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DIRECT AND CONVERSE THEOREMS FOR GENERALIZED BERNSTEIN-TYPE OPERATORS

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Abstract. We establish direct and converse theorems for generalized parameter dependent Bernstein-type operators. The direct estimate is given using a K-functional and the inverse result is a strong converse inequality of type A, in the terminology of [2].

1. Introduction. The following operator was introduced by T. N. T. Goodman and A. Sharma in [4]:

$$(U_n f)(x) \equiv U_n(f,x) =$$

$$= f(0)\, p_{n,0}(x) + f(1)\, p_{n,n}(x) + \sum_{k=1}^{n-1} p_{n,k}(x) \cdot \int_0^1 (n-1)\, p_{n-2,k-1}(t)\, f(t)\, dt,$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $k = \overline{0,n}$.

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It was studied for this operator by P. E. Parvanov and B. D. Popov [5] the relation between the rate of approximation of $U_n f$ and the K-functional

$$K\left(f,\ \frac{1}{n}\right) \equiv K\left(f, \frac{1}{n};\ C[0,1],\ W^2_\infty(\varphi)\right) = \inf_{g\in W^2_\infty(\varphi)} \left\{\|f-g\| + \frac{1}{n}\ \|\varphi^2 g''\|\right\},$$

where $W^2_\infty(\varphi)$ consists of all functions $g : [0,1] \to \mathbf{R}$ such that g' is absolutely continuous on $[0,1]$ and $\|\varphi^2 g''\|$ is finite. Here $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ and $\|\cdot\|$ is the uniform norm on $C[0,1]$. Originally, in the expression of $K(f,1/n)$ was considered $L_\infty[0,1]$ instead of $C[0,1]$. They have proved a direct inequality and a strong converse inequality of type A, in the terminology of [2], that is for every $f \in C[0,1]$ were established

$$\frac{1}{2}\cdot\|U_n f - f\| \ \le\ K\left(f, \frac{1}{n}\right)\ \le\ (6+\sqrt{8})\cdot\|U_n f - f\| \tag{1}$$

The Bernstein-type operator discussed in this paper will be given by

$$U_n^\alpha : C[0,1] \to C[0,1], \quad (U_n^\alpha f)(x) \ \equiv\ U_n^\alpha(f,x)\ =$$

$$=\ f(0)w_{n,0}(x,\alpha) + f(1)w_{n,n}(x,\alpha) + \sum_{k=1}^{n-1} w_{n,k}(x,\alpha)\cdot\int_0^1 (n-1)p_{n-2,k-1}(t)f(t)\,dt$$

(see also [3]), where

$$w_{n,k}(x,\alpha) = \binom{n}{k}\cdot\frac{\Pi_{i=0}^{k-1}\ (x+i\alpha)\ \Pi_{j=0}^{n-k-1}\ (1-x+j\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+(n-1)\alpha)},$$

$k=\overline{0,n}$ and $\alpha \ge 0$ is a parameter which may depend only on the natural number n. In the case $\alpha = 0$, U_n^0 is the Goodman-Sharma operator defined above. The purpose of this paper is to establish for U_n^α direct inequality and strong converse inequality of type A.

2. Direct theorem. The theorem in question can be stated as follows:

Theorem 1. *Let $\alpha \ge 0$ and*

$$K\left(f, \frac{1}{1+\alpha}\cdot\left(\frac{2}{n+1}+\alpha\right)\right) = \inf_{g\in W^2_\infty(\varphi)}\left\{\|f-g\| + \frac{1}{1+\alpha}\cdot\left(\frac{2}{n+1}+\alpha\right)\cdot\|\varphi^2 g''\|\right\}$$

Then for every $f \in C[0,1]$ we have

$$\|U_n^\alpha f - f\|\ \le\ 2\ K\left(f, \frac{1}{1+\alpha}\cdot\left(\frac{2}{n+1}+\alpha\right)\right) \tag{2}$$

