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ESTIMATE FOR THE NUMBER OF ZEROS OF ABELIAN INTEGRALS ON ELLIPTIC CURVES*

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Abstract. For $h \in \left(0, \dfrac{1}{6}\right)$ we obtain an upper bound for the number of zeros of the Abelian integral $h \to I(h) = \displaystyle\int_{\delta(h)} [g(x,y)dx - f(x,y)dy]$, where $\delta(h)$ is the closed connected component of the level curve $\left\{\dfrac{x^2+y^2}{2} - \dfrac{x^3}{3} + axy^2 = h\right\}$, $a \in (0,1)$. The bound depends explicitly on the maximum of degrees of the polynomials f and g.

1. Introduction. Let $H(x,y)$, $(x,y) \in \boldsymbol{R}^2$ be a polynomial of degree m (we call it Hamiltonian) and let $f(x,y)$, $g(x,y)$ be two polynomials with degrees not exceeding n. Let $\Sigma = (\sigma_1, \sigma_2) \subset \boldsymbol{R}$ be a maximal interval of existence of a

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continuous family of ovals (closed connected components $\delta(h)$ of the real algebraic curve $\{H(x,y)=h\}$), such that for $h\in(\sigma_1,\sigma_2)$ the ovals $\delta(h)$ are free of critical points. Define the complete Abelian integral

$$I(h)=\int\limits_{\delta(h)}\left[g(x,y)dx-f(x,y)dy\right],\qquad h\in\Sigma. \tag{1}$$

The infinitesimal Hilbert problem is to give an estimate $Z(m,n)$ of the number of zeros of the Abelian integral $I(h)$,$h\in\Sigma$ only in terms of the degrees of the polynomials H,f,g. In such a form the problem was formulated by V. I. Arnold [1], and investigated also in the paper Yu. Ilyashenko [9].

Let us recall briefly the connection between the infinitesimal Hilbert problem and the second part of the 16th Hilbert problem. Consider the perturbed Hamiltonian system

$$\begin{aligned}\dot{x}&=H_y+\varepsilon f,\\ \dot{y}&=-H_x+\varepsilon g,\end{aligned} \tag{2}$$

where ε is a small parameter. In the context of system (2), the second part of the 16th Hilbert problem is to find the maximal number and the position of Poincare limit cycles of this system for ε small.

The zeros of the displacement function

$$P_\varepsilon(h)-h=\varepsilon I(h)+O(\varepsilon^2)$$

give the limit cycles of (2). The first coefficient $I(h)$ in this expansion in ε is exactly the Abelian integral (1).

The first general result in the direction of solving the infinitesimal Hilbert problem was achieved by A.Varchenko [21] and A. Khovanskii [12], who proved independently the existence of the upper estimate $Z(m,n)$. However, their estimate does not depend explicitly on m and n. Recently, in a series of papers, Yu. Ilyashenko, S. Yakovenko and D. Novikov [10, 11, 14, 15, 22] introduce a new approach for estimating the number of zeros of $I(h)$ in terms of the Hamiltonian H and the degree n of the perturbation. Their best result is that for any Hamiltonian H from a certain open set of "good" Hamiltonians there exists a constant $c(H)<\infty$ such that the number of real isolated zeros of $I(h)$ in Σ does not exceed $\exp(c(H)n))$.

Better estimates for $Z(m,n)$ for certain Hamiltonians of low degree are obtained in the papers [3], [4]–[7], [16]–[20].

The main result of the present paper is the following:

Theorem 1. *For the degenerate cubic Hamiltonians*

$$H(x,y) = \frac{x^2+y^2}{2} - \frac{x^3}{3} + axy^2, a \in (0,1) \tag{3}$$

the number Z of the isolated zeros of $I(h)$ in $\left(0, \frac{1}{6}\right)$ (the maximal interval of existence of $\delta(h)$) satisfies the estimates:

- $Z \le 4\left[\frac{n}{3}\right] + 2$ *for* $n > 1$,
- $Z = 0$ *for* $n = 1$.

