{(0.1(1+b)P_1(t-1))} + 0.4\epsilon_0 + 0.2\epsilon_{(-0.1(1+b)P_1(t-1))}).$$

2. *A model with independent simple net returns.*

As another example for discrete process, described by SSGWMM, we may consider the following price process. The market consists of two assets – a bank account which is used as a discount factor, i.e. $P_0(t) = 1$, and a stock. We consider the space (Ω, A, F, P) with discrete filtration F. For the discounted process of prices $P_1(t)$, we have that it is discrete and is given by the equation

$$P_1(t) = P_1(t-1)(1 + \epsilon(t))$$

for each $t \in N\backslash\{0\}$ where $P_1(0) \in (0, +\infty)$, and $\epsilon(t)$ is a sequence of identically distributed random variables with distribution Q, such that $\epsilon(t)$ is independent of F_{t-1} for any $t \in N\backslash\{0\}$. We suppose also that $\int \mid \epsilon(t) \mid dQ < \infty$ and $Q((0,\infty)) > 0$ and $Q((-\infty, 0)) > 0$. At these conditions the discounted price process $P(t) = (P_0(t), P_1(t))$ has characteristic

$$C_P = (N\backslash\{0\}, (0,0), \epsilon_{(1,P_1(0))}, 0, 0, 0, 0, 0, K_P),$$

where $K(t, G) = \int 1_G(0, P_1(t-1)x)Q(dx)$ for each $t \in N\backslash\{0\}$ and $G \in B^2$ where B is the sigma-algebra of the Borel sets in R. There are no dividends, i. e.

$$C_D = (\oslash, (0, 0), \epsilon_0, 0, 0, 0, 0, 0, 0).$$

Now we will consider some continuous-time models.

3. *Price process with continuous paths.*

As a first example for a continuous model which is a special case of SSG-WMM, we consider the process $P(t)$ with characteristic

$$C_P = (\oslash, (0, 0, 0), F_{(P_0(0), P_1(0), P_2(0))}, x^f_{0P}, 0, 0, c_P, 0, 0),$$

where x^f_{0P} is a predictable process with values in R^3, and c_P is a predictable process with values – symmetric non-negative definite matrices in $R^3 \times R^3$. Such type processes are price processes with continuous paths. Usually we suppose that the prices are discounted, i. e. $P_0(t) = 1$ for each t.

4. *The Black-Sholes model.*

The Black-Sholes model could also be considered as a special case of SS-GWMM. We consider a financial market where a bank account and a stock are traded with prices respectively $\tilde{P}_0(t)$ and $\tilde{P}_1(t)$, for which the stochastic differential equations

$$d\tilde{P}_0(t) = r\tilde{P}_0(t)dt$$

and

$$d\tilde{P}_1(t) = \mu\tilde{P}_1(t)dt + \sigma\tilde{P}_1(t)dB(t)$$

hold. If we denote the respective discounted prices with $P_0(t) = \dfrac{\tilde{P}_0(t)}{\tilde{P}_0(t)} = 1$ and $P_1(t) = \dfrac{\tilde{P}_1(t)}{\tilde{P}_0(t)}$, we have respectively the equations

$$dP_0(t) = 0$$

and

$$dP_1(t) = (\mu - r)P_1(t)dt + \sigma P_1(t)dB(t).$$

Here $\tilde{P}_0(0) = 1$, $\tilde{P}_1(0), \sigma \in (0, \infty), r$ and $\mu \in R$ are given. The characteristic of the discounted price process is

$$C_P = (\oslash, (0, 0), \epsilon_{(1, P_1(0))}, x^f_{0P}, 0, 0, c_P, 0, 0),$$

where $x^f_{0P,0}(t) = 0$, $x^f_{0P,1}(t) = (\mu - r)P_1(t)$, $c_{P,00}(t) = c_{P,01}(t) = c_{P,10}(t) = 0$, and $c_{P,11}(t) = (\sigma P_1(t))^2$. Here the filtration F of the space (Ω, A, F, P) is the canonical filtration of $(\tilde{P}_0(t), \tilde{P}_1(t))$.