Proof. By [6, p. 1180, Lemma 3.1] we have for $\alpha > 0$ and $x \in (0,1)$ the following identity

$$w_{n,k}(x,\alpha) = \binom{n}{k} \cdot \frac{B\left(x\alpha^{-1}+k, (1-x)\alpha^{-1}+n-k\right)}{B\left(x\alpha^{-1}, (1-x)\alpha^{-1}\right)},$$

where $B(\,\cdot\,,\,\cdot\,)$ denotes the Beta function. Consequently, $U_n^\alpha f$ can be represented by means of the Goodman-Sharma operator, as follows

$$(U_n^\alpha f)(x) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}\,(1-t)^{\frac{1-x}{\alpha}-1} \cdot (U_n f)(t)\,dt.$$

Hence, by simple computations and [5, p. 166, (2.1)–(2.4)] we obtain that U_n^α is linear and positive,

$$U_n^\alpha(u-x,x) = 0, \tag{3}$$

$$U_n^\alpha\left((u-x)^2,x\right) = \left(\frac{2}{n+1}+\alpha\right) \cdot \frac{x(1-x)}{1+\alpha} \tag{4}$$

and

$$\|U_n^\alpha f\| \;\le\; \|f\| \tag{5}$$

for every $f \in C[0,1]$. Now, the proof is standard (cf. [1, Chapter 9]): using Taylor's formula

$$g(u) \;=\; g(x) + (u-x)\;g'(x) + \int_x^u (u-v)\;g''(v)\,dv,$$

for $g \in W_\infty^2(\varphi)$, and (3), [1, p. 141, (9.6.1)] and (4), we obtain

$$\begin{aligned} |(U_n^\alpha g)(x) - g(x)| \;&\le\; U_n^\alpha\left(\left|\int_x^u \frac{|u-v|}{\varphi(v)^2}\cdot\varphi(v)^2\;|g''(v)|\;dv\right|,\;x\right) \\ &\le\; \left(\frac{2}{n+1}+\alpha\right)\cdot\frac{1}{1+\alpha}\cdot\|\varphi^2 g''\| \end{aligned}$$

Hence, by (5), we have

$$\begin{aligned} |(U_n^\alpha f)(x) - f(x)| &\le |(U_n^\alpha (f-g))(x) - (f-g)(x)| + |(U_n^\alpha g)(x) - g(x)| \\ &\le 2\,\|f-g\| + \left(\frac{2}{n+1} + \alpha\right)\cdot \frac{1}{1+\alpha}\cdot \|\varphi^2 g''\| \end{aligned}$$

Taking infimum over all $g \in W_\infty^2(\varphi)$, we obtain (2). $\square$

3. Direct and converse theorems. In order to prove our results, we need two lemmas:

Lemma 1. *We have*

$$\sup_{\substack{f\in C[0,1]\\ f\neq linear}} \frac{\|\varphi^2 (U_n f)''\|}{\|U_n f - f\|} = \begin{cases} 0, & if \quad n=1 \\ c_0 n, & if \quad n\ge 2 \end{cases} \tag{6}$$

where $1/2 \le c_0 \le 4 + 3\sqrt{2}$.

Proof. Using [5, p. 175, Lemma 5.2] and the estimate given in the proof of [5, p. 177, Theorem 5] we obtain

$$\begin{aligned} \|\varphi^2 (U_n f)''\| &\le \|\varphi^2 (U_n(f - U_n f))''\| + \|\varphi^2 (U_n^2 f)''\| \\ &\le \sqrt{2}\; n\cdot \|f - U_n f\| + (4+2\sqrt{2})\; n\cdot \|U_n f - f\| \\ &= (4+3\sqrt{2})\; n\; \|U_n f - f\| \end{aligned}$$

So

$$\sup_{\substack{f\in C[0,1]\\ f\neq \text{linear}}} \frac{\|\varphi^2 (U_n f)''\|}{\|U_n f - f\|} \le (4+3\sqrt{2})n.$$