The Hamiltonians of the same type appear as a particular case in [6], with a weaker estimate for Z, namely $Z \le \left[\frac{7n+13}{3}\right]$.

2. Properties of the Abelian integrals. Consider the Hamiltonian system in the plaine:

$$\begin{cases} \dot{x} = H_y, \\ \dot{y} = -H_x, \end{cases} \tag{4}$$

where $H(x,y)$ are the cubic polynomials from (3).

This system has four equilibrium points. Two of them – $M_1(0,0)$ and $M_2(1,0)$ lie correspondingly on the critical levels $\{H{=}h_1{=}0\}$ and $\left\{H{=}h_2{=}\frac{1}{6}\right\}$. The values h_3, h_4 of $H(x,y)$ in the other two equilibrium points $M_3\left(-\frac{1}{2a}, -\frac{1}{2a}\sqrt{\frac{2a+1}{a}}\right)$; $M_4\left(-\frac{1}{2a}, \frac{1}{2a}\sqrt{\frac{2a+1}{a}}\right)$ coincide and are equal to $h_3 = h_4 = \frac{3a+1}{24a^3}$. The point M_1 is a center, and the points M_2, M_3, M_4 are saddle points of the system (4). The phase portrait of the system is given on Figure 1. For $0 < h < 1/6$ the level curve $\{H = h\} \subset \boldsymbol{R}^2$ has a unique closed orbit (an oval) for (4). As above, we will denote this oval by $\delta(h)$. Its interior contains only one equilibrium point of the Hamiltonian system – the center M_1.

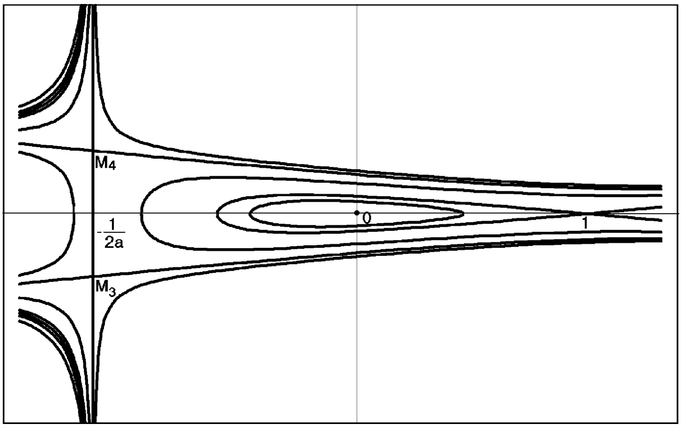

Fig. 1. Phase portrait of system (4).

Let $I(h)$ denote the Abelian integral from (1) with $\Sigma = \left(0, \dfrac{1}{6}\right)$. $I(h)$ can be continued analytically along any path in $\boldsymbol{C} \setminus \{h_1, h_2, h_3\}$. Let $H_1(\Gamma_h, \boldsymbol{Z})$ denote the first homology group of the affine algebraic curve

$$\Gamma_h : \{H(x, y) = h\} \subset \boldsymbol{C}^2, \quad h \in \boldsymbol{C}.$$

We will denote by $\gamma_1(h) \equiv \delta(h)$ the vanishing cycle at the point M_1, by $\gamma_2(h)$ – the vanishing cycle at M_2, and by $\gamma_3(h)$, and $\gamma_4(h)$ – the vanishing cycles at M_3 and M_4. We denote also

$$D_\rho = \{|z| < \rho\} \setminus \{h_1, h_2, h_3\} \subset \boldsymbol{C},$$

where ρ is a fixed sufficiently big real number. Let z_0 be a point on the boundary $\{|z| = \rho\}$ of D_ρ. Suppose that $l^i \in \pi(D_\rho, z_0)$ is a simple loop around h_i, $i = 1, 2$, and l^i_* is the induced by l^i monodromy operator acting on the first homology group

$$l^i_* : H_1(\Gamma_h, \boldsymbol{Z}) \longrightarrow H_1(\Gamma_h, \boldsymbol{Z})$$

of the affine algebraic curve $\Gamma_h : \{H = h\} \subset \boldsymbol{C}^2$.