5. *Generalized Black-Sholes model.*

We can obtain a generalization of the Black-Sholes model when instead of the Wiener process, we consider a more general Lévy process. We consider a bank account with price $\tilde{P}_0(t) = e^{rt}$ and a stock for which the discounted process of the price $P_1(t)$ satisfies the equation

$$dP_1(t) = (\mu - r)P_1(t-)dt + \sigma P_1(t-)dB(t) + \int xP_1(t-)(p-q)(dt, dx),$$

where $P_1(0) \in (0, +\infty)$, $\mu \in R$, $\sigma \in [0, +\infty)$, and p is a homogeneous Poisson random measure with compensator $q = \lambda \otimes H$. For H we suppose that it is a measure in (R, B) for which $\int(\min(| x |^2, | x |)H(dx) < \infty$, $\int | \log(x+1) | 1_{(-1,-\frac{1}{2})}(x)H(dx) < \infty$ and $H((-\infty, -1]) = 0$ (condition which do not allow negative jumps). If we define the random process $X(t) = (\mu - r)t + \sigma B(t) + \int x(p-q)(dt, dx)$, for each $t \in [0, \infty)$ for the differential of the discounted process we have $dP_1(t) = P_1(t-)dX(t)$. The process $(P_0(t), P_1(t))$ then has characteristic

$$C_P = (\oslash, (0,0), \epsilon_{(1,P_1(0))}, x^f_{0P}, 0, 0, c_P, Z_P, 0),$$

where $x^f_{0P,0}(t) = 0$, $x^f_{0P,1}(t) = (\mu - r)P_1(t-)$, $c_{P,00}(t) = c_{P,01}(t) = c_{P,10}(t) = 0$, $c_{P,11}(t) = (\sigma P_1(t-))^2$, $Z_P(t, G) = \int 1_{G\setminus\{0\}}(0, P_1(t-)x)H(dx)$ for each $t \in [0, \infty)$ and for each $G \in B^2$. Again there are no dividends.

6. *A diffusion model.*

Here we will consider one such model and we will show that it is a special case of SSGWMM. We consider a market with two assets – a bank account and a stock. The discounted price process satisfies the equations

$$dP_1(t) = \mu(\sigma(t))P_1(t)dt + \sigma(t)P_1(t)dB(t),$$

$$d\sigma(t) = \alpha(\sigma(t))dt + \beta(\sigma(t))\sigma(t)dB_1(t),$$

where $\mu, \alpha, \beta : R \to R$ are continuous functions, $P_1(0)$ and $\sigma(0)$ are constants in $(0, \infty)$, and $B(t)$ and $B_1(t)$ are Wiener processes with correlation $[B, B_1](t) = st$ for any $t \in [0, +\infty)$. The second equation describes the change of the stochastic volatility of the price of the stock. The process $(P_0(t), P_1(t), \sigma(t))$ has characteristic

$$(\oslash, (0,0,0), \epsilon_{(1,P_1(0),\sigma(0))}, x^f_{0P}, 0, 0, c_P, 0, 0),$$

where $x^f_{0P,0}(t) = 0$, $x^f_{0P,1}(t) = \mu(\sigma(t))P_1(t)$, $x^f_{0P,2}(t) = \alpha(\sigma(t))$, and for the elements of c_P we have

$$c_{P,00}(t) = c_{P,01}(t) = c_{P,02}(t) = c_{P,10}(t) = c_{P,20}(t) = 0,$$

$$c_{P,11}(t) = (\sigma(t)P_1(t))^2,$$

$$c_{P,12}(t) = s\beta(\sigma(t))P_1(t)\sigma^2(t),$$

$$c_{P,21}(t) = s\beta(\sigma(t))P_1(t)\sigma^2(t),$$

$$c_{P,22}(t) = (\beta(\sigma(t))\sigma(t))^2$$

for each $t \in [0, \infty)$.

7. *Keller's model.*

Another example of SSGWMM is Keller's model [17]. In this model we consider a bank account and a stock as basic financial instruments again but there is an attempt for more realistic reflection of the price process. Because the real markets are closed during the night, when there is no trade, the night periods are shrunk to points – integer moments. The overnight price change corresponds to a jump at an integer time.

The price of the bank account is given by $P_0(t) = e^{(r_d t + r_n [t])}$ where r_d and r_n are the intraday and the overnight interest rates. For the price of the stock we have respectively $P_1(t) = P_1(0)e^{(R_d(t)+R_n(t))}$ where $R_d(t)$ is the intraday return process, and for the overnight return process we have $R_n(t) = \sum_{k=1}^{[t]} \Delta R_{nk}$. Here ΔR_{nk} is a sequence of independent and identically distributed random variables with distribution Q for which $\int e^{|x|}Q(dx) < \infty$. The stock prices jump at random times.

