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THE KOEBE DOMAIN FOR CONCAVE UNIVALENT FUNCTIONS

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Abstract. Let D denote the open unit disc and $f : D \to \overline{\mathbf{C}}$ be meromorphic and injective in D. We further assume that f has a simple pole at the point $p \in (0, 1)$ and an expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n, \quad |z| < p.$$

Especially, we consider f that map D onto a domain whose complement with respect to $\overline{\mathbf{C}}$ is convex. It is proved that this implies

$$K := \left\{ w \ : \ \left| w + \frac{p(1+p^2)}{(1-p^2)^2} \right| > \frac{2p^2}{(1-p^2)^2} \right\} \subset f(D)$$

and that for any $c \in \overline{\mathbf{C}} \setminus K$ there exists a function f satisfying the conditions mentioned above such that c does not belong to $f(D)$. This means that K is the exact Koebe domain for the class of functions considered here.

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For a family A of functions f analytic in the open unit disc D the Koebe domain $K(A)$ is defined by

$$K(A) := \bigcap_{f\in A} f(D).$$

There are many families of functions that play an important role in Geometric Function Theory for which the Koebe domains have been determined, for example the family of schlicht functions and the family of convex functions. Since these families are invariant under rotations around the origin the Koebe domains are discs with center in the origin. In the present paper we consider a family which is not invariant under rotations and we shall determine its Koebe domain. We are concerned with the family of concave univalent functions with pole $p \in (0,1)$ denoted by $Co(p)$ here. To be precise, we say that a function $f : D \to \overline{\mathbf{C}}$ belongs to the family $Co(p)$ if and only if:
(1) f is meromorphic in D and has a simple pole at the point $p \in (0,1)$.
(2) f has an expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n, \quad |z| < p.$$

(3) f maps D conformally onto a set whose complement with respect to $\overline{\mathbf{C}}$ is convex.

There are results on $Co(p)$ that resemble very much those on convex functions, for example it has been proved in [3] that $|a_n(f)| > 1$ for $f \in Co(p)$. Other results look very different from the analogous results on convex functions. Results of this type are those on the domains of variability of the Taylor coefficients $a_n(f), f \in Co(p)$ (compare [6], [1], [3], [2] and especially [5] where further references on $Co(p)$ may be found).
Concerning covering theorems on $Co(p)$ it may be deduced from the results of J. Miller (see [6, Theorem 5]) that

$$\left\{\frac{-p}{1+p^2}\right\} = \bigcap_{f\in Co(p)} (\overline{\mathbf{C}} \setminus f(D)).$$

Further, it was proved in [1] and in [2] that for $f \in Co(p), c \in \overline{\mathbf{C}} \setminus f(D)$, the sharp inequalities

$$|c| \geq \frac{p}{(1+p)^2}$$

and

$$-\frac{p}{(1-p)^2} \leq \mathrm{Re}(c) \leq -\frac{p}{(1+p)^2}$$

are valid.
The present paper is devoted to the determination of the Koebe domain of $Co(p)$. We show

Theorem 1. *Let* $p \in (0,1)$. *Then*

$$K(Co(p)) = \left\{ w \ : \ \left| w + \frac{p(1+p^2)}{(1-p^2)^2} \right| > \frac{2p^2}{(1-p^2)^2} \right\}. \tag{1}$$

The proof of (1) will be divided into the two natural steps. First, we prove

Theorem 2. *Let* $p \in (0,1), f \in Co(p)$ *and* $c \in \overline{\mathbf{C}} \setminus f(D)$. *Then the inequality*

$$\left| c + \frac{p(1+p^2)}{(1-p^2)^2} \right| \leq \frac{2p^2}{(1-p^2)^2} \tag{2}$$

is valid.

P r o o f. Since $\overline{\mathbf{C}} \setminus f(D)$ is starlike with respect to c and f is normalized as defined above and has a simple pole at the point p, the function

$$F(z) := \frac{(1-\frac{z}{p})(1-zp)f'(z)}{f(z)-c},$$

resp. its holomorphic continuation from $D \setminus \{p\}$ onto D has a positive real in D (compare [5, Theorem 6]). This and the univalence of f imply that the function G defined by

$$G(z) = \frac{1}{F(z)}, \ z \in D,$$

is holomorphic in D and has a positive real part there. Further, we use

$$G(0) = -c \quad \text{and} \quad G(p) = \frac{p}{1-p^2}.$$

Let $c = x + iy$, $x, y \in \mathbf{R}$. From the properties of the function G we conclude that $x < 0$ and that there exists a function φ holomorphic in D such that $\varphi(D) \subset D$, $\varphi(0) = 0$ and

$$-\frac{G(z)+iy}{x} = \frac{1-\varphi(z)}{1+\varphi(z)}, \ z \in D.$$

Hence, there exists a function Φ holomorphic in D such that $\Phi(D) \subset \overline{D}$,

$$-\frac{G(z)+iy}{x} = \frac{1-z\Phi(z)}{1+z\Phi(z)}, \ z \in D,$$

and

$$-\frac{\frac{p}{1-p^2}+iy}{x} = \frac{1-p\Phi(p)}{1+p\Phi(p)}.$$

This equation together with $\Phi(D) \subset \overline{D}$ yields that for every $c = x+iy \in \overline{\mathbf{C}}\backslash f(D)$ there exists a $\zeta \in \overline{D}$ such that

$$\text{(3)} \qquad -\frac{\frac{p}{1-p^2} + iy}{x} = \frac{1-p\zeta}{1+p\zeta} =: u + iv,$$

where $u + iv$ varies in the disc described by

$$\text{(4)} \qquad \left(u - \frac{1+p^2}{1-p^2}\right)^2 + v^2 \le \left(\frac{2p}{1-p^2}\right)^2.$$

