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OVERDETERMINED STRATA IN GENERAL FAMILIES OF POLYNOMIALS

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ABSTRACT. Denote by $\mathrm{Pol}_{\mathbf{C}^n}$ the space of all complex monic degree n polynomials in one variable and by $\mathcal{PP}_n$ the product space $\mathrm{Pol}_{\mathbf{C}^n} \times \mathrm{Pol}_{\mathbf{C}^{n-1}} \times \ldots \times \mathrm{Pol}_{\mathbf{C}^1}$. Stratify the space $\mathcal{PP}_n$ according to the multiplicities of the roots of the n polynomials and the presence of common roots between any two of them. Define the map $\pi : \mathrm{Pol}_{\mathbf{C}^n} \hookrightarrow \mathcal{PP}_n$ by $P \mapsto (P, P'/n, P''/n(n-1), \ldots, P^{(n-1)}/n!)$. A stratum is called *overdetermined* if its codimension in $\mathcal{PP}_n$ is greater than the codimension of its intersection with $\pi(\mathrm{Pol}_{\mathbf{C}^n})$ in $\pi(\mathrm{Pol}_{\mathbf{C}^n})$. In the paper we give different examples of overdetermined strata.

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1. Introduction. Define the *general family* of polynomials of degree n as the family $\mathcal{Q}_n = x^n + a_1x^{n-1} + \ldots + a_n$. In what follows we assume that $a_1 = 0$ (one can shift the variable x); if, in addition, $a_2 \neq 0$, then one can further normalize the family by setting $a_2 = -1$ (one can change the scale of the x-axis).

In the definitions we follow the same ideas as in [3]. Denote by $\mathrm{Pol}_{\mathbf{C}^n}$ the space of all monic degree n polynomials in one variable with complex coefficients. Denote by $\mathcal{PP}_n$ the product space $\mathrm{Pol}_{\mathbf{C}^n} \times \mathrm{Pol}_{\mathbf{C}^{n-1}} \times \ldots \times \mathrm{Pol}_{\mathbf{C}^1}$. A point of $\mathcal{PP}_n$ is an n-tuple of polynomials $(P_n, P_{n-1}, \ldots, P_1)$ of respective degrees.

One can decompose the space $\mathcal{PP}_n$ according to the multiplicities of the roots of the different polynomials and the presence and multiplicities of their common roots. The combinatorial objects enumerating the strata should be called *coloured partitions* since they are partitions of C^2_{n+1} not necessarily distinct points on $\mathbf{C}$ divided into groups of cardinalities $n, n-1, \ldots, 1$ which we can think of as having different colours (it is easy to check that this decomposition is actually a Whitney stratification).

There is a natural embedding map $\pi : \mathrm{Pol}_{\mathbf{C}^n} \hookrightarrow \mathcal{PP}_n$ sending each monic polynomial P of degree n to $(P, P'/n, P''/n(n-1), \ldots, P^{(n-1)}/n!)$.

Let $\mathcal{L}$ be a coloured partition of C^2_{n+1} coloured points, $St_{\mathcal{L}} \subset \mathcal{PP}_n$ be the corresponding stratum and $\pi(St_{\mathcal{L}}) = St_{\mathcal{L}} \cap \pi(\mathrm{Pol}_{\mathbf{C}^n})$ be its (probably empty) intersection with the embedded space of polynomials $\pi(\mathrm{Pol}_{\mathbf{C}^n})$. We call this intersection a stratum in $\mathcal{Q}_n$. Note that $\dim St_{\mathcal{L}}$ equals the number of parts in $\mathcal{L}$.

Definition 1 (B. Z. Shapiro)**.** *The stratum $St_{\mathcal{L}}$ is called* overdetermined *if the codimension c_1 of $St_{\mathcal{L}}$ in $\mathcal{PP}_n$ is greater than the codimension c_0 of $\pi(St_{\mathcal{L}})$ in $\pi(\mathrm{Pol}_{\mathbf{C}^n})$ in the assumption $\pi(St_{\mathcal{L}}) \neq \emptyset$. We call $\pi(St_{\mathcal{L}})$ an* overdetermined stratum in $\mathcal{Q}_n$.