On the other hand, for

$$f_1(x) = \frac{c_1}{2}\cdot x^2 + x + 1, \qquad x\in[0,1],$$

where $c_1 > 0$ is a given constant, we have $f_1 \in W_\infty^2(\varphi)\{0,1\}$ (see [5, p. 171]) and, in view of [5, p. 171, Lemma 4.2], we get

$$\begin{aligned} \varphi(x)^2 (U_n f_1)''(x) &= (U_n(\varphi^2 f_1''))(x) = c_1\, U_n(u(1-u)\;,\; x) \\ &= c_1 \left(1 - \frac{2}{n+1}\right)\, \varphi(x)^2 \end{aligned}$$

Thus

$$\|\varphi^2 (U_n f_1)''\| = \frac{c_1}{4} \cdot \left(1 - \frac{2}{n+1}\right)$$

Moreover, by [5, p. 173, (4.6)] and [5, p. 171, Lemma 4.2] we obtain

$$\begin{aligned} (U_n f_1)(x) - f_1(x) &= \sum_{k=n+1}^{\infty} \frac{1}{k(k-1)} \cdot \varphi(x)^2 \ (U_k f_1)''(x) \\ &= \sum_{k=n+1}^{\infty} \frac{1}{k(k-1)} \cdot U_k(\varphi^2 f_1'', x) \\ &= \sum_{k=n+1}^{\infty} \frac{1}{k(k-1)} \cdot c_1 \ \left(1 - \frac{2}{k+1}\right) \cdot x(1-x) \end{aligned}$$

Therefore

$$\|U_n f_1 - f_1\| = \frac{c_1}{4} \cdot \sum_{k=n+1}^{\infty} \frac{1}{k(k-1)} \cdot \left(1 - \frac{2}{k+1}\right) = \frac{c_1}{4} \cdot \frac{1}{n} \left(1 - \frac{1}{n+1}\right)$$

Hence

$$\frac{\|\varphi^2 (U_n f_1)''\|}{\|U_n f_1 - f_1\|} = \frac{1 - \frac{2}{n+1}}{\frac{1}{n}\left(1 - \frac{1}{n+1}\right)} = n - 1 \ \geq \ \frac{n}{2}$$

for $n \geq 2$. Thus

$$\sup_{\substack{f \in C[0,1] \\ f \neq \text{linear}}} \frac{\|\varphi^2 (U_n f)''\|}{\|U_n f - f\|} \ \geq \ \frac{n}{2}, \qquad n \geq 2$$

In conclusion, for $n \geq 2$ we get (6) with $c_0 \in \left[1/2, \ 4 + 3\sqrt{2}\right]$.

For $n = 1$ we have $(U_n f)(x) = f(0)(1-x) + f(1)x$. Therefore $\|\varphi^2 (U_n f)''\| = 0$, which implies the conclusion. $\square$

Lemma 2. *We have*

(7) $$\sup_{\substack{f \in C[0,1] \\ f \neq linear}} \frac{\|U_n^{\alpha} f - U_n f\|}{\|\varphi^2 (U_n f)''\|} = \alpha_0 \cdot \frac{\alpha}{1+\alpha},$$

where $1/2 \le \alpha_0 \le 1$ *and* $n \ge 2$.

P r o o f. By Taylor's formula

$$(U_n f)(t) = (U_n f)(x) + (t-x)(U_n f)'(x) + \int\limits_x^t (t-u)\cdot (U_n f)''(u)\, du \tag{8}$$

and

$$\int\limits_0^1 t^{\frac{x}{\alpha}-1}\,(1-t)^{\frac{1-x}{\alpha}-1}\,(t-x)\,dt = 0, \tag{9}$$