We have $R_d(t) = \sum_{l \in N} 1_{\{T_l \le t\}}\theta_l$ where $\{T_l;\ l \in N\}$ is an increasing sequence of stopping times for which almost certainly with respect to P we have $\lim\limits_{l \to \infty} T_l = \infty$ and θ_l is a sequence of random variables with values in $R \backslash \{0\}$. For the distribution of jump times and sizes we have

$$P\{(T_{l+1}, \theta_{l+1}) \mid (T_m, \theta_m);\ m \in \{0, \dots, l\}\} = (\mathrm{Exp}(U_{l+1}) * \epsilon_{T_l}) \otimes N(0, V_{l+1})$$

for any $l \in N$ where $\mathrm{Exp}(U_{l+1})$ is an exponential distribution with parameter U_{l+1} and the processes U_l and V_l are given recursively with the equations

$$U_0 = 0;\ U_{l+1} = a_0 + \alpha\theta_l^2 + \beta U_l,$$

$$V_0 = 0;\ V_{l+1} = b_0 + \gamma(T_l - T_{l-1}) + \delta V_l,$$

where a_0, $b_0 \in (0,\infty)$ and $\alpha,\beta,\gamma,\delta \in [0,\infty)$ are fixed constants, $\theta_0 = 0$ and $T_{-1} = T_0 = 0$. We also suppose that R_n and (T_l, θ_l) are independent. In this model the return is characterized by normally distributed jumps separated by exponential waiting times. The changes in volatility are summarized firstly by the variance U_l of the jump height and secondly by the parameter V_l of the distribution of the waiting time $T_l - T_{l-1}$ between successive jumps. For the characteristic of the discounted price process we have

$$C_P = (N\backslash\{0\}, (0,0), \epsilon_{(1,P_1(0))}, x^f_{0P}, 0, 0, 0, Z_P, K_P),$$

where

$$K(t,G) = \int 1_G(0, P_1(t-)(e^{(x-r_n)} - 1))Q(dx),$$

$$Z(t,G) = \sum_{l\in N} 1_{(T_l,T_{l+1})}(t)V_{l+1} \int 1_G(0, P_1(t-)(e^x - 1))N(0, U_{l+1})(dx),$$

$$x^f_{0P,0}(t) = 0,$$

$$x^f_{0P,1}(t) = \int x_1 Z(t, d(x_1,x_2)) - P_1(t-)r_d$$

for each $t \in [0,\infty)$ and $G \in B^2$.

SSGWMM can be used to model not only stock prices but also prices of any of the financial instruments.

8. *A model for the short-term interest rate.*

As a further example we consider a model of changes of the short-term interest rate. Here the financial instrument with index 0 is a short-term investment with fixed income for which we have $P_0(t) = e^{\int_0^t r(s)ds}$. The interest rate here is a continuous stochastic process which is a solution of the stochastic differential equation

$$dr(t) = \mu(r(t))dt + \sigma(r(t))dB(t),$$

where $\mu : R \to R$ and $\sigma : R \to R$ are given continuous functions and $r(0)$ is fixed. The stochastic process $r(t)$ is with characteristic

$$C_r = (\oslash, 0, \epsilon_{r(0)}, x^f_{0r}, 0, 0, c_r, 0, 0),$$

where $x^f_{0r} = \mu(r(t))$ and $c_r(t) = (\sigma(r(t)))^2$ for each $t \in [0,\infty)$.

9. *A model with random jumps.*

Now we will consider another example where the part with finite variation of the price process has jumps not only at deterministic times of a discrete set Θ but also at the moments of jumps of a Poisson process. Such a process has characteristic for example

$$C_P = (\oslash, (1,1,1), q^{(P_0(0),P_1(0),P_2(0))}, x^f_{0P}, 0, (1,1,1), c_P, 0, 0).$$

Here we have a market of three financial instruments with a vector of the prices $(P_0(t), P_1(t), P_2(t))$ with initial distribution $q^{(P_0(0),P_1(0),P_2(0))}$. Also, the part with finite variation has the representation

$$X^f(t) = \int_0^t x^f_{0P}(s)ds + \int_0^t x^f_2(s)dN(s),$$

where $N(s)$ is a three-dimensional Poisson process with parameter $\lambda = (1,1,1)$ and the process $x^f_2(s) = (1,1,1) = \text{const}$.
We have $\int_0^t x^f_2(s)dN(s) = (N(t), N(t), N(t))$.

4. A special case. Brownian motion with returns to zero.

Following the structural approach to the default problem we consider the probability of reaching the zero level by the process in *HGWMM.*

Definition 4.1. *The probability* $\Psi(u_0) = P\{u_0 + R(t) - C(t) < 0;\ t > 0\}$ *we will call* probability of (theoretical) default (first reaching of the zero) in a HGWMM.