From (3) we get

$$y = -x\,v$$

and

$$\frac{p}{(1-p^2)} = -x\,u.$$

Now, we solve these two equations with respect to u and v and insert this into (4). The resulting inequality is easily shown to be equivalent to (2). □

Remark. An alternative proof of Theorem 2 may be deduced from the fact that the disc described by (2) is the starlike center region for the class of all starlike meromorphic functions with pole p and the above normalization in the origin. The definition and determination of this region may be found in [4].

To end the proof of Theorem 1, it remains to show that for any c satisfying (2) there exists a function $f \in Co(p)$ such that $c \in \overline{\mathbf{C}}\backslash f(D)$. In view of the Koebe theorem for the family of schlicht functions, the natural candidates for such f are the functions $f \in Co(p)$ that map the unit disc onto $\overline{\mathbf{C}}$ minus a segment of a line. These functions may be written in the form

$$\text{(5)} \qquad f_\theta(z) = \frac{z - \frac{p(1-e^{i\theta})}{1-p^2e^{i\theta}} z^2}{\left(1 - \frac{z}{p}\right)(1 - zp)}, \quad z \in D, \quad \theta \in [0, 2\pi) \text{ fixed}.$$

In [6], [1] and [3] it has been proved that these functions belong to $Co(p)$ and that they have the above mentioned mapping property. For our aims, we need detailed information about the slit. These informations are the content of the following theorem.

Theorem 3. *Let $\theta \in [0, 2\pi)$ and f_θ be defined by (5) and $\tau \in [0, 2\pi)$ by*

$$\text{(6)} \qquad e^{i\tau} = \frac{e^{i\theta} - p^2}{1 - p^2 e^{i\theta}}.$$

Then

$$\overline{\mathbf{C}} \setminus f_\theta(D) = f(\partial D) =$$

$$\left\{ c \,:\, c = -\frac{p}{1+p^2}\left(1 + e^{i\frac{\tau}{2}}\,\frac{2p\,\cos\left(\psi + \frac{\tau}{2}\right)}{1+p^2-2p\cos\,\psi}\right), \psi \in [0, 2\pi)\right\}. \tag{7}$$

Let d_k, $k = 1, 2$, *denote the endpoints of the slit* (7). *Both of them lie on the boundary of the disc described by* (2), *i. e.*

$$\left| d_k + \frac{p(1+p^2)}{(1-p^2)^2}\right| = \frac{2p^2}{(1-p^2)^2}, \; k = 1, 2. \tag{8}$$

P r o o f. The verification of (7) is accomplished setting $z = e^{i\psi}$, multiplying nominator and denominator of $f_\theta(e^{i\psi})$ by $e^{-i\psi}$ and using the abbreviation (6). To prove (8) we have to determine the values ψ_k, $k = 1, 2$, such that $d_k = e^{i\psi_k}$. These are the roots of the equation

$$\frac{d}{d\psi}\left(\frac{2p\,\cos\left(\psi + \frac{\tau}{2}\right)}{1+p^2-2p\cos\,\psi}\right) = 0.$$

Hence, we get

$$\sin\left(\psi_k + \frac{\tau}{2}\right) = \frac{2p}{1+p^2}\,\sin\,\frac{\tau}{2}.$$

For the proof of (8) we derive from this

$$\cos\left(\psi_k + \frac{\tau}{2}\right) = \frac{\pm\sqrt{(1+p^2)^2 - 4p^2\sin^2\frac{\tau}{2}}}{1+p^2}$$

and

$$\cos\,\psi_k = \frac{2p\sin^2\frac{\tau}{2} \pm \cos\frac{\tau}{2}\sqrt{(1+p^2)^2-4p^2\sin^2\frac{\tau}{2}}}{1+p^2},$$

where, as in the following, the upper sign belongs to ψ_1 and the lower one to ψ_2. Using these formulae and the abbreviation $d_k = x_k + iy_k$, $x_k, y_k \in \mathbf{R}$, $k = 1, 2$, and some elementary computation, we get

$$x_k + \frac{p(1+p^2)}{(1-p^2)^2} = \frac{2p^2}{(1+p^2)(1-p^2)^2}\left(2p\sin^2\frac{\tau}{2} \mp \cos\frac{\tau}{2}\sqrt{(1+p^2)^2-4p^2\sin^2\frac{\tau}{2}}\right)$$

and

$$y_k = \frac{2p^2}{(1+p^2)(1-p^2)^2}\left(-2p\sin\frac{\tau}{2}\cos\frac{\tau}{2} \mp \sin\frac{\tau}{2}\sqrt{(1+p^2)^2-4p^2\sin^2\frac{\tau}{2}}\right).$$

Adding the squares of the right sides reveals the truth of (8). This ends the proof of Theorem 3. □

The formulae (7) and (8) show that for any c in the disc defined by (2) there exists a function f_θ such that $c \in \overline{\mathbf{C}} \setminus f_\theta(D)$. This fact together with Theorem 2 proves Theorem 1.

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