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OPTIMALITY CONDITIONS FOR D.C. VECTOR OPTIMIZATION PROBLEMS UNDER D.C. CONSTRAINTS

N. Gadhi, A. Metrane

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Abstract. In this paper, we establish necessary optimality conditions and sufficient optimality conditions for D.C. vector optimization problems under D.C. constraints. Under additional conditions, some results of [9] and [15] are also recovered.

1. Introduction. Many authors studied optimality conditions for vector optimization problems where the objectives are defined by single-valued mappings and obtained optimality conditions in terms of Lagrange-Kuhn-Tucker multipliers. Lin [19] has given optimality conditions for differentiable vector optimization problems by using the Motzkin's theorem. Censor [2] gives optimality conditions for differentiable convex vector optimization by using the theorem of Dubovitskii-Milyutin. When the objective functions are locally Lipschitzian, Minami [21] obtained Kuhn-Tucker type or Fritz-John type optimality conditions for weakly efficient solutions in terms of the generalized gradient.

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In this paper, we are concerned with the multiobjective optimization problem

$$(P) \qquad \begin{cases} Y^{+} - \text{Minimize } f(x) - g(x) \\ \text{subject to : } \; h(x) - k(x) \notin -int\,(Z^{+}) \end{cases}$$

where $f, g : X \to Y \cup \{+\infty\}$ are Y^{+}-convex and lower semi-continuous mappings and $h, k : X \to Z \cup \{+\infty\}$ are Z^{+}-convex and continuous mappings.

Such a problem has been discussed by several authors at various levels of generality [7, 8, 9, 11, 12, 15, 20, 23, 28]. Our approach consists of using a special scalarization function introduced in optimization by Hiriart-Urruty [11] to detect necessary and sufficient optimality conditions for (P). Several intermediate optimization problems are introduced to help us in our investigation. One the other hand, considering the reverse convex case which is a particular problem of (P), one obtains Gadhi, Laghdir and Metrane's results [9] and extends another result of Laghdir [15] to the vector valued case.

The rest of the paper is written as follows : Section 2 contains basic definitions and preliminary material. Sections 3 and 4 are devoted to necessary and sufficient optimality conditions for the optimization problem (P).

2. Preliminaries. Throughout this paper, $X,\ Y,\ Z$ and W are Banach spaces whose topological dual spaces are $X^*,\ Y^*,\ Z^*$ and W^* respectively. Let $Y^{+} \subset Y$ (resp. $Z^{+} \subset Z$) be a pointed $(Y^{+} \cap (-Y^{+}) = \{0\})$, convex and closed cone with nonempty interior introducing a partial order in Y (resp. in Z) defined by

$$y_1 \leq_Y y_2 \Leftrightarrow y_2 \in -y_1 + Y^{+}$$

We adjoin to Y two artificial elements $+\infty$ and $-\infty$ such that

$$-\infty = -(+\infty),\ (+\infty) - (+\infty) = +\infty,\ 0\,(+\infty) = 0,$$

$$y_1 - \infty \leq_Y y_2 \text{ for all } y_1,\ y_2 \in Y,$$

and

$$y_2 \leq_Y y_1 + \infty = +\infty \text{ for all } y_1,\ y_2 \in Y \cup \{+\infty\}.$$

The negative polar cone $(Y^{+})^{\circ}$ of Y^{+} is defined as

$$\left(Y^{+}\right)^{\circ} = \left\{y^* \in Y^* : \langle y^*, y\rangle \leq 0 \text{ for all } y \in Y^{+}\right\},$$

where $\langle .,. \rangle$ is the dual pairing.

$\overline{x} \in C$ is an efficient (resp. weak efficient) solution of (P) if $(f-g)(\overline{x})$ is a Pareto (resp. weak Pareto) minimal vector of $(f-g)(C)$.
The point $\overline{x} \in C$ is a local efficient (resp. weak local efficient) solution of (P_1) if there exists a neighborhood V of $\overline{x}$ such that $(f-g)(\overline{x})$ is a Pareto (resp. weak Pareto) minimal vector of $(f-g)(C \cap V)$.
Since convexity plays an important role in the following investigations, recall the concept of cone-convex mappings.
The mapping $f : X \to Y \cup \{+\infty\}$ is said to be Y^+-convex if for every $\alpha \in [0,1]$ and $x_1, x_2 \in X$

$$\alpha f(x_1) + (1-\alpha) f(x_2) \in f(\alpha x_1 + (1-\alpha) x_2) + Y^+.$$

Definition 2.1. *A mapping $h : X \to Y \cup \{+\infty\}$ is said to be Y^+-D.C. if there exists two Y^+-convex mappings f and g such that:*

$$h(x) = f(x) - g(x) \quad \forall x \in X.$$

Let us recall the definition of the lower semi-continuity of a mapping. For more details on this concept, we refer the interested reader to [5, 22].