Definition 4.2. *The probability* $\Psi(u_0, t_1, t_2) = P\{u_0 + R(t) - C(t) < 0; t \in (t_1, t_2)\}$ *we will call* probability of (theoretical) default (first reaching of the zero) in the interval (t_1, t_2) *in HGWMM.*

We use the term *theoretical*, because a real default is registered only at times of claims.

We consider firstly the case where we have only a portfolio of financial instruments for which there are no dividends, i.e. $U(t; u_0) = A(t)P(t)$ and $U(0; u_0) = A(0)P(0) = u_0$. If we suppose that $A(t)P(t) = w(t) \in N(u_0, \sigma^2 t)$, i. e. $w(t)$ has a normal distribution, the HGWMM process in this case is a Wiener process. We have the following lemma.

Lemma 4.1. *Let the wealth process in a HGWMM have the form* $U(t; u_0) = w(t)$ *where* $w(0) = u_0$ *and* $w(t) \in N(u_0, \sigma^2 t)$. *Then the probability for (theoretical) default for the interval* $(0, T)$ *is given by the formula*

$$\Psi(u_0, 0, T) = \frac{2}{\sqrt{2\pi\sigma^2 T}} \int_{-\infty}^{-u_0} e^{-\frac{x^2}{2\sigma^2 T}} dx. \tag{13}$$

Lemma 4.2. *Let the wealth process in a HGWMM consist only of the portfolio process, i. e. it has the form* $U(t;u_0) = w(t)$ *where* $w(0) = u_0$ *and* $w(t) \in N(u_0, \sigma^2 t)$. *Then the probability for (theoretical) default until the stopping time* t_1 *of the first special event is given by*

$$\Psi(u_0, 0, t_1) = \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} \frac{1}{\sqrt{y}} \int_{-\infty}^{-u_0} e^{-\frac{x^2}{2\sigma^2 y}} dx dF_{\Delta}(y), \tag{14}$$

where $F_{\Delta}(y)$ *is the cumulative distribution function of the time interval between* 0 *and* t_1.

Let us now try to find the distribution of the time of default in the same special case, i. e. when the wealth process in a HGWMM is $U(t;u_0) = w(t)$. Let T_0 be *the moment of first reaching of* 0 *by* $w(t)$.

The following lemma holds:

Lemma 4.3. *The distribution of the time* T_0 *is given by the formula*

$$P\{T_0 < t\} = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{-\frac{u_0}{\sigma\sqrt{t}}} e^{-\frac{y^2}{2}} dy.$$

The proof of Lemma 4.1., Lemma 4.2., and Lemma 4.3. can be seen in [9].

Let us define $1_I(t) = 1,\ t \in I$ and $1_I(t) = 0$ outside of I where I is an interval in the set R of the real numbers. Let us consider the process

$$w(t) = \sum_i B_i(t - t_i) 1_{[t_i, t_{i+1})}(t)$$

where $B_i(t)$ are copies of the standard Brownian motion. We consider two cases. The first one is where $\sigma(B_i(t)) = \sigma(B_j(t))$ for $i \neq j$ and for any $t \geq 0$. Here $\sigma(X(t))$ denotes the sigma-algebra generated by $X(t)$. The second one is where $B_i(t)$ are independent Brownian motions. We introduce two definitions corresponding to these cases.

Definition 4.3. *Let* $0 = t_0 < t_1 < t_2 < \cdots < \cdots$ *be positive real numbers interpreted as times. A Wiener process (Brownian motion) with returns to the zero at times* t_i *and parts with identical filtration (PIF) we call the process*

$$w(t) = \sum_i B_i(t - t_i) 1_{[t_i, t_{i+1})}(t)$$

defined for $t \in [0, +\infty)$ *where* $B_i(t)$ *are copies of the standard Brownian motion for which* $\sigma(B_i(t)) = \sigma(B_j(t))$ *for* $i \neq j$ *and for any* $t \geq 0$.

Definition 4.4. *Let $0 = t_0 < t_1 < t_2 < \cdots < \cdots$ be positive real numbers interpreted as times.* A Wiener process (Brownian motion) with returns to the zero at times t_i *and independent parts (IP) we call the process*

$$w(t) = \sum_i B_i(t - t_i) 1_{[t_i, t_{i+1})}(t)$$

defined for $t \in [0, +\infty)$ where $B_i(t)$ are independent standard Brownian motions.

