Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **30** (2004), 95–102

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

SEQUENCES OF MAXIMAL DEGREE VERTICES IN GRAPHS

Nickolay Khadzhiivanov, Nedyalko Nenov

Communicated by V. Drensky

Abstract. Let $\Gamma(M)$ where $M \subset V(G)$ be the set of all vertices of the graph G adjacent to any vertex of M. If $v_1, \dots, v_r$ is a vertex sequence in G such that $\Gamma(v_1, \dots, v_r) = \emptyset$ and v_i is a maximal degree vertex in $\Gamma(v_1, \dots, v_{i-1})$, we prove that $e(G) \leq e(K(p_1, \dots, p_r))$ where $K(p_1, \dots, p_r)$ is the complete r-partite graph with $p_i = |\Gamma(v_1, \dots, v_{i-1}) \setminus \Gamma(v_i)|$.

We consider only finite non-oriented graphs without loops and multiple edges. The vertex set and the edge set of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. We call *p-clique* of a graph G a set of p pairwise adjacent vertices. A set of vertices of a graph G is said to be *independent*, if every two of them are not adjacent. We shall use also the following notations:

$e(G) = |E(G)|$ – the number of the edges of G;

$G[M]$ – the subgraph of G induced by M, where $M \subset V(G)$;

$\Gamma_G(M)$ – the set of all vertices of G adjacent to any vertex of M;

$d_G(v) = |\Gamma_G(v)|$ – the degree of a vertex v in G;

2000 *Mathematics Subject Classification:* 05C35.
Key words: Maximal degree vertex, complete s-partite graph, Turan's graph.

K_n and $\overline{K}_n$ – the complete and discrete n-vertex graphs, respectively.

Let $G_i = (V_i, E_i)$ be graphs such that $V_i \cap V_j = \emptyset$ for $i \neq j$. We denote by $G_1 + \cdots + G_s$ the graph $G = (V, E)$ with $V = V_1 \cup \cdots \cup V_s$ and $E = E_1 \cup \cdots \cup E_s \cup E'$, where E' consists of all couples $\{u, v\}$, $u \in V_i$, $v \in V_j$; $1 \leq i < j \leq s$. The graph $\overline{K}_{p_1} + \cdots + \overline{K}_{p_s}$ will be denoted by $K(p_1, \ldots, p_s)$ and will be called a complete s-partite graph with partition classes $V(\overline{K}_{p_1})$, …,$V(\overline{K}_{p_s})$. If $p_1 + \cdots + p_s = n$ and $p_1, \ldots, p_s$ are as equal as possible (in the sense that $|p_i - p_j| \leq 1$ for all pairs $\{i, j\}$, then $K(p_1, \ldots, p_s)$ is denoted by $T_s(n)$ and is called the s-partite n-vertex Turan's graph.

Clearly, $e(K(p_1, \ldots, p_r)) = \sum\{p_i p_j |\, 1 \leq i < j \leq r\}$. If $p_1 - p_2 \geq 2$, then $K(p_1 - 1, p_2 + 1, p_3, \ldots, p_r) - K(p_1, p_2, \ldots, p_r) = p_1 - p_2 - 1 > 0$. This observation implies the following elementary proposition, we shall make use of later:

Lemma. *Let n and r be positive integers and $r \leq n$. Then the inequality*

$$\sum\{p_i p_j |\, 1 \leq i < j \leq r\} \leq e(T_r(n))$$

holds for each r-tuple $(p_1, \ldots, p_r)$ of nonnegative integers p_i such that $p_1 + \cdots + p_r = n$. The equality occurs only when $K(p_1, \ldots, p_r) = T_r(n)$.

In our articles [6, 7] we introduced the following concept:

Definition 1. *Let G be a graph, $v_1, \ldots, v_r \in V(G)$ and $\Gamma_i = \Gamma_G(v_1, \ldots, v_i)$, $i = 1, 2, \ldots, r-1$. The sequence $v_1, \ldots, v_r$ is called **α-sequence**, if the following conditions are satisfied: v_1 is a maximal degree vertex in G and for $i \geq 2$, $v_i \in \Gamma_{i-1}$ and v_i has a maximal degree in the graph $G_{i-1} = G[\Gamma_{i-1}]$.*

In [6, 7] we proved the following

Theorem 1. *Let $v_1, \ldots, v_r$ be an α-sequence in the n-vertex graph G and there is no $(r+1)$-clique containing all members of the sequence. Then*

$$e(G) \leq e(K(n - d_1, d_1 - d_2, \ldots, d_{r-2} - d_{r-1}, d_{r-1}))$$

where $d_1 = d_G(v_1)$ and $d_i = d_{G_{i-1}}(v_i)$, $i = 2, \ldots, r$. The equality holds if and only if $G = K(n - d_1, d_1 - d_2, \ldots, d_{r-2} - d_{r-1}, d_{r-1})$.