Example 2. *If a coloured partition contains the condition that a multuple root of some P_i should coincide with a root of P_{i+1}, then each such nonempty stratum is overdetermined since a multiple root of $P^{(i)}$ is automatically a root of $P^{(i+1)}$.*

Definition 3. *An overdetermined stratum is called* non-trivial *if the difference $c_1 - c_0$ is due not only to the presence of multiple roots (in P and in its derivatives).*

The aim of the present paper is to present different examples of non-trivial overdetermined strata.

Definition 4. *A real polynomial is called* hyperbolic *(resp.* strictly hyperbolic*) if it has only real roots (resp. only real and distinct roots).*

In previous papers (see [1] and [2]) the non-trivial strata in the case of hyperbolic polynomials of degree ≤ 5 were fully classified (the word "non-trivial" is omitted there). Overdetermined strata of the family $\mathcal{Q}_n$ in the case of real polynomials are defined in the same way as in the complex case.

Remark 5. A polynomial P such that there are $> n-2$ equalities between roots of P, P', ..., $P^{(n-1)}$ belongs to an overdetermined stratum in $\mathcal{Q}_n$. Indeed, the latter depends on $n-2$ parameters (after the normalization $a_1 = 0$, $a_2 = -1$).

2. Examples of overdetermined strata in the case of complex polynomials. The following example shows how to construct overdetermined strata on the basis of polynomials divisible by their derivatives of order k, $k > 1$.

Example 6. *For $l, k \in \mathbf{N}$, $1 < l \leq k < n$, $l \leq n/2$, set $k = ql + r$, $n = q_1 l + r_1$, $q, r, q_1, r_1 \in \mathbf{N} \cup 0$, $r, r_1 \leq l - 1$. Consider the set $\mathcal{S}$ of complex polynomials P such that $\kappa P(x) = x^r \Phi(x^l) P^{(k)}(x)$ $(*)$ for some monic polynomial Φ of degree q and $\kappa = n!/(n-k)!$.*

One checks directly (by comparing the coefficients from left and right in $(*)$) that P must be of the form $x^{r_1}\Psi(x^l)$ where Ψ is a monic polynomial of degree q_1. It is clear that $P^{(i)}(0) = 0$ $(**)$ if $i - r_1$ is not a multiple of l.

Lemma 7. *For $\delta := n - k + q_1(l-1) + r_1 - 1 > n - 2$ $(***)$ the points (i.e. the polynomials) of the set $\mathcal{S}$ belong to overdetermined strata in $\mathcal{Q}_n$.*

Remarks 8. 1) We do not claim that the polynomials from $\mathcal{S}$ belong to one and the same stratum because for some of them additional equalities between roots of P and of its derivatives might hold, hence, the polynomial will belong to a stratum of lower dimension.

2) Condition $(***)$ can be achieved by fixing k and l and by choosing n sufficiently large w.r.t. k (one has $\delta = (n(2l-1)/l) - k + (r_1/l) - 1$).

P r o o f. Condition $(*)$ provides $n - k$ equalities between the roots of P and of $P^{(k)}$, and condition $(**)$ provides $q_1(l-1) + r_1$ such equalities; when defining δ we subtract 1 because P and $P^{(k)}$ might have a common zero root and one condition might be a corollary of the others. There remains to be applied Remark 5. $\square$

Example 9. *Use the notation from Example* 6. *Consider the set* $\mathcal{T}$ *of complex polynomials* P *such that* $\kappa P(x) = (x^k + ax + b)P^{(k)}(x)$ $(****)$, $a, b \in \mathbf{C}^*$ *or* $a, b \in \mathbf{R}^*$.