Definition 2.2 [22]**.** *A mapping $f : X \to Y \cup \{+\infty\}$ is said to be lower semicontinuous at $\overline{x} \in X$, if for any neighborhood V of zero and for any $b \in Y$ satisfying $b \leq_Y f(\overline{x})$, there exists a neighborhood U of $\overline{x}$ in X such that*

$$f(U) \subset b + V + (Y^+ \cup \{+\infty\}).$$

Definition 2.3 [24, 27]**.** *Let $f : X \to Y \cup \{+\infty\}$ be a Y^+-convex mapping. The vectorial subdifferential of f at $\overline{x} \in domf$ is given by*

$$\partial^v f(\overline{x}) = \{T \in L(X,Y) \; : \; T(h) \leq_Y f(\overline{x}+h) - f(\overline{x}) \quad \forall h \in X\}.$$

Here, $domf$ and $L(X,Y)$ denote respectively the domain of f and the set of all continuous linear mappings between X and Y.

Remark 2.1. When f is a convex function, $\partial^v f(\overline{x})$ reduces to the well known subdifferential (of convex analysis)

$$\partial f(\overline{x}) = \{x^* \in X^* : f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x} \rangle \quad \text{for all} \;\; x \in X\}.$$

Remark 2.2 [8]. Let $f : X \to Y \cup \{+\infty\}$ be a Y^+-convex mapping. If f is also continuous at $\overline{x}$, then

$$\partial^v f(\overline{x}) \neq \emptyset.$$

The next concept was introduced in [6] in finite dimension. We give it in the infinite dimensional case.

Definition 2.4. *Let U be a nonempty subset of Y. A functional $g : U \to \mathbb{R} \cup \{+\infty\}$ is called Y^+-increasing on U, if for each $y_0 \in U$*

$$y \in \left(y_0 + Y^+\right) \cap U \text{ implies } g(y) \geq g(y_0).$$

In [16], and using the separation Hahn-Banach geometric theorem, B. Lemaire set the following proposition which generalize both Gol'shtein's result [10] and Levin's result [18]. He used, for a simple function $h : Y \to \mathbb{R} \cup \{+\infty\}$, and another function which is Y^+-increasing $g : Y \to \mathbb{R} \cup \{+\infty\}$, the convention that

$$g \circ h(x) = g(h(x)) \text{ if } h(x) \in dom\ g \text{ and } g(+\infty) = +\infty.$$

Consequently, $g \circ h$ is a function from X into $\mathbb{R} \cup \{+\infty\}$ and its effective domain is given by

$$dom\,(g \circ h) = h^{-1}(dom\ g).$$

Proposition 2.1 [16]. *Let X and Y be two real Banach spaces. Consider a mapping h from X into $Y \cup \{+\infty\}$ and a function g from Y into $\mathbb{R} \cup \{+\infty\}$. If*
i) h is Y^+-convex,
ii) g is convex, Y^+-increasing and continuous in some point of $h(X)$.
Then, for all $x \in dom\,(g \circ h)$, on has

$$\partial(g \circ h)(x) = \bigcup_{y^* \in \partial g(h(x))} \partial(y^* \circ h)(x).$$

In the sequel, we shall need the following result of [5]. Under the nonemptiness of the set $\{x \in X : h(x) \in -\operatorname{int} Y^+\}$, one has

$$\partial(\delta_{-Y^+} \circ h)(\overline{x}) = \bigcup_{\substack{y^* \in \left(-Y^+\right)^\circ \\ \langle y^*, h(\overline{x})\rangle = 0}} \partial(y^* \circ h)(\overline{x}), \tag{2.1}$$

for all $x \in dom\,(g \circ h)$. Here, the symbol $\langle\ ,\ \rangle$ denotes the bilinear pairing between Y and Y^*, and δ_S is the indicator function of S.