Here we will give some properties of the Wiener processes with returns to zero. The following lemmas are related to the concept *PIF Brownian motion with returns to zero.* The Theorem 4.1. is related to the concept *IP Brownian motion with returns to zero.*

It is easy to prove that $B(\min(t, t_1))$ (the standard Brownian motion stopped at time t_1) is a martingale. The process

$$B(t) 1_{[0,t_1)}(t)$$

may be presented as

$$B(t) 1_{[0,t_1)}(t) = B(\min(t, t_1)) - B(t_1) 1_{[t_1,+\infty)}(t).$$

It is easy to see that this process is not a martingale. Indeed, let $0 < s < t_1 < t$. Then we have

$$E[B(t) 1_{[0,t_1)}(t) \mid F_s] = 0 \neq B(s) = B(s) 1_{[0,t_1)}(s)$$

in the general case.

Lemma 4.4. *The process $B(t) 1_{[0,t_1)}(t)$ is a semimartingale.*

Proof. We have

$$B(t) 1_{[0,t_1)}(t) = B(\min(t, t_1)) - B(t_1) 1_{[t_1,+\infty)}(t).$$

But the first term $B(\min(t, t_1))$ is a martingale, and therefore – a local martingale. The second term $-B(t_1) 1_{[t_1,+\infty)}(t)$ is a càdlàg process. Its variation is

$$Var(-B(t_1) 1_{[t_1,+\infty)}(t)) =$$

$$= \sup_{0=s_0<s_1<\ldots<s_n=t} \sum_{i=1}^{n} \mid -B(t_1) 1_{[t_1,+\infty)}(s_i) + B(t_1) 1_{[t_1,+\infty)}(s_{i-1}) \mid =$$

$$= \mid B(t_1) \mid < \infty.$$

We see that the variation is finite, and it follows that $B(t) 1_{[0,t_1)}(t)$ is a semimartingale. $\square$

Lemma 4.5. *The process* $B_i(t-t_i)1_{[t_i,t_{i+1})}(t)$ *is a semimartingale.*

Proof. This lemma stems immediately from the above lemma and from the fact that $B(\min(t-t_i, t_{i+1}-t_i))$ is a martingale. $\square$

Lemma 4.6. *Let* $B(t)$ *be the standard Brownian motion with its natural filtration* F^B *in the probability space* (Ω, A, P). *Let* $w(t)$ *be a PIF Wiener process with returns to zero at times*

$$t_1, t_2, \dots, t_n$$

and $\sigma(B_i(t)) = \sigma(B_j(t)) = \sigma(B(t))$ *for* $i \neq j$ *and for any* $t \geq 0$. *Then for the natural filtration* F^w *of* $w(t)$ *we have*

$$F_t^w = F^B_{r_{[t]}(t-t_{[t]})}$$

where

$$[t] = \max_{1\leq i\leq n}\{i : t_i \leq t\},$$

and

$$r_i(a) = \max\{t_1-t_0, t_2-t_1, \dots, t_i-t_{i-1}, a\}.$$

Proof. (sketch) The natural filtration F^w of $w(t)$ is defined by $F^w = \{F_t^w;\ t\in[0,\infty)\}$ where $F_t^w = \sigma(w(s); s<t)$. Because of the returns to zero at the points $t_i,\ i\geq 1$, inside each interval $[t_i, t_{i+1}),\ i\geq 1$, the random variables $w(t)$ are copies of the random variables in the longest interval before t_i. Hence, the enlargement of the σ-algebras F_t^w will occur only if $(t-t_{[t]})$ is greater than the length of the longest current interval between two returns to zero. $\square$

Lemma 4.7. *The PIF Wiener process with returns to zero is a semimartingale.*

Proof. By the presentation

$$w(t) = \sum_i B_i(t-t_i)1_{[t_i,t_{i+1})}(t)$$

we see that the Wiener process with returns to zero is a sum of semimartingales with a local-martingale part which is a martingale. Therefore, the process is a semimartingale. $\square$

Lemma 4.8. *Let* σ *be a Markov process with respect to the filtration* $F=\{F_t; t\geq 0\}$ *in the probability space* (Ω, A, P). *Let also* $P\{\sigma\leq T\}=1$ *where* $T<\infty$. *Let* $\tilde{B}(t) = B(\min(t,\sigma))$ *and* $\tilde{F}_t = F_{\min(t,\sigma)}$ *where* $B(t)$ *is the standard*

Brownian motion adapted to F. Then $\tilde{B}(t)$ is a martingale with respect to $\tilde{F}(t)$ and

$$E(\tilde{B}(t) - \tilde{B}(s) \mid \tilde{F}_s) = 0,$$

$$E[(\tilde{B}(t) - \tilde{B}(s))^2 \mid \tilde{F}_s) =$$

$$= E[\min(t, \sigma) - \min(s, \sigma) \mid \tilde{F}_t];\ t \geq s.$$

The proof of this lemma can be seen in [21].