The Picard-Lefschetz formula reads:

$$l^i_*(\gamma) = \gamma + \langle \gamma_i \circ \gamma \rangle \gamma_i, \qquad i = 1, 2. \tag{5}$$

where γ is an arbitrary cycle from $H_1(\Gamma_h, \boldsymbol{Z})$, and $\langle \gamma_i \circ \gamma \rangle$ is the intersection index of the cycles γ_i and γ.

Since for $h \to h_3$ we have two vanishing cycles – γ_3 and γ_4, then the monodromy operator, induced by the loop around the point h_3, is a composition of two monodromy operators – the first one, induced by γ_3, and the second, induced by γ_4.

The Picard-Lefschetz formula implies some formulas describing the branching of the Abelian integral near the critical values. Namely, if ω is a meromorphic one-form without residua, then in a neighborhood of h_2 in the complex domain we have:

$$\int\limits_{\gamma_1(h)} \omega = \frac{\log(h - h_2)}{2\pi i} \int\limits_{\gamma_2(h)} \omega + \Phi(h). \tag{6}$$

$\Phi(h)$ is a holomorphic single-valued function near h_2. Similar formulas are valid for the integrals on γ_j, $j = 2, 3, 4$ near $h_1 = 0$. In a neighborhood of h_3 one has:

$$\int\limits_{\gamma_1(h)} \omega = \frac{\log(h - h_3)}{2\pi i} \int\limits_{\gamma_3(h)+\gamma_4(h)} \omega + \Phi_1(h), \tag{7}$$

with $\Phi_1(h)$ a holomorphic single-valued function in a neighborhood of h_3.

We will denote by $J(h)$ the derivative of $I(h)$:

$$J(h) = \frac{d}{dh} I(h).$$

Since $I(h)$ is an integral over the cycle vanishing at $h = 0$, then $I(0) = 0$. Therefore the number of zeros of $I(h)$ in the interval $(0, \frac{1}{6})$ is less or equal to the number of zeros of $J(h)$ in the same interval.

For any $n \in \boldsymbol{N}$, define the numbers μ, ν, ρ by:

$$\mu = \left[\frac{n-1}{3}\right];$$

$$\nu = \left[\frac{n-2}{3}\right];$$

$$\rho = \left[\frac{n}{3}\right] - 1.$$

The following decomposition of the integrals $J(h)$ is proved in [6]:

$$J(h) = P_\mu(h)J_0(h) + Q_\nu(h)J_1(h) + R_\rho(h)J_2(h),$$

where

$$J_j(h) = \int\limits_{\delta(h)} \tilde{\omega}_j, j = 0, 1, 2$$

with

$$\tilde{\omega}_j = \frac{x^j dx}{y(2ax+1)}, \qquad j = 0, 1, 2.$$

and $P_\mu(h)$; $Q_\nu(h)$; $R_\rho(h)$ are polynomials on h of the corresponding degrees (if the degree is negative, the polynomial is taken to be identically zero). The polynomials $P_\mu(h)$; $Q_\nu(h)$; $R_\rho(h)$ have real coefficients.

We denote:

$$F(h) = \frac{J(h)}{J_0(h)}.$$

In order to estimate the number of zeros of $F(h)$, one can use the argument principle for a suitable complex domain.

The proofs of Theorem 2, Lemma 1 – Lemma 5, given below, are similar to the proofs of the corresponding statements in [4] and will be omitted here.