Remark 2.3. Notice that the function $y \to \delta_{-Y^+}(y)$ is Y^+-increasing. Moreover for any Y_+-convex mapping $h : X \to Y \cup \{+\infty\}$, the composite function $\delta_{-Y^+} \circ h$ is also convex.

For a subset S of Y, we consider the function

$$\Delta_S(y) = \begin{cases} d(y,S) & \text{if } \ y \in Y \backslash S, \\ -d(y, Y\backslash S) & \text{if } \ y \in S, \end{cases}$$

where $d(y,S) = \inf\{\|u-y\| : u \in S\}$. This function was introduced by Hiriart-Urruty [11] (see also [13]), and used after by Ciligot-Travain [3], and Amahroq and Taa [1].
The next proposition has been established by Hiriart-Urruty [11]

Proposition 2.2 [11]. *Let $S \subset Y$ be a closed convex cone with non-empty interior and $S \neq Y$. The function Δ_S is convex, positively homogeneous, 1-Lipschitzian, decreasing on Y with respect to the order introduced by S. Moreover* $(Y\backslash S) = \{y \in Y : \Delta_S(y) > 0\}$, $\operatorname{int}(S) = \{y \in Y : \Delta_S(y) < 0\}$ *and the boundary of S* : $\operatorname{bd}(S)\{y \in Y : \Delta_S(y) = 0\}$.
It is easy to verify the following lemma.

Lemma 2.3. *The function $\Phi : Y \to \mathbb{R}$ defined by*

$$\Phi(y) = \Delta_{-\operatorname{int}(Y^+)}(y)$$

is $(Y^+)-$increasing on Y.
Let K be a closed convex subset of X. The normal cone $N_K(\overline{x})$ of K at $\overline{x}$ is denoted

$$N_K(\overline{x}) = \{x^* \in X^* : 0 \geq \langle x^*, x - \overline{x}\rangle \quad \text{for all} \ \ x \in K\}.$$

This cone can be also written as

$$N_K(\overline{x}) = \partial\delta_K(\overline{x}),$$

where δ_K is the indicator function of K. Properties of the subdifferential and the normal cone can be found in Rockafellar [25].
As a direct consequence of Proposition 2.2, one has the following result.

Proposition 2.4 [3]. *Let $S \subset Y$ be a closed convex cone with a nonempty interior. For all $y \in Y$, one has*

$$0 \notin \partial\Delta_S(y).$$

Lemma 2.5. *Let C be convex cone of Y, then*

$$(Y \setminus int\,(C)) - C \subset Y \setminus int\,(C)\,.$$

P r o o f. Suppose that there exists $y \in (Y \setminus int\,(C)) - C$ such that

$$y \notin Y \setminus int\,(C)\,.$$

It follows that there exist $a \in Y \setminus int\,(C)$ and $b \in C$ such that $y = a - b$. Consequently,

$$a = y + b \in int\,(C) + C \subset int\,(C)\,;$$

which contradicts the fact that $a \in Y \setminus int\,(C)$. $\square$

3. Necessary optimality conditions. In this section, we conserve the notations previously given. In order to give necessary optimality conditions for the optimization problem (P), we consider the following intermediate problem

$$(P_1): \quad \begin{cases} Y^+ - \text{Minimize } f(x) - g(x) \\ \text{subject to :} \quad \begin{array}{c} x \in X \setminus S \\ x \in C, \end{array} \end{cases}$$

where $C \subset X$ is a closed set and $S \subset X$ is an open convex set.
For all the sequel, we assume that $\overline{x} \in dom\,(f) \cap dom\,(g)$. The following lemma will play a crucial role in our investigation.

Lemma 3.1. *If $\overline{x} \in C$ is a local weak minimal solution of (P_1) with respect to Y^+, then for all $T \in \partial^v g\,(\overline{x})$, $\overline{x}$ solves the following scalar convex minimization problem*

$$\begin{cases} \textit{Minimize} & \Delta_{-\operatorname{int}(Y^+)}\,(f(x) - f(\overline{x}) - T(x - \overline{x})) \\ \textit{Subject to} & x \in C \cap X \setminus S. \end{cases}$$

P r o o f. Suppose the contrary. There exist $x_0 \in C \cap X \setminus S$ such that

$$\Delta_{-\operatorname{int}(Y^+)}\,(f(x_0) - f(\overline{x}) - T(x_0 - \overline{x})) < \Delta_{-\operatorname{int}(Y^+)}\,(0) = 0.$$