Let us consider now a HGWMM where $U(t; u_0) = d(t) - w(t) - p(t)$. Here we suppose that $d(t)$ is a strongly monotone deterministic function of income, defined for $t \in [0, +\infty)$, for which $d(0) = u_0$, $0 < u_0 < 1$ and $\lim_{t\to+\infty} d(t) = \infty$. Also, $w(t)$ is an IP Wiener process with returns to zero at the moments $t_i = d^{-1}(i)$ for the non-negative integers $i \geq 1$. The process models the fluctuations of the price of the financial instruments held in the portfolio. These changes are added to the function of the income. The Wiener process with returns to the zero models situation where by using suitable options the portfolio is transformed to have values at the moments $t_i = d^{-1}(i)$ which are exactly equal to the predicted values for the function of the income $d(t)$.

We suppose also that $p(t) = \sum_{i=1}^{N(t)} 1 = N(t)$ is a homogeneous Poisson process with an intensity $\lambda = 1$ and with jumps 1. It models equal costs as a result of special events. The processes $w(t)$ and $p(t)$ are independent.

We are searching for the probability of default during the interval $[0, t_k]$ in such a wealth process. Let $B(t)$ be the standard Brownian motion.

The non-default probability until time t_1 is

$$1 - \Psi(u_0, 0, t_1) = P(\{U(t; u_0) > 0,\ t \in [0, t_1)\}) =$$

$$P(\{d(t) > \max_{t\in[0,t_1)} (p(t) + w(t))\}) \geq$$

$$\geq P(\{N(t_1) = 0\} \bigcap \{\max_{t\in[0,t_1)} (w(t)) \leq d(0)\}) =$$

$$= P(\{N(t_1) = 0\})(1 - P(\{\max_{t\in[0,t_1)} (w(t)) \geq d(0)\})) =$$

$$= (1 - 2P(\{B(t_1) > u_0\}))P(\{N(t_1) = 0\}). \tag{15}$$

By analogy for the non-default probability until t_2 we obtain the following boundary

$$1-\Psi(u_0,0,t_2)=P(\{U(t;u_0)>0;\ t\in[0,t_2)\})=$$
$$=P(\{d(t)>\max_{t\in[0,t_2)}(p(t)+w(t))\})\geq$$
$$\geq P(\{N(t_1)=0\}\bigcap\{N(t_2)=0\}\bigcap$$
$$\bigcap\{\max_{t\in[0,t_1)}(w(t))\leq d(0)\}\bigcap\{\max_{t\in[t_1,t_2)}(w(t))\leq 1\})=$$
$$=P(\{N(t_1)=0\}\bigcap\{N(t_2)=0\})(1-P(\{\max_{t\in[0,t_1)}(w(t))\geq u_0\}))\times$$
$$\times(1-P(\{\max_{t\in[t_1,t_2)}(w(t))\geq 1\}))=$$
$$=(1-2P(\{B(t_1)>u_0\}))(1-2P(\{B(t_2-t_1)>1\}))P(\{N(t_1)=0\}\bigcap$$
$$\bigcap\{N(t_2)=0\}). \tag{16}$$

For the non-default probability until t_3 we obtain

$$1-\Psi(u_0,0,t_3)=P(\{U(t;u_0)>0,\ t\in[0,t_3)\})=$$
$$=P(\{d(t)>\max_{t\in[0,t_3)}(p(t)+w(t))\})\geq$$
$$\geq P(\{N(t_1)=0\}\bigcap\{N(t_2)=0\}\bigcap\{N(t_3)\leq 1\}\bigcap$$
$$\bigcap\{\max_{t\in[0,t_1)}(w(t))\leq u_0\}\bigcap\{\max_{t\in[t_1,t_2)}(w(t))\leq 1\}\bigcap$$
$$\bigcap\{\max_{t\in[t_2,t_3)}(w(t))\leq 1\})=$$
$$=P(\{N(t_1)=0\}\bigcap\{N(t_2)=0\}\bigcap\{N(t_3)\leq 1\})(1-P(\{\max_{t\in[0,t_1)}(w(t))\geq u_0\}))\times$$
$$\times(1-P(\{\max_{t\in[t_1,t_2)}(w(t))\geq 1\}))(1-P(\{\max_{t\in[t_2,t_3)}(w(t))\geq 1\}))=$$

$$= (1-2P(\{B(t_1) > u_0\}))(1-2P(\{B(t_2-t_1) > 1\}))(1-2P(\{B(t_3-t_2) > 1\}))\times$$

$$\times P(\{N(t_1) = 0\} \bigcap \{N(t_2) = 0\} \bigcap \{N(t_3) \le 1\}). \tag{17}$$