Later α-sequences appear in [1, 2, 3, 4] under the name "degree-greedy algorithm". In [6] we obtained the following corollary of Theorem 1: If G is an n-vertex graph and $e(G) \geq e(T_r(n))$, then either $G = T_r(n)$ or each maximal (in the sense of inclusion) α-sequence in G has length $\geq r + 1$. Later this corollary is published in [1] and [2]. Some other results about α-sequences and its generalizations are given in our papers [8, 9].

R. Faudree in [5] introduces the following modification of α-sequences:

Definition 2. *The sequence of vertices $v_1, \ldots, v_r$ in a graph G is called **β-sequence**, if the following conditions are satisfied: v_1 is a maximal degree vertex in G, and, for $i \geq 2$, $v_i \in \Gamma_{i-1} = \Gamma_G(v_1, \ldots, v_{i-1})$ and $d_G(v_i) = \max\{d_G(v) | v \in \Gamma_{i-1}\}$.*

Obviously, the set of vertices of each β-sequence is a clique. In the discrete graph $\overline{K}_n$ each β-sequence consists of one vertex only. If G is not a discrete graph, then there are β-sequences of length 2. Let for $K(p_1, p_2, \ldots, p_r)$ we have $p_1 \leq p_2 \leq \cdots \leq p_r$ and $v_i \in V(\overline{K}_i)$, $i = 1, \ldots, r$. Then $v_1, \ldots, v_r$ is a β-sequence. If $v_1, \ldots, v_s$ is a β-sequence in a graph G, then it may be extended to a maximal in the sense of inclusion β-sequence $v_1, \ldots, v_s, \ldots, v_r$. It is clear that this extension is not contained in an $(r+1)$-clique and in this case $\Gamma(v_1, \ldots, v_r) = \emptyset$.

Definition 3. *Let $v_1, \ldots, v_r$ be a β-sequence in a graph G and*

$$V_1 = V(G) \backslash \Gamma(v_1), \quad V_2 = \Gamma(v_1) \backslash \Gamma(v_2),$$

$$V_3 = \Gamma(v_1, v_2) \backslash \Gamma(v_3), \ldots, V_r = \Gamma(v_1, \ldots, v_{r-1}) \backslash \Gamma(v_r).$$

*We shall call the sequence $V_1, \ldots, V_r$ a **stratification of** G induced by the β-sequence $v_1, \ldots, v_r$ and V_i will be called the i-th stratum. The number of vertices in the i-th stratum will be denoted by p_i.*

Let us note that $V_i = \Gamma_{i-1} \backslash \Gamma_i$, where by Γ_0 we understand $V(G)$, and $\Gamma_i = \Gamma_G(v_1, \ldots, v_i)$, $i = 1, \ldots, r$. Hence $V_i \subset \Gamma_{i-1}$ and $V_i \cap \Gamma_i = \emptyset$, thus $V_i \cap V_j = \emptyset$ for $i \neq j$. In addition $v_i \in \Gamma_{i-1} \backslash \Gamma(v_i)$ implies $v_i \in V_i$. From $V_i \cap \Gamma(v_i) = \emptyset$ it follows that the vertex v_i is not adjacent to any vertex of V_i. Therefore $d(v_i) \leq n - p_i$. But $d(v) \leq d(v_i)$ for every vertex $v \in V_i$. Consequently,

$$d(v) \leq n - p_i, \text{ for each } v \in V_i. \tag{1}$$

It is clear that

$$V = \bigcup_{i=1}^{r} V_i \cup \Gamma_r,$$

so that

$$\sum_{i=1}^{r} p_i + |\Gamma_r| = n. \tag{2}$$

Analogically to Theorem 1 we obtain the following

Theorem 2. *Let $v_1, \ldots, v_r$ be a β-sequence in an n-vertex graph G, which is not contained in an $(r+1)$-clique. If V_i is the i-th stratum of the stratification induced by this sequence and $p_i = |V_i|$, then*

$$e(G) \leq e(K(p_1, \ldots, p_r)) \tag{3}$$

and the equality occurs if and only if

$$d(v) = n - p_i \quad \textit{for each } v \in V_i, \; i = 1, \dots, r. \tag{4}$$