This implies with Proposition 2.4

$$f(x_0) - f(\overline{x}) - T(x_0 - \overline{x}) \in -\operatorname{int}\left(Y^+\right). \tag{3.1}$$

By assumption, we have $T \in \partial^v g(\overline{x})$, Then

$$\langle T, x_0 - \overline{x} \rangle \in (g(x_0) - g(\overline{x})) - Y^+ \tag{3.2}$$

From (3.1), (3.2) and the fact that $\operatorname{int}(Y^+) + Y^+ \subset \operatorname{int}(Y^+)$, one has

$$f(x_0) - g(x_0) - (f(\overline{x}) - g(\overline{x})) \in -\operatorname{int}\left(Y^+\right),$$

witch contradicts the fact that $\overline{x}$ is a local weak minimal solution of (P_1). □
We shall need to assume that for two subsets A and B of X and $\overline{x} \in A \cap B$, the condition

$$d(x, A \cap B) \leq k\left[d(x, A) + d(x, B)\right]$$

holds for some $k > 0$ and each x in some neighborhood of $\overline{x}$. Conditions ensuring this inequality are given in Jourani [14, Proposition 3.1]. See also [1] and the references therein.

Theorem 3.2. *Assume that f is finite and continuous at $\overline{x}$ and that the condition*

$$d(x, C \cap (X \setminus S)) \leq k\left[d(x, C) + d(x, (X \setminus S))\right] \tag{3.3}$$

holds for some $k > 0$ and all x in some neighborhood of $\overline{x}$. If $\overline{x}$ is a local weak minimal solution of (P_1) then for all $T \in \partial^v g(\overline{x})$ there exist $y^* \in (-Y^+)^{\circ} \setminus \{0\}$ such that

$$y^* \circ T \in \partial(y^* \circ f)(\overline{x}) + N_C^c(\overline{x}) - N_S(\overline{x}) + N_{dom(f)}^c(\overline{x}).$$

Proof. Set $H(.) = f(.) - f(\overline{x}) - T(. - \overline{x})$.

- On the one hand, as $\Delta_{-\operatorname{int}(Y^+)}$ is Y^+-increasing and H is Y^+-convex, the function $\Delta_{-\operatorname{int}(Y^+)} \circ H$ is convex.
- On the second hand, as $\Delta_{-\operatorname{int}(Y^+)}$ and H are continuous, the function $\Delta_{-\operatorname{int}(Y^+)} \circ H$ is continuous.

Combining the above facts, we deduce that $\Delta_{-\operatorname{int}(Y^+)} \circ H$ is locally Lipschitz at $\overline{x}$. Consequently, there exists $\alpha > 0$ such that $\Delta_{-\operatorname{int}(Y^+)} \circ H$ is α-Lipschitzian around $\overline{x}$.
By Lemma 3.1, $\overline{x}$ minimize the function $\Delta_{-\operatorname{int}(Y^+)} \circ H(.) + \alpha d(., C \cap (X \setminus S))$ over $dom(f)$. Using inequality (3.3), $\overline{x}$ minimize the function

$$\Delta_{-\operatorname{int}(Y^+)} \circ H(.) + \alpha k d(x, C) + \alpha k d(x, (X \setminus S))$$

over $dom\,(f)$. Then,

$$0 \in \partial^c \left(\Delta_{-\operatorname{int}(Y^+)}\left(H\left(.\right)\right) + \alpha k d\left(., C\right) + \alpha k d\left(., \left(X \setminus S\right)\right) + \delta_{dom(f)}\right)\left(\overline{x}\right).$$

Then, applying the sum rule [4], we obtain

$$0 \in \partial^c \left(\Delta_{-\operatorname{int}(Y^+)}\left(H\left(.\right)\right)\right)\left(\overline{x}\right) + \alpha k \partial^c d\left(., C\right)\left(\overline{x}\right) + \alpha k \partial^c d\left(., \left(X \setminus S\right)\right)\left(\overline{x}\right) + N^c_{dom(f)}\left(\overline{x}\right).$$

Since H is Y^+-convex and $\Delta_{-\operatorname{int}(Y^+ \times Z^+)}\,(.)$ is convex, continuous in 0 and Y^+-increasing, then from Proposition 2.1, there exist $y^* \in \partial\Delta_{-\operatorname{int}(Y^+)}\,(0)$ such that

$$0 \in \partial\left(y^* \circ H\right)\left(\overline{x}\right) + N^c\left(C, \overline{x}\right) + N^c_{X \setminus S}\left(\overline{x}\right) + N^c_{dom(f)}\left(\overline{x}\right).$$