In the general case we obtain:

$$1 - \Psi(u_0, 0, t_k) = P(\{U(t; u_0) > 0,\ t \in [0, t_k)\}) =$$

$$= P(\{d(t) > \max_{t\in[0,t_k)} (p(t) + w(t))\}) \ge$$

$$\ge P(\{N(t_1) = 0\} \bigcap \{N(t_2) = 0\} \bigcap \{N(t_3) \le 1\} \bigcap \ldots \bigcap$$

$$\bigcap \{N(t_{k-1}) \le k-3\} \bigcap \{N(t_k) \le k-2\} \bigcap$$

$$\bigcap \{\max_{t\in[0,t_1)} (w(t)) \le u_0\} \bigcap \{\max_{t\in[t_1,t_2)} (w(t)) \le 1\} \bigcap$$

$$\bigcap \{\max_{t\in[t_2,t_3)} (w(t)) \le 1\} \bigcap \ldots \bigcap \{\max_{t\in[t_{k-1},t_k)} (w(t)) \le 1\}) =$$

$$= P(\{N(t_1) = 0\} \bigcap \{N(t_2) = 0\} \bigcap \{N(t_3) \le 1\} \bigcap \ldots \bigcap$$

$$\bigcap \{N(t_{k-1}) \le k-3\} \bigcap \{N(t_k) \le k-2\})(1 - P(\{\max_{t\in[0,t_1)} (w(t)) \ge u_0\}))\times$$

$$\times(1 - P(\{\max_{t\in[t_1,t_2)} (w(t)) \ge 1\}))(1 - P(\{\max_{t\in[t_2,t_3)} (w(t)) \ge 1\}))\times$$

$$\times(1 - P(\{\max_{t\in[t_{k-1},t_k)} (w(t)) \ge 1\})) =$$

$$= (1-2P(\{B(t_1) > u_0\}))(1-2P(\{B(t_2-t_1) > 1\}))(1-2P(\{B(t_3-t_2) > 1\}))\times$$

$$\times \ldots \times (1 - 2P(\{B(t_k - t_{k-1}) > 1\}))P(\{N(t_1) = 0\} \bigcap$$

$$\bigcap \{N(t_2) = 0\} \bigcap \{N(t_3) \le 1\} \bigcap \ldots \bigcap$$

$$\bigcap \{N(t_{k-1}) \le k-3\} \bigcap \{N(t_k) \le k-2\}). \tag{18}$$

We have prooved the following

Theorem 4.1. *Let $U(t; u_0) = d(t) - w(t) - p(t)$ be the wealth process in a HGWMM. Let $d(t)$ be a strongly monotone deterministic function of income defined for $t \in [0, +\infty)$ for which $d(0) = u_0$, $0 < u_0 < 1$ and $\lim\limits_{t\to+\infty} d(t) = \infty$. We suppose that $w(t)$ is an IP Wiener process with returns to the zero at the moments $t_i = d^{-1}(i)$ for the non-negative integers $i \geq 1$. We also suppose that $p(t) = \sum_{i=1}^{N(t)} 1 = N(t)$ is a homogeneous Poisson process with an intensity $\lambda = 1$ and with jumps 1. The processes $w(t)$ and $p(t)$ are independent. With $B(t)$ we denote the standard Brownian motion. Then the non-default probability for the interval $[0, t_k)$ has the following boundary:*

$$1-\Psi(u_0, 0, t_k) \geq C_k(1-2P(\{B(t_1) > u_0\})) \prod_{i=1}^{k-1}(1-2P(\{B(t_{i+1}-t_i) > 1\})) \tag{19}$$

where

$$C_k = P(\{N(t_1) = 0\} \bigcap \{N(t_2) = 0\} \bigcap \{N(t_3) \leq 1\} \bigcap \ldots \bigcap$$

$$\bigcap \{N(t_{k-1}) \leq k-3\} \bigcap \{N(t_k) \leq k-2\}).$$

Since we have

$$\{N(t_1) = 0\} \bigcap \{N(t_2) = 0\} = \{N(t_2) = 0\},$$

for C_k we obtain

$$C_k = P(\bigcap_{j=1}^{k-1} \{N(t_{j+1}) \leq j-1\}).$$

If we introduce $s_j = t_{j+1}$ for $j = 1, \ldots, k-1$, we obtain

$$C_k = P(\bigcap_{j=1}^{k-1} \{N(s_j) \leq j-1\}). \tag{20}$$

The close formula for the last probability can be seen in [13].

