Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **30** (2004), 195–206

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

EQUIMULTIPLE LOCUS OF EMBEDDED ALGEBROID SURFACES AND BLOWING–UP IN CHARACTERISTIC ZERO

R. Piedra-Sánchez* and J. M. Tornero**

Communicated by V. Brînzănescu

Abstract. The smooth equimultiple locus of embedded algebroid surfaces appears naturally in many resolution processes, both classical and modern. In this paper we explore how it changes by blowing–up.

1. Introduction. During all this paper, we will consider K an algebraically closed field of characteristic 0 and $\mathcal{S} = \operatorname{Spec}(K[[X,Y,Z]]/(F))$ an embedded algebroid surface which, with no loss of generality, is considered to be defined by a Weierstrass equation

$$F(Z) = Z^n + \sum_{k=0}^{n-1} a_k(X,Y) Z^k,$$

2000 *Mathematics Subject Classification:* 14B05, 32S25.
Key words: Resolution of surface singularities, blowing–up, equimultiple locus.
*Supported by FQM 304 and BFM 2000–1523.
**Supported by FQM 218 and BFM 2001–3207.

where n is the multiplicity of $\mathcal{S}$, that is $\operatorname{ord}(a_k) \geq n-k$ for all $k=0,\ldots,n-1$. After the well–known Tchirnhausen transformation $Z \longmapsto Z - a_{n-1}/n$ we can get a Weierstrass equation of the form

$$F(Z) = Z^n + \sum_{k=0}^{n-2} a_k(X,Y) Z^k.$$

From now on, by a Weierstrass equation we will mean an equation like this. Observe that, if we denote the initial form of F by $\overline{F}$, the affine variety defined by $\overline{F}$ (that is, the tangent cone of $\mathcal{S}$) is a plane if and only if it is the plane $Z=0$.

In this situation the equimultiple locus of $\mathcal{S}$ is

$$\mathcal{E}(\mathcal{S}) = \left\{ P \in \operatorname{Spec}(K[[X,Y,Z]]) \mid F \in P^{(n)} \right\},$$

which is never empty, as $M=(X,Y,Z)$ always lies in $\mathcal{E}(\mathcal{S})$. Note that $Z \in P$ for all $P \in \mathcal{E}(\mathcal{S})$.

Geometrically speaking, the equimultiple locus represents points at where the multiplicity is the same as in the origin; hence they are the "closest" points to the origin in (coarse) terms of singularity complexity. We will denote by $\mathcal{E}_0(\mathcal{S})$ the subset of smooth elements of $\mathcal{E}(\mathcal{S})$.

Our aim is studying the set $\mathcal{E}(\mathcal{S})$ and, specifically, how its elements change by blowing–up, in order to have a better understanding of the evolution of $\mathcal{E}_0(\mathcal{S})$ through a resolution process. In this environment, our main result explains how can we deduce "geometrically" $\mathcal{E}_0\left(\mathcal{S}^{(1)}\right)$ from $\mathcal{E}_0(\mathcal{S})$, where $\mathcal{S}^{(1)}$ is the result of blowing–up $\mathcal{S}$ with center in an element of $\mathcal{E}_0(\mathcal{S})$.

The interest of this relies in the fact that the equimultiple locus contains important information for desingularization purposes. For instance, if $\mathcal{S}$ has normal crossing singularities, blowing–up centers lying in $\mathcal{E}_0(\mathcal{S})$ of maximal dimension resolves the singularity at the origin. This is the famous Levi–Zariski theorem on the resolution of surface singularities, stated by Levi ([3]) and proved by Zariski ([11]).

Concerning the extension of the results of this paper to the arbitrary characteristic case, we must say that, no matter which the characteristic is, the equimultiple locus has some very interesting properties, some of which we state below:

(a) It is hyperplanar, that is, there exists a regular parameter which lies in every element of $\mathcal{E}(\mathcal{S})$. This was proved by Mulay ([4]) after some previous work of Abhyankar ([2]) and Narasimhan ([5]).

(b) As proved by Abhyankar ([1]), the Levi–Zariski theorem remains true in positive characteristic.

However, the techniques used cannot be applied to the general case, as it is not straightforward (actually, it is not known) that the same regular parameter can define the tangent cone and a hyperplane containing the equimultiple locus. In particular, Mulay's proof of the hyperplanarity is not constructive, so a different approach may be needed to handle with characteristic $p > 0$ (see [7] for some instances).