Lemma 10. *If* k *is large enough, then the set* $\mathcal{T}$ *consists of polynomials belonging to overdetermined strata.*

Proof. For k large enough many of the coefficients of P must be 0 (for such k one can say that $x^k + ax + b$ is a *fewnomial*). Namely, if $P = x^n + a_1x^{n-1} + \ldots + a_n$, then one must have $a_i = 0$ for

$$i = 1, 2, \ldots, k-2, k+1, k+2, \ldots, 2k-3, 2k+1, 2k+2, \ldots, 3k-4, 3k+1, 3k+2, \ldots,$$

i.e. for $k_0 := (k-2) + (k-1) + \ldots + 1 = (k-2)(k-1)/2$ indices i. This means that k_0 of the derivatives of P vanish at 0 which implies that there are at least $k_0 - 1$ equalities between roots of P and its derivatives.

Equality $(****)$ provides $n - k$ equalities between the roots of P and $P^{(k)}$ of which at most one is a corollary of the previously found equalities (some roots might equal 0). Hence, there are at least $s := n - k + (k-2)(k-1)/2 - 2$ equalities between roots of P and its derivatives and for $s > n-2$ (i.e. for $k \geq 5$) the set $\mathcal{T}$ consists of points belonging to overdetermined strata, see Remark 5. $\square$

Remark 11. The above example is not valid for hyperbolic polynomials because the fewnomial $x^k + ax + b$ is not hyperbolic for $k > 3$. (If it were hyperbolic, then so would be its derivative $kx^{k-1} + a$ which is the case only for $a = 0$; hence, one must also have $b = 0$.)

Further we use the *Gauss lemma*, see [4]: *For a complex polynomial* P *of one complex variable the roots of* P' *belong to the convex hull of the roots of* P. *A root of* P' *can belong to the border of this convex hull only if it is also a root of* P *(hence, a multiple root of* P*).*

Lemma 12. *Use again the notation from Example* 6. *If* $l = 2$, *if the polynomials are real, and if* $\Phi(.)$ *is supposed to have only positive real roots, then the set* $\mathcal{S}$ *contains only points belonging to overdetermined strata consisting of hyperbolic polynomials (as* $l = 2$, *they are even or odd together with* n*).*

Proof. Indeed, suppose that P has a complex non-real root x_0, of multiplicity m_0. One can choose x_0 to be a vertex of the convex hull of the set of roots of P. Hence, x_0 is a root of P' of multiplicity $m_0 - 1$. If $m_0 = 1$, then x_0 remains outside the convex hull of the set of zeros of P' and (by induction on k)

of $P^{(k)}$ for $1 \leq k < n$. This means that equality $(*)$ is impossible (recall that all roots of $x^r\Phi(x^l)$ are real). If $m_0 > 1$, then in a similar way one shows that x_0 is a root of $P^{(k)}$ of multiplicity $< m_0$ or a non-root which makes $(*)$ impossible again. □

3. Obtaining new examples by integrating old ones. There is an almost evident way to obtain new overdetermined strata in $\mathcal{Q}_{n+1}$ by integrating polynomials of degree n (which define overdetermined strata in $\mathcal{Q}_n$) and by rescaling the x-axis. In such a way all equalities between roots of $\mathcal{Q}_n$ and its derivatives give rise to equalities between the roots of $\mathcal{Q}'_{n+1}$ and its derivatives.

The codimensions c_0 and c_1 are preserved except for special values of the constant of integration C for which there are new equalities, involving roots of $\mathcal{Q}_{n+1}$. Hence, for such values of C the polynomial belongs to a stratum of lower dimension than for generic values. For these special values of C, however, one still has $c_1 - c_0 > 0$, i.e. the polynomial belongs to an overdetermined stratum. If the stratum in $\mathcal{Q}_n$ which is being integrated is non-trivial, then so are the strata in $\mathcal{Q}_{n+1}$ obtained from it by integration.