Lemma 1. *$J_0(h) \neq 0$ in $\boldsymbol{C} \setminus \{h_2, h_3\}$.*

Lemma 2. *$F(h)$ is a holomorphic single-valued function in $\boldsymbol{C} \setminus [h_2, +\infty)$.*

The integral $J_j(h)$; $j = 0, 1, 2$ (and similarly $F(h)$) has two different analytic continuation across the cutting $[h_2, +\infty)$ depending on the way h approaches this cut-from the upper or from the lower complex half-plane. Let us denote the continuation of $J_j(h)$ (resp. $F(h)$) from the upper half-plane by $J_j^+(h)$ ($F^+(h)$), and from the lower half-plane – by $J_j^-(h)$ ($F^-(h)$).

The increasing of the argument of $F(h)$ along the intervals $(h_2, h_3) \cup (h_3, +\infty)$ can be estimated (using an idea of Petrov) by counting the number of zeros of $ImF(h)$ in these intervals.

We have the following:

Lemma 3.

$$ImJ_j^{\pm}(h) = \begin{cases} \pm\dfrac{i}{2}\displaystyle\int\limits_{\gamma_2(h)} \tilde{\omega}_j; & h \in (h_2, h_3)\,, \\ \pm\dfrac{i}{2}\displaystyle\int\limits_{\gamma_2+\gamma_3+\gamma_4} \tilde{\omega}_j; & h \in (h_3, +\infty)\,, \quad j = 0, 1, 2. \end{cases}$$

Below the following Wronskians will play an important role:

$$W_{\gamma_i,\gamma_j}(\omega_k, \omega_l) = \begin{vmatrix} \int\limits_{\gamma_i} \omega_k & \int\limits_{\gamma_j} \omega_k \\ \int\limits_{\gamma_i} \omega_l & \int\limits_{\gamma_j} \omega_l \end{vmatrix},$$

where $\omega_k, \omega_l \in H^1(\Gamma_h, \boldsymbol{C})$ are arbitrary one-forms and $\gamma_i, \gamma_j \in H_1(\Gamma_h, \boldsymbol{Z})$ are one-cycles. Let us introduce the notation

$$\tilde{\omega} = P_\mu(h)\tilde{\omega}_0 + Q_\nu(h)\tilde{\omega}_1 + R_\rho(h)\tilde{\omega}_2.$$

The function F can be continued on the interval (h_2, ∞) but its values depend on the path along which we make the continuation. More precisely, take a point $h' \in (-\infty, h_2)$, and a point $h'' \in (h_2, \infty)$. If we connect h' and h'' with a path $h(s)$, $s \in [0, 1]$, $h(0) = h'$, $h(1) = h''$ and $Imh(s) > 0$, $s \in (0, 1)$ then we denote the result by $F^+(h'')$. Similarly we define $F^-(h'')$.

The following statement holds:

Theorem 2.

$$ImF^{\pm}(h) = \begin{cases} \dfrac{\pm i}{2|J_0(h)|^2} W_{\gamma_1,\gamma_2}(\tilde{\omega}, \tilde{\omega}_0), & h \in (h_2, h_3), \\ \dfrac{\pm i}{2|J_0(h)|^2} W_{\gamma_1,\gamma_2+\gamma_3+\gamma_4}(\tilde{\omega}, \tilde{\omega}_0), & h \in (h_3, +\infty)\,. \end{cases}$$

We adopt the notations:

$$G_1(h) = iW_{\gamma_1,\gamma_2}(\tilde{\omega}, \tilde{\omega}_0); G_2(h) = iW_{\gamma_1,\gamma_2+\gamma_3+\gamma_4}(\tilde{\omega}, \tilde{\omega}_0).$$

To estimate the number of zeros of $G_1(h)$ and $G_2(h)$, we will apply the argument principle to the analytic continuation of these functions in a suitable complex domain.

We denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the following domains: $\mathcal{D}_1 = \boldsymbol{C}\setminus[h_3,+\infty)$; $\mathcal{D}_2 = \boldsymbol{C}\setminus[h_2,h_3]$.

Lemma 4. *The functions $G_1(h)$ and $G_2(h)$ are holomorphic and single-valued in the domains $\mathcal{D}_1$ and $\mathcal{D}_2$ correspondingly.*

Define $G_1^{\pm}$ on the interval (h_3,∞) and $G_2^{\pm}$ on the interval (h_2,h_3) in a similar way as we did it for $F^{\pm}$.