P r o o f. The assumption that $\{v_1, \dots, v_r\}$ is not contained in an $(r+1)$-clique implies that $\Gamma_r = \Gamma(v_1, \dots, v_r) = \emptyset$ and by (2) we have $\sum_{i=1}^{r} p_i = n$. On the other hand,

$$2e(G) = \sum_{v \in V(G)} d(v) = \sum_{i=1}^{r} \sum_{v \in V_i} d(v)$$

and by (1) it follows that

$$2e(G) \le \sum_{i=1}^{r} p_i(n - p_i) = 2e(K(p_1, \dots, p_r)). \tag{5}$$

So, (3) is proved. We have an equality in (3) if and only if there is an equality in (5), i.e. when (4) is available.

Theorem 2 is proved. □

By (1) it follows immediately

Proposition. *If* $v_1, \dots, v_r$ *is a* β*-sequence in an* n*-vertex graph* G*, which is not contained in an* $(r+1)$*-clique, then*

$$\sum_{i=1}^{r} d_G(v_i) \le (r-1)n.$$

Note that the equality in (3) is possible also when $G \ne K(p_1, \dots, p_r)$.

Example. *Let* Π *be the graph which is the 1-skeleton of a triangular prism. Let* $[a_1, a_2, a_3]$ *be the lower base,* $[b_1, b_2, b_3]$ *– the upper base and* $[a_i, b_i]$ *– the sideedges. Then* a_1, b_1 *is a* β*-sequence in the graph* Π*. There is no 3-clique containing both* a_1 *and* b_1*. The stratification induced by this sequence is* $V_1 = \{a_1, b_2, b_3\}$*,* $V_2 = \{b_1, a_2, a_3\}$*. Therefore* $p_1 = p_2 = 3$*. So we have* $e(\Pi) = 9$ *and* $e(K(3,3)) = 9$*, but* $\Pi \ne K(3,3)$*.*

Now we shall prove that in case of equality in (3), we may state something stronger than (4), but under an additional assumption about the β-sequence from Theorem 2.

Definition 4. *Let v_1 be a vertex of maximal degree in a graph G. The symbol $l_G(v_1)$ denotes the maximal length of β-sequences with first member v_1.*

Theorem 3. *Let $v_1, \ldots, v_r$ be a β-sequence such that $r = l_G(v_1)$. Then, in the notation of Theorem 2, if $e(G) = e(K(p_1, \ldots, p_r))$, then we have $G = K(p_1, \ldots, p_r)$.*

P r o o f. It is clear that the assumption $r = l_G(v_1)$ guarantees that in G there is no $(r+1)$-clique containing the members of the sequence $v_1, \ldots, v_r$, so that $\Gamma_r = \emptyset$ and (3) is valid (c.f. Theorem 2). Then the equality in (3) implies (4), we shall make use of below.

In order to prove Theorem 3, we shall proceed by induction on r. For $r = 1$ we have one stratum V_1 with $|V_1| = p_1 = n$ and by assumption $e(G) = e(K(p_1)) = 0$. Consequently G is a discrete graph, i.e. $G = K(p_1)$.

Let $r \geq 2$ and suppose the statement to be valid for $r - 1$.

We shall prove first that $V_r = \Gamma_{r-1} \setminus \Gamma_r = \Gamma_{r-1}$ is an independent vertex set. Suppose the contrary and let u_1 and u_2 be two adjacent vertices of V_r. We shall prove that $v_1, \ldots, v_{r-1}, u_1, u_2$ is a β-sequence in G.