we obtain

$$\begin{aligned}
|(U_n^\alpha f)(x) - (U_n f)(x)| \;&= \\
&= \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)}\cdot \left| \int\limits_0^1 t^{\frac{x}{\alpha}-1}\,(1-t)^{\frac{1-x}{\alpha}-1}\, [(U_n f)(t) - (U_n f)(x)]\, dt \right| \\
&= \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)}\cdot \left| \int\limits_0^1 t^{\frac{x}{\alpha}-1}\,(1-t)^{\frac{1-x}{\alpha}-1} \cdot \right. \\
&\quad \left. \cdot \left[(t-x)\cdot (U_n f)'(x) + \int\limits_x^t (t-u)(U_n f)''(u)\, du\right] dt \right| \\
&\le \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)}\cdot \int\limits_0^1 t^{\frac{x}{\alpha}-1}\,(1-t)^{\frac{1-x}{\alpha}-1}\cdot \left| \int\limits_x^t (t-u)\cdot (U_n f)''(u)\, du \right| dt \\
&\le \frac{\|\varphi^2 (U_n f)''\|}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)}\cdot \int\limits_0^1 t^{\frac{x}{\alpha}-1}\,(1-t)^{\frac{1-x}{\alpha}-1}\cdot \left| \int\limits_x^t \frac{|t-u|}{u(1-u)}\, du \right| dt
\end{aligned}$$

Using again [1, p. 141, (9.6.1)] and

$$\int\limits_0^1 t^{\frac{x}{\alpha}-1}\,(1-t)^{\frac{1-x}{\alpha}-1}\,(t-x)^2\,dt \;=\; \frac{\alpha x(1-x)}{1+\alpha}\cdot B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right), \tag{10}$$

we have

$$|(U_n^\alpha f)(x) - (U_n f)(x)| \le$$

$$\le \frac{\|\varphi^2 (U_n f)''\|}{\varphi(x)^2 \cdot B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1} (1-t)^{\frac{1-x}{\alpha}-1} \cdot (t-x)^2 \, dt$$

$$= \frac{\alpha}{1+\alpha} \cdot \|\varphi^2 (U_n f)''\|$$

Hence

$$\|U_n^\alpha f - U_n f\| \le \frac{\alpha}{1+\alpha} \cdot \|\varphi^2 (U_n f)''\| \tag{11}$$

and therefore

$$\sup_{\substack{f \in C[0,1] \\ f \ne \text{linear}}} \frac{\|U_n^\alpha f - U_n f\|}{\|\varphi^2 (U_n f)''\|} \le \frac{\alpha}{1+\alpha}$$

On the other hand, for $f_1(x) = (c_1/2) \cdot x^2 + x + 1$ (see Lemma 1) we have, as above

$$\|\varphi^2 (U_n f_1)''\| = \frac{c_1}{4} \cdot \left(1 - \frac{2}{n+1}\right)$$

and

$$(U_n f_1)''(x) = c_1 \left(1 - \frac{2}{n+1}\right)$$

Hence, by (8), (9) and (10), replacing f with f_1, we obtain

$$(U_n^\alpha f_1)(x) - (U_n f_1)(x) =$$

$$= \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1} (1-t)^{\frac{1-x}{\alpha}-1} \cdot \left\{ \int_x^t (t-u) \, (U_n f_1)''(u) \, du \right\} dt$$

$$= \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1} (1-t)^{\frac{1-x}{\alpha}-1} \cdot \left\{ \int_x^t (t-u) \cdot c_1 \left(1 - \frac{2}{n+1}\right) du \right\} dt$$

$$= \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1} (1-t)^{\frac{1-x}{\alpha}-1} \cdot c_1 \left(1 - \frac{2}{n+1}\right) \cdot \frac{1}{2} \, (t-x)^2 \, dt$$

$$= \frac{\alpha}{1+\alpha} \cdot \frac{c_1}{2} \cdot \left(1 - \frac{2}{n+1}\right) \cdot x(1-x)$$

So

$$\|U_n^\alpha f_1 - U_n f_1\| = \frac{\alpha}{1+\alpha} \cdot \frac{c_1}{8} \cdot \left(1 - \frac{2}{n+1}\right)$$

and

$$\frac{\|U_n^\alpha f_1 - U_n f_1\|}{\|\varphi^2 (U_n f_1)''\|} = \frac{1}{2} \cdot \frac{\alpha}{1+\alpha}$$