If one considers the case of hyperbolic polynomials, then one has to check whether the given polynomial has a hyperbolic primitive which is not always the case.

Notation 13. 1) *For a continuous function* W *we set* $W^{(-1)} = \int_0^x W(t)dt$.

2) *For a hyperbolic polynomial* P *of degree* ≤ 5 *we denote the roots of* P, $\ldots$, $P^{(4)}$ *respectively by* $x_1 \leq \ldots \leq x_5$, $f_1 \leq \ldots \leq f_4$, $s_1 \leq s_2 \leq s_3$, $t_1 \leq t_2$ *and* l_1. *Their arrangement can be described by a* configuration vector *whose components indicate the relative positions of the roots (coinciding roots are put in square brackets). E.g., the configuration vector* $(0, f, s, [0t], f, [0sl], f, [t0], s, f, 0)$ *defines the arrangement*

$$(AR)\ :\ x_1 < f_1 < s_1 < x_2 = t_1 < f_2 < x_3 = s_2 = l_1 < f_3 < t_2 = x_4 < s_3 < f_4 < x_5$$

Example 14. *The polynomial* $W = x^5 - x^3 + 9x/100 = x(x^2 - 1/10)(x^2 - 9/10)$ *is divisible by* $W''' = 60(x^2 - 1/10)$. *It defines an overdetermined stratum realizing the arrangement* (AR). *One has* $W^{(-1)} = x^2(x^4/6 - x^2/4 + 9/200)$ *which is hyperbolic (to be checked directly) and has a double root (a local minimum) at* 0 *and simple roots elsewhere; hence, for* $b > 0$ *small enough the polynomial* $W^{(-1)} -$

b is strictly hyperbolic. It belongs to an overdetermined stratum (because so does W and there hold the same equalities between roots of W and its derivatives, hence, between roots of the derivatives of $W^{(-1)}$). However, for different values of b different arrangements, hence, different strata might be defined. Indeed, denote the roots of $W^{(-1)}$ by $p_1 \leq \cdots \leq p_6$.

A priori one can have exactly one of the three conditions (remember that b can be varied):

a) $p_2 < f_1$ and $p_5 > f_4$;

b) $p_2 = f_1$ and $p_5 = f_4$;

c) $p_2 > f_1$ and $p_5 < f_4$.

In case c) one has a priori also three possibilities:

c1) $p_2 < s_1$ and $p_5 > s_3$;

c2) $p_2 = s_1$ and $p_5 = s_3$;

c3) $p_2 > s_1$ and $p_5 < s_3$.

One has also to study the three possibilities $p_3 < f_2$ and $p_4 > f_3$; $p_3 = f_2$ and $p_4 = f_3$; $p_3 > f_2$ and $p_4 < f_3$ and see how they interact with the possibilities listed above.

We do not claim that all these possibilities are realized but only that they a priori can exist. Notice that when one has equalities, then the corresponding overdetermined stratum is of smaller dimension than the strata in which there are inequalities, see part 1) of Remarks 8.

Remark 15. The above example can be given in the case of complex polynomials as well (one has to forget about hyperbolicity and the inequalities with $>$ or $<$ have to be replaced by inequalities with $\neq$). In Example 14 (in the case of hyperbolic polynomials) we were lucky because there exist values of $b > 0$ for which $W^{(-1)} - b$ is strictly hyperbolic. This is not always the case, see part 1) of the next example.

Definition 16. *For $n \geq 3$ call* Gegenbauer's polynomial *the unique polynomial P with first three coefficients equal to* 1, 0, -1 *which is divisible by its second derivative.*

Remark 17. It turns out that P is even or odd together with n and strictly hyperbolic. For $n \geq 4$ it defines an overdetermined stratum because all its derivatives which are odd polynomials vanish at 0 (and P'' divides P).