Lemma 5.

$$ImG_1^{\pm} = \pm\frac{1}{2}W_{\gamma_3+\gamma_4,\gamma_2}(\tilde{\omega},\tilde{\omega}_0);\;\; h\in(h_3,+\infty).$$

$$ImG_2^{\pm} = \pm\frac{1}{2}W_{\gamma_2,\gamma_3+\gamma_4}(\tilde{\omega},\tilde{\omega}_0);\;\; h\in(h_2,h_3).$$

The cycles γ_2, γ_3 and γ_4 are homologous over $\overline{\Gamma}_h$. Denote by ∞_1, ∞_2, ∞_3 the tree infinite points on $\overline{\Gamma}_h$. Calculating the residues of $\tilde{\omega}_1$ and $\tilde{\omega}_2$ in these points, we obtain:

$$Res\tilde{\omega}_1|_{x=\infty_1} = -\sqrt{\frac{3}{2a}};\; Res\tilde{\omega}_1|_{x=\infty_2} = \sqrt{\frac{3}{2a}};\; Res\tilde{\omega}_1|_{x=\infty_3} = 0;$$

$$Res\tilde{\omega}_2|_{x=\infty_1} = -\sqrt{6a}(1-3a);\; Res\tilde{\omega}_2|_{x=\infty_2} = \sqrt{6a}(1-3a);\; Res\tilde{\omega}_2|_{x=\infty_3} = 0.$$

Then

$$(8)\quad W_{\gamma_3+\gamma_4,\gamma_2}(\tilde{\omega},\tilde{\omega}_0) = l\pi i\left[\sqrt{\frac{3}{2a}}Q_\nu(h) + \sqrt{6a}(1-3a)R_\rho(h)\right]\left(\int\limits_{\gamma_2(h)}\tilde{\omega}_0\right).$$

The value of l is one of the numbers 2, -2, 4, -4 and depends on the position of the three infinite points with respect to the cycles γ_2, γ_3, γ_4.

We introduce the notations:

$$I_j(h) = \int\limits_{\omega} x^j y dx, \qquad j = 0,1,2.$$

where ω is an arbitrary one-cycle on the Riemann surface Γ_h. Let $I_j'(h)$ denote the derivatives of the integrals above for $j = 0,1,2$.

We will prove:

Lemma 6. *The integrals $I_0'(h)$; $I_1'(h)$; $I_2'(h)$ have the following expansions near infinity:*

$$I_0'(h) = h^{-\frac{2}{3}} p_0(h) + h^{-2} q_0(h) + h^{-\frac{1}{3}} r_0(h);$$
$$I_1'(h) = p_1(h) + h^{-\frac{2}{3}} q_1(h) + h^{-\frac{1}{3}} r_1(h);$$
$$I_2'(h) = p_2(h) + h^{\frac{1}{3}} q_2(h) + h^{-\frac{1}{3}} r_2(h)$$

where $p_j(h)$; $q_j(h)$; $r_j(h)$; $j = 0; 1; 2$ are single-valued holomorphic functions defined in a neighborhood of the infinity.

Proof. Using the almost unchanged arguments from [7], one can prove that the integrals $I_j(h)$; $j = 0, 1, 2$ satisfy the following Picard-Fuchs equations:

$$\begin{cases} -{I_1}' + (a+1){I_2}' - 6ah{I_0}' + 4aI_0 = 0; \\ (-12a^2h + a + 1){I_1}' + (2a^2 - a - 1){I_2}' + \\ a(1-a)I_0 + 12a^2 I_1 = 0; \\ 2a(6h-1){I_2}' + (6h-1){I_1}' + (a+1)I_0 + \\ (3a-7)I_1 - 16aI_2 = 0. \end{cases}$$