Since $\Delta_{-\operatorname{int}(Y^+)}\,(.)$ is a convex function and $\Delta_{-\operatorname{int}(Y^+)}\,(0) = 0$ we have for all $y \in Y$

$$\Delta_{-\operatorname{int}(Y^+)}\left(y\right) \geq \langle y^*, y\rangle$$

and hence for all $y \in -Y^+$

$$\langle y^*, y\rangle \leq \Delta_{-\operatorname{int}(Y^+)}\left(y\right) = -d\left(y, Y \setminus -\operatorname{int}\left(Y^+\right)\right) \leq 0.$$

That is $y^* \in \left(-Y^+\right)^\circ$. From proposition 2.4, we have that $y^* \neq 0$.
Thus there exist $y^* \in \left(-Y^+\right)^\circ \setminus \{0\}$ such that

$$0 \in \partial\left(y^* \circ f + \langle -y^* \circ T, x - \overline{x}\rangle\right)\left(\overline{x}\right) + N^c_C\left(\overline{x}\right) + N^c_{X \setminus S}\left(\overline{x}\right) + N^c_{dom(f)}\left(\overline{x}\right).$$

Finally, for all $T \in \partial^v g\left(\overline{x}\right)$, there exist $y^* \in \left(-Y^+\right)^\circ \setminus \{0\}$ such that

$$y^* \circ T \in \partial\left(y^* \circ f\right)\left(\overline{x}\right) + N^c_C\left(\overline{x}\right) + N^c_{X \setminus S}\left(\overline{x}\right) + N^c_{dom(f)}\left(\overline{x}\right). \tag{3.4}$$

Since S is an open convex subset, it is also epi-Lipschitz at $\overline{x}$ [26]. By a result of Rockafellar [26], we conclude that

$$N^c_{X \setminus S}\left(\overline{x}\right) = -N_S\left(\overline{x}\right). \tag{3.5}$$

Combining (3.4) and (3.5), we get the result. □

Remark 3.1. Theorem 3.5 gives necessary optimality conditions for (P). It uses the result obtained in Theorem 3.2.
Set

$$S = \left\{x \in X : h\left(x\right) - k\left(\overline{x}\right) \in -\operatorname{int}\left(Z^+\right)\right\},$$

and

$$C = \left\{x \in X : k\left(x\right) - k\left(\overline{x}\right) \in -Z^+\right\}.$$

Lemma 3.3. *If $\overline{x}$ is a local weak minimal solution of (P), then $\overline{x}$ is a local weak minimal solution of the following problem*

$$\begin{cases} Y^{+} - \textit{Minimize } f(x) - g(x) \\ \textit{subject to :} \quad \begin{matrix} x \in X \setminus S \\ x \in C. \end{matrix} \end{cases}$$

Proof. Set $F := \{x \in X : h(x) - k(x) \notin int(Z^{+})\}$.
Since $\overline{x} \in ((X \backslash S) \cap C) \cap F$, it suffices to prove that

$$(X \backslash S) \cap C \subset F.$$

Taking $x \in (X \backslash S) \cap C$, one has

$$h(x) \in k(\overline{x}) + Z \setminus -int\left(Z^{+}\right)$$

and

$$k(\overline{x}) \in k(x) + Z^{+},$$

which means,

$$h(x) - k(x) \in Z \setminus -int\left(Z^{+}\right) + Z^{+}.$$

From Lemma 2.5, we obtain that $x \in F$. The proof is thus complete. □

We shall need the following lemma.

Lemma 3.4. *Denoting by $\overline{S}$ the norm topological closure in X of the subset S, we have*

$$\overline{S} := \left\{x \in X : h(x) \in -Z^{+}\right\}.$$

Proof. From the continuity assumption of h and the fact that the cone Y^{+} is closed,

$$\overline{S} \subset \left\{x \in X : h(x) \in -Z^{+}\right\}.$$

Conversely, let $x \in X$ such that $h(x) \in -Z^{+}$. From the nonemptiness of S, there exists $a \in X$ such that

$$h(a) \in -int\left(Z^{+}\right).$$

Setting $x_n := \dfrac{1}{n}a + \left(1 - \dfrac{1}{n}\right)x$ for any $n \geq 1$, the sequence $(x_n)_{n \geq 1}$ converges to x. Since h is convex, one has

$$h(x_n) \in \frac{1}{n}h(a) + \left(1 - \frac{1}{n}\right)h(x) - Z^{+} \in -int\left(Z^{+}\right) - Z^{+} \subset -int\left(Z^{+}\right);$$

which means that $x_n \in S$. Then, $\{x \in X : h(x) \in -Z^+\} \subset \overline{S}$. $\square$