Example 18. 1) *Consider the case when n is even. Observe that for Gegenbauer's polynomial one has*

$$mP^{(-1)} = (x^2 - \alpha)P' - 2xP, \quad m = n(n-1) - 2, \quad \alpha = (4n-6)/n(n-1).$$

Hence, $P^{(-1)}$ is the only primitive of P which is hyperbolic (and it is not strictly hyperbolic). Indeed, $P^{(-1)}(0) = 0$ and $P^{(-1)}$ changes sign at the non-zero roots of P except at the biggest and the smallest one (they equal $\pm\sqrt{\alpha}$) where $P^{(-1)}$ has double roots. The left of these double roots is a local maximum and the right one is a local minimum. Hence, for any $b \in \mathbf{R}^*$ the polynomial $P^{(-1)} + b$ is not hyperbolic.

2) *If n is odd, then $P_1 := ((x^2 - \alpha)P' - 2xP)/m$ is a primitive of P (different from $P^{(-1)}$), with $P_1(0) < 0$ for $n = 4n_1 + 1$ and $P_1(0) > 0$ for $n = 4n_1 - 1$; P_1 changes sign at the roots of P except at $\pm\sqrt{\alpha}$ where it has double roots which are local minima. Hence, for $b > 0$ small enough the polynomial $P_1 - b$ is strictly hyperbolic.*

3) *The polynomials $P^{(-1)}$ from 1) and P_1 from 2) (for $n \geq 5$) belong to overdetermined strata.*

Lemma 19. *Any natural power $k > 1$ of Gegenbauer's polynomial P with $n \geq 3$ defines a non-trivial overdetermined stratum in $\mathcal{Q}_{kn}$.*

Proof. The polynomial P^k has n k-fold roots which gives $n(k-1)$ equalities between roots of P^k and its derivatives. There are $n-2$ roots in common between $(P^k)^{(k+1)}$ and P^k. Indeed, $(P^k)^{(k+1)}$ is a sum of products of k factors; each factor is either P or one of its derivatives. In each product the sum of the orders of derivation equals $k+1$. Hence, each product contains at least one factor P or P''. Thus the common roots of P and P'' add $n-2$ more equalities and there remain to be added the equalities due to the vanishing of the derivatives of odd order at 0. (These equalities make the stratum non-trivial.) Hence, there are more than $kn-2$ independent equalities between roots of Q_{kn} and its derivatives. There remains to use Remark 5. □

Open questions 20. 1) *Characterize all overdetermined strata in $\mathcal{Q}_n$.*

2) *Call an overdetermined stratum in $\mathcal{Q}_n$* new *if it is not obtained from an overdetermined stratum in $\mathcal{Q}_{n-1}$ as a result of integration. Is it true or not that in the case of strictly hyperbolic polynomials all new overdetermined strata contain only polynomials which are even or odd together with n ?*

4. Another example. The following example shows that overdetermined strata are not always defined as in Examples 6 and 9, i.e. when P is divisible by $P^{(k)}$ for some $1 < k$ (we nevertheless exclude the trivial case when $k = n-1$ and n is odd), or by integrating such examples, see Example 18. In order to better visualize the arrangement of the roots of a polynomial of degree 6 and of its derivatives we represent them in a triangle. We denote the roots of P, P', $\ldots$, $P^{(5)}$ respectively by x_i, f_i, s_i, t_i, F_i, l (to match "first", "second", "third", "fourth" and "last"). The places of the roots in the triangle reveal the fact that the roots of $P^{(k)}$ are between the roots of $P^{(k-1)}$:

$$\begin{array}{ccccccccccc} & & & & & l & & & & & \\ & & & & F_1 & & F_2 & & & & \\ & & & t_1 & & t_2 & & t_3 & & & \\ & & s_1 & & s_2 & & s_3 & & s_4 & & \\ & f_1 & & f_2 & & f_3 & & f_4 & & f_5 & \\ x_1 & & x_2 & & x_3 & & x_4 & & x_5 & & x_6 \end{array}$$

Lemma 21. *Set $P = x^6 - x^4 + ax^2 + b$, $a, b \in \mathbf{R}$. The coefficients a, b can be chosen such that P be strictly hyperbolic and*

a) $t_1 = x_2$, $t_3 = x_5$, $s_2 = x_3$, $s_3 = x_4$;

b) for $0 \leq i < j \leq 4$, $P^{(i)}$ is not divisible by $P^{(j)}$.