As for the extension of our result to higher dimensions is concerned, we cannot be very optimistic. First of all, there is the additional difficulty that, in positive characteristic, the equimultiple locus is not hyperplanar anymore, as shown by Narasimhan ([6]). Secondly, the Levi–Zariski resolution process as stated does not work, as proved by Spivakovski ([9]), which, in any case, makes results on this line less interesting. However, a less coarse version of the Levi–Zariski theorem for a more general type of varieties will surely mean a great achievement and it will need results of this sort towards a more thorough understanding of the evolution of the equimultiple locus by sucessive blowing–ups.

2. Notation and technical results. For the sake of completeness, we recall here basic facts and technical results related to quadratic and monoidal transformations that will be of some help in the sequel.

For all what follows, let $\mathcal{S}$ be an embedded algebroid surface of multiplicity n,

$$F = Z^n + \sum_{k=0}^{n-2} \left(\sum_{i,j} a_{ijk} X^i Y^j \right) Z^k = Z^n + \sum_{k=0}^{n-2} a_k(X,Y) Z^k$$

a Weierstrass equation of $\mathcal{S}$. We will denote

$$N(F) = \left\{ (i,j,k) \in \mathbf{N}^3 \mid a_{ijk} \neq 0 \right\}.$$

Definition. *The elements of $\mathcal{E}(\mathcal{S})$ different from M will be called equimultiple curves. The elements of $\mathcal{E}_0(\mathcal{S})$ other than M will be called permitted curves.*

Remark. The notion of permitted curves coincides with the one derived from normal flatness in the work of Hironaka.

Remark. In particular, note that we can assume $P \in \mathcal{E}_0(\mathcal{S})$ to be, for instance (Z, X), after a suitable change of variables in $K[[X, Y]]$. Plainly, (Z, X) is permitted if and only if $i + k \geq n$ for all $(i, j, k) \in N(F)$.

Under these circumstances, the monoidal transform of $\mathcal{S}$, centered in (X, Z), in the point corresponding to the direction $(\alpha : 0 : \gamma)$ (say $\alpha \neq 0$) of the exceptional divisor is the surface $\mathcal{S}^{(1)}$ defined by the equation

$$F^{(1)} = \left(Z_1 + \frac{\gamma}{\alpha}\right)^n + \sum_{(i,j,k)\in N(F)} a_{ijk} X_1^{i+k-n} Y_1^j \left(Z_1 + \frac{\gamma}{\alpha}\right)^k .$$

Observe that this only makes sense (that is, gives a non–unit) whenever $\overline{F}(\alpha, 0, \gamma) = 0$. The homomorphism

$$\begin{array}{rcl} \pi^P_{(\overline{\alpha}:0:\gamma)} : K[[X,Y,Z]] & \longrightarrow & K[[X_1,Y_1,Z_1]] \\ X & \longmapsto & X_1 \\ Y & \longmapsto & Y_1 \\ Z & \longmapsto & X_1\left(Z_1 + \dfrac{\gamma}{\alpha}\right) \end{array}$$

will be called the homomorphism associated to the monoidal transformation in $(\overline{\alpha} : 0 : \gamma)$ or, in short, the equations of the monoidal transformation. The overline is because one must privilege a non-zero coordinate, but all the possibilities define associated equations.

As for quadratic transforms (that is, blowing–ups with center M) is concerned: the quadratic transform of $\mathcal{S}$ in the point corresponding to the direction $(\alpha : \beta : \gamma)$ (say $\alpha \neq 0$) of the exceptional divisor is the surface $\mathcal{S}^{(1)}$ defined by the equation

$$F^{(1)} = \left(Z_1 + \frac{\gamma}{\alpha}\right)^n + \sum_{(i,j,k)\in N(F)} a_{ijk} X_1^{i+j+k-n} \left(Y_1 + \frac{\beta}{\alpha}\right)^j \left(Z_1 + \frac{\gamma}{\alpha}\right)^k .$$

Again this only makes sense whenever $\overline{F}(\alpha, \beta, \gamma) = 0$. Analogously, the homomorphism

$$\begin{array}{rcl} \pi^M_{(\overline{\alpha}:\beta:\gamma)} : K[[X,Y,Z]] & \longrightarrow & K[[X_1,Y_1,Z_1]] \\ X & \longmapsto & X_1 \\ Y & \longmapsto & X_1\left(Y_1 + \dfrac{\beta}{\alpha}\right) \\ Z & \longmapsto & X_1\left(Z_1 + \dfrac{\gamma}{\alpha}\right) \end{array}$$

will be called the homomorphism associated to the quadratic transformation in $(\overline{\alpha} : \beta : \gamma)$ or the equations of the quadratic transformation.