Applying the general theory of Fuchsian singularities ([23]), one can express the solutions of this system as series, converging near $h = \infty$ as follows:

$$I_0(h) = h^{\frac{1}{3}} \tilde{p}_0(h) + \tilde{q}_0(h) + h^{\frac{2}{3}} \tilde{r}_0(h);$$
$$I_1(h) = h^{\frac{1}{3}} \tilde{p}_1(h) + h\tilde{q}_1(h) + h^{\frac{2}{3}} \tilde{r}_1(h);$$
$$I_2(h) = h^{\frac{4}{3}} \tilde{p}_2(h) + h\tilde{q}_2(h) + h^{\frac{2}{3}} \tilde{r}_2(h),$$

where $\tilde{p}_j(h)$; $\tilde{q}_j(h)$; $\tilde{r}_j(h)$; $j = 0; 1; 2$ are single-valued holomorphic functions near the infinite point. In order to complete the proof of the theorem, it remains to differentiate the expressions for $I_j(h)$; $j = 0, 1, 2$.

Lemma 7. *$J_0(h)$ decreases at infinity not faster than $h^{-\frac{2}{3}}$.*

The proof of the Lemma 7 is similar to the proof of the corresponding lemma in [4].

3. Upper bounds for the number of zeros of $G_1(h)$, $G_2(h)$, $F(h)$. Denote by D_1 the domain obtained from the circle centered at 0 with radius R ($R > 0$ is sufficiently big) by removing the circle with center in h_3 and radius r ($r > 0$ – sufficiently small) and cutting along the interval on the real axis lying between the points $h_3 + r$ and R, and by D_2 – the domain obtained from the circle $\{|h| \le R\}$ ($R > 0$ – sufficiently big) by removing the circles $\{|h - h_2| \le r_1\}$ and $\{|h - h_3| \le r_2\}$ (r_1, $r_2 > 0$ – sufficiently small) and cutting along the interval $[h_2 + r_1, h_3 - r_2]$.

Denote by m_1 the number of zeros of the polynomial

$$\sqrt{\frac{3}{2a}}Q_\nu(h) + \sqrt{6a}(1-3a)R_\rho(h)$$

in the interval $(h_3, +\infty)$, and by m_2 – the number of its zeros in the interval (h_2, h_3). (This polynomial is taken from (8).)

First we need to estimate the number of zeros of G_1, G_2.

Suppose that Z_1 is the number of zeros of $G_1(h)$ in the domain D_1, and Z_2 – the number of zeros of $G_2(h)$ in the domain D_2.

Theorem 3.

- *If* $\sqrt{\frac{3}{2a}}Q_\nu(h) + \sqrt{6a}R_\rho(h) \not\equiv 0$, *then*

  $Z_j \le \left[\frac{n-2}{3}\right] + m_j + 1$ *for* $n = 3k$ *or* $n = 3k+1, k \ge 1$; $j = 1, 2$ *and*

  $Z_j \le \left[\frac{n-2}{3}\right] + m_j$ *for* $n = 3k+2, k \ge 0$; $j = 1, 2$.

- *If* $\sqrt{\frac{3}{2a}}Q_\nu(h) + \sqrt{6a}R_\rho(h) \equiv 0$, *then*

  $Z_j \le \left[\frac{n-2}{3}\right] + 1$ *for* $n = 3k$ *or* $n = 3k+1, k \ge 1$; $j = 1, 2$ *and*

  $Z_j \le \left[\frac{n-2}{3}\right]$ *for* $n = 3k+2, k \ge 0$; $j = 1, 2$.

P r o o f. The theorem is proved by the use of the argument principle.
Proof for Z_2.
The function $G_2(h)$ has the following expansion in a neighborhood of h_2:

$$G_2(h) = p_1(h-h_2)\ln(h-h_2) + q_1(h-h_2),$$

with $p_1(h-h_2), q_1(h-h_2)$ – holomorphic single-valued functions near h_2. If $[p_1(0)]^2+[q_1(0)]^2>0$, then the increase of the argument of $G_2(h)$ on $\{|h-h_2|=r_1\}$ can be made arbitrary small for $r_1>0$ – sufficiently small. If both $p_1(h-h_2)$ and $q_1(h-h_2)$ have zeros of multiplicity k in the point h_2, then the argument of $G_2(h)$ on $\{|h-h_2|=r_1\}$ decreases with about $2\pi k$. Therefore, the increase of the argument of $G_2(h)$ in the positive direction on the considered circle can be made arbitrary small for sufficiently small $r_1>0$.