Theorem 3.5. *Assume that f is finite and continuous at $\overline{x}$, that there exists $a \in X$ satisfying*

$$k(a) - k(\overline{x}) \in -\operatorname{int}\left(Z^+\right),$$

and that the condition

$$d(x, C \cap (X \setminus S)) \leq k\left[d(x, C) + d(x, (X \setminus S))\right] \tag{3.6}$$

holds for some $k > 0$ and all x in some neighborhood of $\overline{x}$. If $\overline{x}$ is a local weak minimal solution of (P) then for all $T \in \partial^v g(\overline{x})$ there exist $y^* \in (-Y^+)^\circ \setminus \{0\}$, $z_1^* \in (-Z^+)^\circ$ and $z_2^* \in (-Z^+)^\circ$ such that $\langle z_2^*, h(\overline{x}) - k(\overline{x})\rangle = 0$ and

$$y^* \circ T \in \partial(y^* \circ f)(\overline{x}) + \partial(z_1^* \circ k)(\overline{x}) - \partial(z_2^* \circ h)(\overline{x}) + N^c_{dom(f)}(\overline{x}).$$

Proof. Let $T \in \partial^v g(\overline{x})$. Applying Lemma 3.3 and Theorem 3.2, there exist $y^* \in (-Y^+)^\circ \setminus \{0\}$ such that

$$y^* \circ T \in \partial(y^* \circ f)(\overline{x}) + N^c_C(\overline{x}) - N_S(\overline{x}) + N^c_{dom(f)}(\overline{x}).$$

On the one hand, using (2.1),

$$N_C(\overline{x}) = \partial(\delta_{-Z^+} \circ k)(\overline{x}) = \bigcup_{z^* \in (-Z^+)^\circ} \partial(z^* \circ k)(\overline{x}) \tag{3.7}$$

On the other hand, from Lemma 3.4,

$$\delta_{\overline{S}} = \delta_{-Z^+} \circ h.$$

Since $N(S, \overline{x}) = N\left(\overline{S}, \overline{x}\right)$, one obtains (due to (2.1))

$$N(S, \overline{x}) = \partial\delta_{\overline{S}}(\overline{x}) = \partial(\delta_{-Z^+} \circ h)(\overline{x}) = \bigcup_{\substack{z^* \in \left(-Z^+\right)^\circ \\ \langle z^*, h(\overline{x}) - k(\overline{x})\rangle = 0}} \partial(z^* \circ h)(\overline{x}). \tag{3.8}$$

Combining (3.7) and (3.8), we get the result. $\square$

Remark 3.2. Obviously, Condition (3.6) is fulfilled for $C = X$.

Consider the reverse-convex optimization problem

$$(P') : \begin{cases} Y^+ - \text{Minimize } f(x) - g(x) \\ \text{subject to : } h(x) \notin -int\,(Z^+), \end{cases}$$

where $f, g : X \to Y \cup \{+\infty\}$ are Y^+-convex and lower semi-continuous mappings and $h : X \to Z \cup \{+\infty\}$ is a Z^+-convex and continuous mapping.
As a consequence of Theorem 3.5, one obtains a result of [9].

Corollary 3.6 [9]**.** *Assume that f is finite and continuous at $\overline{x}$. If $\overline{x}$ is a local weak minimal solution of (P') then for all $T \in \partial^v g(\overline{x})$, there exist $y^* \in (-Y^+)^\circ \setminus \{0\}$ and $z^* \in (-Z^+)^\circ$ such that*

$$\langle z^*, h(\overline{x})\rangle = 0$$

and

$$y^* \circ T \in \partial (y^* \circ f)(\overline{x}) - \partial (z^* \circ h)(\overline{x}) + N^c_{dom(f)}(\overline{x}).$$

Remark 3.3. When $Y = \mathbb{R}$ and g is strictly Hadamard differentiable at $\overline{x}$, the above corollary extends a result of Laghdir [15].