Remark. In the previous situation, consider a change of variables in $K[[X,Y,Z]]$ given by

$$\begin{cases} \varphi(X) &= a_1X' + a_2Y' + a_3Z' + \varphi_1(X',Y',Z') \\ \varphi(Y) &= b_1X' + b_2Y' + b_3Z' + \varphi_2(X',Y',Z') \\ \varphi(Z) &= c_1X' + c_2Y' + c_3Z' + \varphi_3(X',Y',Z') \end{cases},$$

with $\operatorname{ord}(\varphi_i) \geq 2$.

Assume also that

$$\begin{cases} \alpha &= a_1\alpha' + a_2\beta' + a_3\gamma' \\ \beta &= b_1\alpha' + b_2\beta' + b_3\gamma' \\ \gamma &= c_1\alpha' + c_2\beta' + c_3\gamma' \end{cases}$$

with (say) $\gamma' \neq 0$. Then there is a unique change of variables $\psi : K[[X_1,Y_1,Z_1]] \longrightarrow K[[X_1',Y_1',Z_1']]$ such that

$$\psi\pi^M_{(\overline{\alpha}:\beta:\gamma)} = \pi^M_{(\alpha':\beta':\overline{\gamma'})}\varphi.$$

Both this remark and its monoidal counterpart (which will not be used in this paper) are easy, although rather long, so we skip the proofs. The interested reader may consult [8] and [10] for the complete details.

Definition. *Let $Q \in \mathcal{E}(\mathcal{S})$, with $Q = (Z, G(X,Y))$. Then for $\underline{u} \in \mathbf{P}^2(K)$, the ideal*

$$\varpi^M_{\underline{u}}(Q) = \left(Z_1, \frac{\pi^M_{\underline{u}}(G(X,Y))}{X_1^{\operatorname{ord}(G)}}\right)$$

is called the quadratic transform of Q in the point corresponding to $\underline{u}$.

Obviously, this definition makes sense only if the quadratic transform in the direction $\underline{u}$ does. There is a natural version of monoidal transform of Q with center P, for all $P \in \mathcal{E}_0(\mathcal{S})$.

Notation. *We will denote by ν the natural isomorphism*

$$\nu : K[[X,Y,Z]] \longrightarrow K[[X_1,Y_1,Z_1]]$$

sending X to X_1, Y to Y_1 and Z to Z_1.

3. The theorem. As our result is inspired by the resolution process, we will restrict ourselves to the case which is interesting for desingularization issues:

that where $\mathcal{S}$ and $\mathcal{S}^{(1)}$ have the same multiplicity. In particular, this avoids some possibilities.

Lemma. *If the tangent cone of $\mathcal{S}$ is not a plane, the multiplicity of any monoidal transform of $\mathcal{S}$ is strictly less than n.*

Proof. If the tangent cone is not a plane, mind there is only one possible element in $\mathcal{E}_0(\mathcal{S})$. After a suitable change of variables on $K[[X,Y]]$, let this curve be (Z,X). Then $\overline{F}$ cannot depend on Y,

$$\overline{F} = Z^n + \sum_{i+k=n} a_{i0k} X^i Z^k = \prod_{l=1}^{n} (Z - \alpha_l X).$$

That is, the directions of the exceptional divisor are $(1 : 0 : \alpha_l)$ for $l = 1, \ldots, n$; not all of them equal. Then, the equation for one of these monoidal transforms is

$$F^{(1)} = (Z_1 + \alpha_{l_0})^n + \sum a_{ijk} X_1^{i+k-n} Y_1^j (Z_1 + \alpha_{l_0})^k$$

where we find the monomial

$$\left(\prod_{\alpha_{l_0} \neq \alpha_l} (\alpha_{l_0} - \alpha_l) \right) Z^m,$$

with $m = \#\{l \mid \alpha_{l_0} = \alpha_l\}$. This monomial cannot cancel in any case. So there is a monomial in $F^{(1)}$ of order strictly smaller than n and we are done. $\square$