Note that $G_2^+(h_2+r_1)=\overline{G_2^-}(h_2+r_1)$.

Similarly, since near the point h_3 the function $G_1(h)$ can be expressed in the following way:

$$G_1(h)=p_2(h-h_3)\ln(h-h_3)+q_2(h-h_3)$$

with $p_2(h-h_3), q_2(h-h_3)$ – holomorphic and single-valued near h_3, then the increase of $argG_1(h)$ in a positive direction on $\{|h-h_3|=r_2\}$ can be made arbitrary small for $r_2>0$ sufficiently small.

Similarly $G_1^+(h_3+r_2)=\overline{G_1^-}(h_3+r_2)$.

If $n=3k$ or $n=3k+1$, $k\geq 1$, then on the big circle the argument of $G_2(h)$ increases with $\left[\dfrac{n-2}{3}\right]2\pi\pm\varepsilon$ (here $\varepsilon>0$ can be made arbitrary small for R – sufficiently big).

If $n=3k+2$, $k\geq 0$, then on $\{|h|=R\}$, the $argG_2(h)$ increases with no more than $\left[\dfrac{n-2}{3}\right]2\pi-\dfrac{2\pi}{3}\pm\varepsilon_1$ (again $\varepsilon_1>0$ is arbitrary small for R sufficiently big).

- The case $\sqrt{\dfrac{3}{2a}}Q_\nu(h)+\sqrt{6a}(1-3a)R_\rho(h)\not\equiv 0$.
  On the part of the contour of D_2, consisting on the small circles and the boards of the cuts, $argG_2(h)$ increases with no more than $2\pi(m_2+1)$. Therefore, when h runs over the whole contour of D_2 in the positive direction, the argument of $G_2(h)$ increases with no more than
  $2\pi\left\{\left[\dfrac{n-2}{3}\right]+m_2+1\right\}+\varepsilon$ if $n=3k$ or $n=3k+1$, $k\geq 1$, and with no more than $2\pi\left\{\left[\dfrac{n-2}{3}\right]+m_2\right\}+\dfrac{\pi}{3}+\varepsilon_1$ if $n=3k+2$ with $k\geq 0$.

- The case $\sqrt{\dfrac{3}{2a}}Q_\nu(h)+\sqrt{6a}(1-3a)R_\rho(h)\equiv 0$.
  Then $G_2(h)$ can be continued up to a holomorphic single-valued function

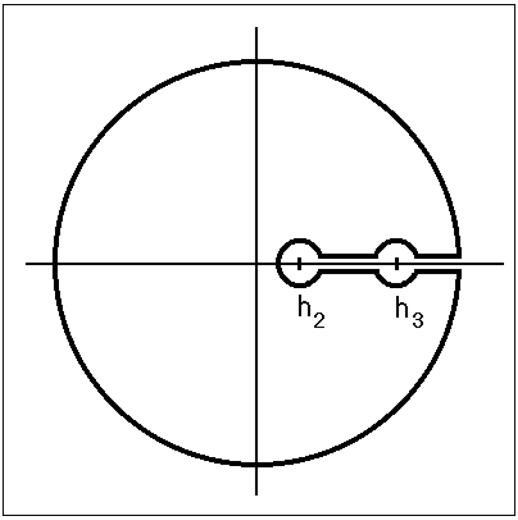

Fig. 2. The domain D

in $\boldsymbol{C}\setminus\{h_2, h_3\}$. In this case instead of D_2 one can consider the domain D_2' obtained by removing from $\{|h| \leq R\}$ two small circles $\{|h-h_2| \leq r_1\}$ and $\{|h-h_3| \leq r_2\}$.