4. Sufficient optimality conditions. In order to give sufficient optimality conditions for the optimization problem (P), we shall prove the following preliminary results.
Consider the intermediate problem (P_2) defined as follows

$$(P_2): \quad \begin{cases} Y^+ - \text{Minimize } f(x) - g(x) \\ \quad \text{subject to : } x \in X\backslash\Omega, \end{cases}$$

where Ω is an open convex subset of X.

Proposition 4.1. *Suppose that there exists $y^* \in (-Y^+)^\circ \setminus \{0\}$ such that*

$$\partial_\varepsilon (y^* \circ g)(\overline{x}) + N_\Omega(\overline{x}) \subset \partial_\varepsilon (y^* \circ f)(\overline{x}) \quad \text{for all } \varepsilon > 0. \tag{4.1}$$

Then $\overline{x}$ is a local weak minimal solution of (P_2).

P r o o f. Since Ω is an open convex subset, it is also epi-Lipschitz at $\overline{x}$ [26]. By a result of Rockafellar [26], we conclude that

$$N^c_{X\backslash\Omega}(\overline{x}) = -N_\Omega(\overline{x}).$$

Since $\partial^c d(., X \setminus \Omega)(\overline{x}) \subset N^c_{X\backslash\Omega}(\overline{x})$, inclusion (4.1) becomes

$$\partial_\varepsilon (y^* \circ g)(\overline{x}) - \partial^c d(., X \setminus \Omega)(\overline{x}) \subset \partial_\varepsilon (y^* \circ f)(\overline{x}), \quad \text{for all } \varepsilon > 0.$$

Consequently,

$$\partial_{\varepsilon}(y^{*}\circ g)(\overline{x})+\partial d(.,\Omega)(\overline{x})-\partial^{c}d(.,X\setminus\Omega)(\overline{x})\subset\partial_{\varepsilon}(y^{*}\circ f)(\overline{x})+\partial d(.,\Omega)(\overline{x}),$$

for all $\varepsilon>0$. As $\partial\Delta_{\Omega}(\overline{x})\subset\partial d(.,\Omega)(\overline{x})-\partial^{c}d(.,X\setminus\Omega)(\overline{x})$, we get

$$\partial_{\varepsilon}(y^{*}\circ g)(\overline{x})+\partial\Delta_{\Omega}(\overline{x})\subset\partial_{\varepsilon}(y^{*}\circ f)(\overline{x})+\partial d(.,\Omega)(\overline{x})\quad\text{for all }\varepsilon>0,$$

which yields that

$$\partial_{\varepsilon}(y^{*}\circ g)(\overline{x})+\partial\Delta_{\Omega}(\overline{x})\subset\partial_{\varepsilon}(y^{*}\circ f+d(.,\Omega))(\overline{x})\quad\text{for all }\varepsilon>0. \tag{4.2}$$

Since Δ_{Ω} is convex continuous then

$$\partial_{\varepsilon}(y^{*}\circ g+\Delta_{\Omega})(\overline{x})=\partial_{\varepsilon}(y^{*}\circ g)(\overline{x})+\partial\Delta_{\Omega}(\overline{x})\quad\text{for all }\varepsilon>0. \tag{4.3}$$

From (4.2) and (4.3), we obtain

$$\partial_{\varepsilon}(y^{*}\circ g+\Delta_{\Omega})(\overline{x})\subset\partial_{\varepsilon}(y^{*}\circ f+d(.,\Omega))(\overline{x})\quad\text{for all }\varepsilon>0.$$

By the classical Hiriart-Urruty [12] sufficient conditions, $\overline{x}$ minimize the function

$$y^{*}\circ f(x)-y^{*}\circ g(x)+d(x,X\setminus\Omega).$$

We conclude that $\overline{x}$ is a minimum of the problem

$$\begin{cases}\text{Minimize } y^{*}\circ(f(x)-g(x))\\ \quad\text{subject to : } x\in X\setminus\Omega.\end{cases}$$

Finally, due to $y^{*}\in(-Y^{+})^{\circ}\setminus\{0\}$, $\overline{x}$ is a local weak minimal solution of (P_2). □

Lemma 4.2. *Let $T\in\partial^{v}k(\overline{x})$. If $\overline{x}$ is a local weak minimal solution of*

$$\begin{gathered}Y^{+}-\textit{Minimize } f(x)-g(x)\\ \textit{subject to : } h(x)-k(\overline{x})-T(x-\overline{x})\notin-\operatorname{int}(Z^{+})\end{gathered}$$

then $\overline{x}$ is a local weak minimal solution of (P).