It is clear that $v_1, \ldots, v_{r-1}$ is a β-sequence in G. By $u_1 \in V_r = \Gamma_{r-1}$ and the fact that all vertices of V_r have the same degree $(= n - p_r)$ it follows that u_1 has a maximal degree among the vertices of Γ_{r-1}. Clearly, $[u_1, u_2] \in E(G)$ and $u_2 \in V_r = \Gamma_{r-1}$ imply that $u_2 \in \Gamma(v_1, \ldots, v_{r-1}, u_1)$, besides that, u_2 has a maximal degree in $\Gamma(v_1, \ldots, v_{r-1}, u_1)$. Hence, $v_1, \ldots, v_{r-1}, u_1, u_2$ is a β-sequence in G. This contradicts the equality $l_G(v_1) = r$.

So, V_r is an independent vertex set.

Let $G' = G[V_1 \cup \cdots \cup V_{r-1}]$. Since V_r is an independent set and $d(v) = n - p_r$ for each $v \in V_r$, then $G = G' + \overline{K}_{p_r}$ and $e(G) = e(G') + p_r(n - p_r)$. It follows from this equality and $e(K(p_1, \ldots, p_r)) = e(K(p_1, \ldots, p_{r-1})) + p_r(n - p_r)$ that

$$e(G') = e(K(p_1, \ldots, p_{r-1})). \tag{6}$$

Obviously, if $M \subset V(G')$, then $\Gamma_G(M) = \Gamma_{G'}(M) \cup V_r$. Thus, if $v \in V(G')$, then $d_G(v) = d_{G'}(v) + p_r$.

It is clear then, that $v_1, \ldots, v_{r-1}$ is a β-sequence in G'. Hence, $s = l_{G'}(v_1) \geq r - 1$. We shall prove that $s = r - 1$. Suppose the contrary and let $v_1, u_2, \ldots, u_s$ be a β-sequence in G' with $s \geq r$. We shall show that $v_1, u_2, \ldots, u_s, v_r$ is a β-sequence in G.

First of all, $v_1, u_2, \ldots, u_s$ is obviously a β-sequence in G as well. From $s = l_{G'}(v_1)$ it follows that $\Gamma_{G'}(v_1, u_2, \ldots, u_s) = \emptyset$ and therefore $\Gamma_G(v_1, u_2, \ldots, u_s) = V_r$. This implies that $v_r \in \Gamma(v_1, u_2, \ldots, u_s)$ and has a maximal degree in G among the vertices of $\Gamma(v_1, u_2, \ldots, u_s)$. Consequently $v_1, u_2, \ldots, u_s, v_r$ is a β-sequence in G. This contradicts the equality $l_G(v_1) = r$, since the length of the last sequence is greater than r.

So, $l_{G'}(v_1) = r-1$. Note that the i-th stratum of the stratification induced by this sequence in G' is identical with the i-th stratum of the stratification of G induced by the β-sequence $v_1, \ldots, v_r$, $i = 1, 2, \ldots, r-1$. Taking into account (6), from the induction hypothesis we deduce that $G' = K(p_1, \ldots, p_{r-1})$ and since $G = G' + \overline{K}_{p_r}$, then $G = K(p_1, \ldots, p_r)$.

Theorem 3 is proved. □

Corollary 1. *Let G be an n-vertex graph, v_1 be a maximal degree vertex and $l_G(v_1) = r$. Then $e(G) \leq e(T_r(n))$ and the equality appears if and only if $G = T_r(n)$.*

Proof. Let $v_1, \ldots, v_r$ be a β-sequence and p_i denote the number of vertices in the i-th stratum of the stratification generated by this sequence. According to Theorem 2, $e(G) \leq e(K(p_1, \ldots, p_r))$. Lemma 1 implies that $e(K(p_1, \ldots, p_r)) \leq e(T_r(n))$. From the above two inequalities we get $e(G) \leq e(T_r(n))$.
Let now $e(G) = e(T_r(n))$. Then $e(G) = e(K(p_1, \ldots, p_r))$ and $e(K(p_1, \ldots, p_r)) = e(T_r(n))$. From the first equality, according to Theorem 3, it follows that $G = K(p_1, \ldots, p_r)$. The second equality, according to Lemma 1, implies $K(p_1, \ldots, p_r) = T_r(n)$. In this way we conclude that $G = T_r(n)$.