Hence

$$\sup_{\substack{f \in C[0,1] \\ f \neq \text{linear}}} \frac{\|U_n^\alpha f - U_n f\|}{\|\varphi^2 (U_n f)''\|} \geq \frac{1}{2} \cdot \frac{\alpha}{1+\alpha}$$

In conclusion we get (7) with $\alpha_0 \in [1/2,\ 1]$. □

Now we can prove the following result:

Theorem 2. *If* $\alpha = \alpha(n)$ *and* $c_0\ \alpha_0 \cdot (n\ \alpha)/(1+\alpha) \leq \alpha_1 < 1$ *for* $n = 1, 2, \ldots$ *then for every* $f \in C[0,1]$ *we have*

$$(12) \qquad (1-\alpha_1)\ \|U_n f - f\| \leq \|U_n^\alpha f - f\| \leq (1+\alpha_1)\ \|U_n f - f\|$$

and

$$(13) \qquad \frac{1-\alpha_1}{6+\sqrt{8}} \cdot K\left(f, \frac{1}{n}\right) \leq \|U_n^\alpha f - f\| \leq 2\ (1+\alpha_1) \cdot K\left(f, \frac{1}{n}\right)$$

P r o o f. By (7) and (6) we have

$$\|U_n^\alpha f - U_n f\| \leq \alpha_0 \cdot \frac{\alpha}{1+\alpha} \cdot \|\varphi^2 (U_n f)''\| \leq c_0\ \alpha_0 \cdot \frac{n\alpha}{1+\alpha} \cdot \|U_n f - f\|$$

$$\leq \alpha_1 \cdot \|U_n f - f\|$$

Hence

$$\|U_n f - f\| \leq \|U_n^\alpha f - U_n f\| + \|U_n^\alpha f - f\| \leq \alpha_1\ \|U_n f - f\| + \|U_n^\alpha f - f\|$$

or

$$(1-\alpha_1)\ \|U_nf-f\| \ \le\ \|U_n^\alpha f-f\|$$

and

$$\|U_n^\alpha f-f\| \ \le\ \|U_n^\alpha f-U_nf\|+\|U_nf-f\| \ \le\ (1+\alpha_1)\ \|U_nf-f\|,$$

respectively. So we obtain (12). For the second statement we use (1) and (12) obtaining (13), which completes the proof. □

Furthermore, we have the following property:

Theorem 3. *If* $\alpha=\alpha(n)$ *and* $(n^2\alpha)/(1+\alpha)\le\alpha_2<1$ *for* $n=1,2,\dots$ *then for every* $f\in C[0,1]$ *we have*

$$(14)\quad (1-\alpha_2)\ \|U_nf-U_{n-1}f\|\le\|U_n^\alpha f-U_{n-1}f\|\le(1+\alpha_2)\ \|U_nf-U_{n-1}f\|$$

P r o o f. By (11) and [5, p. 169, Lemma 4.1] we obtain

$$\|U_n^\alpha f-U_nf\| \ \le\ \frac{\alpha}{1+\alpha}\cdot n(n-1)\cdot\|U_nf-U_{n-1}f\| \ \le\ \alpha_2\cdot\|U_nf-U_{n-1}f\|$$

Then we have

$$\begin{aligned}\|U_nf-U_{n-1}f\| &\le \|U_n^\alpha f-U_{n-1}f\|+\|U_n^\alpha f-U_nf\|\\ &\le \|U_n^\alpha f-U_{n-1}f\|+\alpha_2\cdot\|U_nf-U_{n-1}f\|\end{aligned}$$

or

$$(1-\alpha_2)\ \|U_nf-U_{n-1}f\| \ \le\ \|U_n^\alpha f-U_{n-1}f\|$$

and

$$\begin{aligned}\|U_n^\alpha f-U_{n-1}f\| &\le \|U_n^\alpha f-U_nf\|+\|U_nf-U_{n-1}f\|\\ &\le (1+\alpha_2)\cdot\|U_nf-U_{n-1}f\|,\end{aligned}$$

respectively. Thus we have proved (14). □

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