Proof. Let $T\in\partial^{v}k(\overline{x})$.
Setting $\Omega:=\{x\in X: h(x)-k(\overline{x})-T(x-\overline{x})\in-\operatorname{int}(Z^{+})\}$,

- Let us prove that $F\subset(X\setminus\Omega)$. Indeed, let $x\in F$. By definition,

$$h(x)\in k(x)+Z\setminus-\operatorname{int}\left(Z^{+}\right). \tag{4.4}$$

As $T \in \partial^v k(\overline{x})$, then

$$k(x) \in k(\overline{x}) + T(x - \overline{x}) + Z^+, \tag{4.5}$$

Combining (4.4) and (4.5), one has

$$h(x) - k(\overline{x}) - T(x - \overline{x}) \in \left(Z \setminus -\operatorname{int}\left(Z^+\right)\right) + Z^+.$$

From Lemma 2.5, we obtain that $x \in (X \backslash \Omega)$.

- Since $\overline{x} \in X \backslash \Omega$, one concludes that the proof is thus complete. □

Theorem 4.3. *Suppose that $f, g : X \to Y \cup \{+\infty\}$ are convex, proper and lower semicontinuous, and that $h : X \to Z \cup \{+\infty\}$ is continuous. If there exist $T \in \partial^v k(\overline{x})$, $a \in X$ and $y^* \in (-Y^+)^\circ \setminus \{0\}$ such that*

$$h(a) - k(\overline{x}) - T(a - \overline{x}) \in -\operatorname{int}\left(Z^+\right), \tag{4.6}$$

and

$$\partial_\varepsilon(y^* \circ g)(\overline{x}) + \partial(z^* \circ h)(\overline{x}) - z^* \circ T \subset \partial_\varepsilon(y^* \circ f)(\overline{x}), \tag{4.7}$$

for all $\varepsilon > 0$ and $z^* \in \left\{z^* \in (-Z^+)^\circ : \langle z^*, h(\overline{x}) - k(\overline{x})\rangle = 0\right\}$ then, $\overline{x}$ is a local weak minimal solution of (P).

Proof. As previously, relation (2.1) yields

$$N_\Omega(\overline{x}) = \partial\delta_{\overline{\Omega}}(\overline{x}) = \partial(\delta_{-Z^+} \circ H)(\overline{x}) = \bigcup_{\substack{z^* \in \left(-Z^+\right)^\circ \\ \langle z^*, H(\overline{x})\rangle = 0}} \partial(z^* \circ H)(\overline{x}),$$

where $H(x) = h(x) - k(\overline{x}) - T(x - \overline{x})$. Consequently, from inclusion (4.7), one has

$$\partial_\varepsilon(y^* \circ g)(\overline{x}) + N_\Omega(\overline{x}) \subset \partial_\varepsilon(y^* \circ f)(\overline{x}) \quad \text{for all } \varepsilon > 0.$$

Applying Proposition 4.1, we obtain the result. □

The following result gives sufficient optimality conditions for the reverse-convex optimization problem (P').

Corollary 4.4 [9]**.** *Suppose that $f, g : X \to Y \cup \{+\infty\}$ are convex, proper and lower semicontinuous, and that $h : X \to Z \cup \{+\infty\}$ is continuous. If there exist $a \in X$ and $y^* \in (-Y^+)^\circ \setminus \{0\}$ such that $h(a) \in -\operatorname{int}(Z^+)$ and*

$$\partial_\varepsilon(y^* \circ g)(\overline{x}) + \partial(z^* \circ h)(\overline{x}) \subset \partial_\varepsilon(y^* \circ f)(\overline{x})$$

for all $\varepsilon > 0$ *and* $z^* \in \left\{z^* \in (-Z^+)^\circ : \langle z^*, h(\overline{x})\rangle = 0\right\}$
then $\overline{x}$ *is a local weak minimal solution of* (P').

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N. Gadhi
Cadi Ayyad University
B.P. 3536 Amerchich
Marrakech, Morocco
e-mail: `n.gadhi@ucam.ac.ma`

A. Metrane
Université Cadi Ayyad
Faculté des Sciences Samlalia
Département de Mathématiques
Marrakech, Morocco
e-mail: `metrane@ucam.ac.ma`

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