By Remark 5, for the given a, b the polynomial P belongs to an overdetermined stratum. Indeed, except the four equalities from a) all derivatives of odd order vanish at 0 which adds two more equalities.

Proof of Lemma 21. 1^0. One has $P''' = 120x(x^2 - 1/5)$. Hence, $t_{1,3} = \pm 1/\sqrt{5}$ and one has $t_1 = x_2$, $t_3 = x_5$ only if $P(\pm 1/\sqrt{5}) = 0$, i.e. only if $b = 4/125 - a/5$ (A).

2^0. Next, one has $P'' = 30x^4 - 12x^2 + 2a$ and

$$P = (x^2/30 - 1/50)P'' + R \;\text{ where }\; R = (14a/15 - 6/25)x^2 + b + a/25 \;.$$

The roots $s_{2,3}$ must be roots of R as well as of P''. Hence, one must have

$$s_{2,3}^2 = -\frac{b + a/25}{14a/15 - 6/25} = \frac{30a - 6}{175a - 45} \;\text{ (we use } (A) \text{ here).}$$

The condition $P''(s_{2,3}) = 0$ is equivalent to

$$15(30a-6)^2 - 6(30a-6)(175a-45) + a(175a-45)^2 = 0$$

3^0. Set $c = 5a$. The last equation takes the form

$$U(c) := 49c^3 - 270c^2 + 441c - 216 = 0$$

One has $U'(x) > 0$ for $x \in [0,1]$. Indeed, for $x \in [0,1]$ one has $U'' < 0$; as $U'(1) > 0$, one has $U'(x) > 0$ for $x \in [0,1]$. As $U(0) < 0$, $U(1) > 0$, the last equation has a single real root in $(0,1)$. For this root one has $a \in (0,1/5)$. For the given value of a (hence, of a, b)

– the polynomials P'' and P' are strictly hyperbolic (follows from $a \in (0,1/5)$);

– conditions a) hold.

4^0. As $x_2 = t_1 < s_2 = x_3 < s_3 = x_4 < t_3 = x_5$, all roots x_2, x_3, x_4, x_5 are real. The polynomial P is even and $x_2 = -x_5$, $x_3 = -x_4$ because P'' is even and P''' is odd. So either x_1, x_2 are real, one has $x_1 < x_2$, $x_5 < x_6$ and P is strictly hyperbolic (equalities are impossible because $x_2 = t_1 < x_3$, $x_4 > t_3 = x_5$), or $x_{1,6}$ are complex conjugate. The last possibility must be excluded because in this case x_1 and x_6 must be purely imaginary, x_2 and x_3 must be vertices of the convex hull of the set of roots of P and f_1, f_5 must lie outside this convex hull which contradicts the Gauss lemma. Hence, P is strictly hyperbolic.

5^0. There remains to check that condition b) also holds. One cannot have $P^{(j)}|P^{(i)}$ if j is odd and i is even. It is also clear from condition a) that one does not have $P''|P$, $P''|P'$ or $P'''|P'$. One has $P^{(4)}|P''$ (resp. $P^{(4)}|P'$) only if $a = 1/3 \notin (0,1/5)$ (resp. $a = 3/25$, hence, $c = 3/5$, which is not a root of U). One cannot have $P^{(4)}|P$ because either $F_1 = x_2$, $F_2 = x_5$ (impossible by a)) or $F_1 = x_3 = s_2$, $F_2 = x_4 = s_3$ and $P^{(4)}|P''$ which possibility is already excluded. □

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