The increase of $argG_2(h)$ on the small circles can be made arbitrary small for sufficiently small r_1 and r_2.

Therefore $Z_2 \leq \left[\dfrac{n-2}{3}\right]$ for $n=3k$ or $n=3k+1$, $k \geq 1$ and $Z_2 \leq \left[\dfrac{n-2}{3}\right] - 1$ for $n = 3k+2$, $k \geq 0$.

The estimates for Z_1 can be proved in a similar way, which completes the proof of the theorem.

P r o o f o f T h e o r e m 1. We consider the domain D, obtained by removing from the circle $\{|h| \leq R\}$, with $R > 0$ sufficiently big, the circles $\{|h-h_2| \leq r_1\}$ and $\{|h-h_3| \leq r_2\}$ with $r_1, r_2 > 0$ – sufficiently small, and cutting it along two real intervals: the interval $[h_2+r_1, h_3-r_2]$, and the interval $[h_3+r_2, R]$. The domain D is shown on Figure 2.

Let us denote by $\tilde{Z}$ the number of zeros of $F(h)$ in the domain above. Since $Z \leq \tilde{Z}$, it is sufficient to estimate $\tilde{Z}$ from above using the argument principle.

One can suppose that $\lim\limits_{h\to h_2} F(h) \neq 0$ and $\lim\limits_{h\to h_3} F(h) \neq 0$. Indeed, it is easy to see that this is the generic case. If some, or both, of these limits are zero, then, varying the coefficients of the polynomials f and g, one can approximate $F(h)$ by functions, having nonzero limits in h_2 and h_3 and the needed estimate can be obtained by passing to the limit.

Under the assumptions above the increase of $argF(h)$ along $\{|h-h_2|=r_1\}$ and along both arcs of $\{|h-h_3|=r_2\}$ can be made arbitrary small for $r_1>0$ and $r_2>0$ sufficiently small. It is important to note that the images of h_2+r_1 under $F^+(h)$, respectively $F^-(h)$, are complex conjugate, and the same is true for the images of h_3-r_2 and h_3+r_2 under these functions.

- The case $n=3k$ or $n=3k+1, k\geq 1$.
  Along the circle $\{|h|=R\}, argF(h)$ increases with no more than $\left[\frac{n}{3}\right]2\pi \pm \varepsilon_2 (\varepsilon_2>0$ can be made arbitrarily small for R sufficiently big). Along the rest of the contour of $D\, argF(h)$ increases with no more than $2\pi\left\{3\left[\frac{n-2}{3}\right]+4\right\}\mp\varepsilon_2$.

  Therefore, when h runs along the contour of D in the positive direction, $argF(h)$ increases with no more than
  $$2\pi\left\{\left[\frac{n}{3}\right]+3\left[\frac{n-2}{3}\right]+4\right\}=2\pi\left\{4\left[\frac{n}{3}\right]+1\right\}.$$

- The case $n=3k+2, k\geq 0$.
  Along the big circle, $argF(h)$ increases with no more than $2\pi\left\{\left[\frac{n-2}{3}\right]+\frac{2}{3}\right\}\pm\varepsilon_3$ ($\varepsilon_3>0$ arbitrarily small for R sufficiently big). On the rest part of the contour of D the argument of $F(h)$ increases with no more than $2\pi\left\{3\left[\frac{n-2}{3}\right]+\frac{4}{3}\right\}\mp\varepsilon_3$. Therefore when h runs in the positive direction over the contour of D, $argF(h)$ increases with no more than
  $$2\pi\left\{4\left[\frac{n-2}{3}\right]+2\right\}=2\pi\left\{4\left[\frac{n}{3}\right]+2\right\}.$$

- The case $n=1$.
  Then $Q_\nu(h)=R_\rho(h)\equiv 0$ and $F(h)=P_\mu(h)=const\neq 0$.

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