As a simple example for the application of the proved theorem we consider an insurance company which begins its activity with an initial capital of 900 000 \$. If we work in arbitrary currency units for which holds that one unit is equal to 1 000 000 \$, the initial capital is $0 < u_0 = d(0) = 0.9 < 1$. The company holds 800 000 \$ (0.8 units) in cash and invests 100 000 \$ (0.1 units) in stocks. The premiums which the company receives and the dividends from the stocks are given by the process $d(t) = 0.1t + 0.9$ for $t \in [0, 1]$ and $d(t) = t^2$ for $t \geq 1$. The stocks prices process is a Wiener process $w_1(t) = N(0.1,\ \sigma^2 t)$, where σ^2 is the coefficient of the variation. The total capital is $w_2(t) = 0.8 + w_1(t) = u_0 + w_3(t) \in N(0.9, \sigma^2 t)$, where $w_3(t) \in N(0, \sigma^2 t)$. We have $w_3(t) = \sigma B(t)$, where $B(t)$ is the standard Brownian motion. For the sake of simplicity we assume that $\sigma = 1$. Let the time unit be a year. We assume that the company buys a put option on the stocks which it holds. The time to maturity of the option is $t = 1$ and the strike price is 100 000 \$ (0.1 units). The company uses the option and after that it buys stocks for 100 000 \$. Then it buys a put option on the stocks with the same strike price and time to maturity $\sqrt{2} - 1$ years. The company uses the second option too. Here we assume that the price of the put option is included in the process $d(t)$. The fluctuations of the capital in this case are described by a Wiener process with jumps (returns) to zero at the moments $t_i = d^{-1}(i),\ i = 1, 2$. If the claims are described by a standard Poisson process, we have that the probability not to occur default in the interval $[0, \sqrt{2})$ is

$$1 - \Psi(0.9,\ 0,\ \sqrt{2}) \geq C_2(1 - 2P(\{B(1) > 0.9\}))(1 - 2P(\{B(\sqrt{2} - 1) > 1\})),$$

where

$$C_2 = P(\{N(t_1) = 0\} \bigcap \{N(t_2) = 0\}) = P(\{N(t_2) = 0\}) =$$

$$= e^{-t_2} = e^{-d^{-1}(2)} = e^{-\sqrt{2}}.$$

For the probability $P(\{B(1) > 0.9\})$ we have

$$P(\{B(1) > 0.9\}) = 1 - P(\{B(1) < 0.9\}) = 1 - 0.8159 = 0.1841.$$

For the probability $P(\{B(\sqrt{2} - 1) > 1\})$ we have

$$P(\{B(\sqrt{2} - 1) > 1\}) = 1 - P(\{B(\sqrt{2} - 1) < 1\}) = 1 - 0.9394 = 0.0606.$$

Finally, for the probability not to occur default in the interval $[0, \sqrt{2})$ we obtain

$$1 - \Psi(0.9,\ 0,\ \sqrt{2}) \geq e^{-\sqrt{2}}(0.6318)(0.8788) = (0.1536)(0.8788) = 0.1349.$$

5. Conclusion. In this article we introduce the class of *Wealth Motion Models* which gives different generalizations of many models describing the wealth change of an agent in an economy. This class includes discrete-time as well as continuous-time models. The wealth may be homogenuous as well as distributed on the instruments of a financial market. At the basis of the *Generalized Structured Wealth Motion Models* lies the concept *generalized Lévy process* introduced here to give a more precise description of the typical changes in the wealth. We give some properties of the model and consider some its special cases (including the classic Black-Sholes model).

In the second part of the article we consider the credit risk within the framework of the *Homogenuous Wealth Motion Models*. Our main contributions here are two. The first one is the introduction of the Wiener process with returns to zero which is very useful when we model some kinds of financial processes. The second one is the estimate for the default probability (Theorem 4.1.) where we also consider a simple example.

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