Corollary 1 is proved. □

Corollary 2 ([6]). *Let G be a graph with n vertices and v_1 be a maximal degree vertex, which is not contained in an $(r+1)$-clique, $r \leq n$. Then $e(G) \leq e(T_r(n))$ and the equality occurs only for the Turan's graph $T_r(n)$.*

Proof. Let $l_G(v_1) = s$. Then in G there is a s-clique containing v_1, so $s \leq r$ and therefore $e(T_s(n)) \leq e(T_r(n))$, where the equality is available only when $s = r$ (see Lemma 1). Corollary 1 implies $e(G) \leq e(T_s(n))$ and we have equality only for $G = T_s(n)$. The above two inequalities imply the desired inequality $e(G) \leq e(T_r(n))$. It is clear that we have equality in the last inequality if and only if $s = r$ and $G = T_s(n)$.

Corollary 2 is proved. □

Evidently, Corollary 2 is a generalization of

Turan's Theorem ([10]). *Let G be an n-vertex graph without $(r+1)$-cliques. Then $e(G) \leq e(T_r(n))$ and equality occurs only for $G = T_r(n)$.*

Corollary 3. *Let G be an n-vertex graph and $v_1, \ldots, v_m$ be a β-sequence, which is not contained in an $(r+1)$-clique, $r \leq n$. Then $e(G) \leq e(T_r(n))$*

and in case $m = r$, we have $e(G) = e(T_r(n))$ only when all the strata of the stratification induced by the sequence have almost equal number of vertices and the vertex degrees of any stratum are equal to the number of vertices out of the stratum.

P r o o f. We shall extend the β-sequence $v_1, \ldots, v_m$ to a β-sequence $v_1, \ldots, v_m, \ldots, v_q$ not contained in a $(q+1)$-clique. Obviously, $m \le q \le r$. Let $V_1, \ldots, V_q$ be the strata of the β-sequence $v_1, \ldots, v_q$ and $p_i = |V_i|$. It follows from Theorem 2 that

$$e(G) \le e(K(p_1, \ldots, p_q)) \tag{7}$$

and we have equality in (7) if and only if $d(v) = n - p_i$ for each $v \in V_i$, $i = 1, \ldots, q$. On the other hand, according to Lemma 1 we have

$$e(K(p_1, \ldots, p_q)) \le e(T_q(n)) \le e(T_r(n)) \tag{8}$$

where equality occurs only when $q = r$ and p_i are almost equal.

Thus $e(G) \le e(T_r(n))$. Equality appears here only when there is an equality in (7) and (8). The equality in (7) implies that $d(v) = n - p_i$ for any $v \in V_i$ and the equality in (8) occurs if and only if $q = r$ and $|p_i - p_j| \le 1$ for each i, j.

The extremal case in the proposition follows immediately from the above reasoning.

Corollary 3 is proved. □

REFERENCES

[1] B. Bollobas. Turan's theorem and maximal degrees. *J. Combin. Theory Ser. B* **75** (1999), 160–164.

[2] B. Bollobas. Modern Graph Theory. Springer Verlag, New York, 1998.

[3] B. Bollobas, A.Thomason. Random graphs of small order. *Ann. Discrete Math.* **28** (1985), 47–97.

[4] J. A. Bondy. Large dense neighborhoods and Turan's theorem. *J. Combin. Theory Ser. B* **34** (1983), 109–111.

[5] R. Faudree. Complete subgraphs with large degree sums. *J. Graph Theory* **16** (1992), 327–334.

[6] N. Khadzhiivanov, N. Nenov. Extremal problems for s-graphs and the theorem of Turan. *Serdica Bulg. Math. Publ.* **3** (1977), 117–125 (in Russian).

[7] N. Khadzhiivanov, N. Nenov. The maximum of the number of edges of a graph. *C. R. Acad. Bulgare Sci.* **29** (1976), 1575–1578 (in Russian).

[8] N. Khadzhiivanov, N. Nenov. p-sequences of graphs and some extremal properties of Turan's graphs. *C. R. Acad. Bulgare Sci.* **30** (1977), 475–478 (in Russian).

[9] N. Khadzhiivanov, N. Nenov. On generalized Turan's Theorem and its extension. *Ann. Sof. Univ. Fac. Math.* **77** (1983), 231–242 (in Russian).

[10] P. Turan. An extremal problem in graph theory. *Mat. Fiz. Lapok* **48** (1941), 436–452 (in Hungarian).

Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, Blvd. James Bourchier
1164, Sofia, Bulgaria
e-mail: `hadji@fmi.uni-sofia.bg`
e-mail: `nenov@fmi.uni-sofia.bg`

Received November 4, 2003