Theorem. *Let $\mathcal{S}$ be an algebroid surface and $\mathcal{S}^{(1)}$ a quadratic or monoidal transform of $\mathcal{S}$ with the same multiplicity.*
(a) *If $\mathcal{S}^{(1)}$ is the monoidal transform of $\mathcal{S}$ with center $P \in \mathcal{E}_0(\mathcal{S})$, then, either $\mathcal{E}_0\left(\mathcal{S}^{(1)}\right) = \nu\left(\mathcal{E}_0(\mathcal{S})\right)$ or $\mathcal{E}_0\left(\mathcal{S}^{(1)}\right) = \nu\left(\mathcal{E}_0(\mathcal{S}) \setminus \{P\}\right)$.*
(b) *Let $\mathcal{S}^{(1)}$ be the quadratic transform of $\mathcal{S}$ in the point corresponding to $\underline{u}$. Then:*

(b.1) *If the tangent cone is not a plane then $\mathcal{E}_0\left(\mathcal{S}^{(1)}\right) = \nu\left(\mathcal{E}_0(\mathcal{S})\right)$.*

(b.2) *If the tangent cone is a plane, then in $\mathcal{E}_0\left(\mathcal{S}^{(1)}\right)$ we can find three types of curves:*

(i) *The exceptional divisor of the transform.*

(ii) *Primes $\varpi_{\underline{u}}^M(Q)$, with $Q \in \mathcal{E}(\mathcal{S}) \setminus \mathcal{E}_0(\mathcal{S})$, which are tangent to the exceptional divisor.*

(iii) *Primes $\varpi_{\underline{u}}^M(Q)$, with $Q \in \mathcal{E}_0(\mathcal{S})$, where both $\nu(Q)$ and $\varpi_{\underline{u}}^M(Q)$ are transversal to the exceptional divisor.*

Moreover, if there is any prime of type (ii)*, it also appears the type* (i) *prime.*

Proof. We will do the proof case by case, although some arguments are common to various instances. In what follows let F be, as usual, a Weierstrass equation of $\mathcal{S}$.

Case (a)

From the previous lemma, we can assume that the tangent cone is the plane $Z = 0$. The basic tool for this situation is the following:

Remark. A monoidal transformation which does not imply a descent of the multiplicity cannot create new permitted curves.

This can be proved easily as follows: under the hypothesis of case (a), let (Z, G) be a permitted curve. Then $\mathcal{S}^{(1)}$, the monoidal transform of $\mathcal{S}$ with center at (Z, G), is given by

$$F^{(1)} = Z_1^n + \sum_{k=0}^{n-2} \frac{a_k(X_1, Y_1)}{G(X_1, Y_1)^{n-k}} Z_1^k.$$

This consists simply on taking (Z, G) to (Z, X) by a change of variables (say φ) on $K[[X, Y]]$, applying the transform (the only point in the exceptional divisor in this case is the point corresponding to the direction $(1 : 0 : 0)$) and taking φ^{-1} on $K[[X_1, Y_1]]$. The result follows directly.

This remark clearly implies case (a) of the theorem.

Case (b.2)

Some arguments in this case will be used for the other, so we will begin for it. Let us start for the direction $(1 : 0 : 0)$ (the direction $(0 : 1 : 0)$ is obviously symmetric). If

$$F = Z^n + \sum_{(i,j,k) \in N(F)} a_{ijk} X^i Y^j Z^k,$$

then

$$F^{(1)} = Z_1^n + \sum_{(i,j,k) \in N(F)} a_{ijk} X_1^{i+j+k-n} Y_1^j Z_1^k.$$

Note that, as F is a Weierstrass equation and the multiplicity does not change, $F^{(1)}$ is a Weierstrass equation for $\mathcal{S}^{(1)}$, hence all the elements of $\mathcal{E}\left(\mathcal{S}^{(1)}\right)$ are contained in $Z_1 = 0$, therefore all permitted curves in $\mathcal{E}\left(S^{(1)}\right)$ can be assumed to be of the form $P = (Z_1, \gamma X_1 + \delta Y_1 + G(X_1, Y_1))$, with $\mathrm{ord}(G) \geq 2$.

Let us prove now that, if a permitted curve transversal to the exceptional divisor appears in $\mathcal{E}_0\left(\mathcal{S}^{(1)}\right)$, it comes from a permitted curve in $\mathcal{E}_0(\mathcal{S})$ which was also transversal to the exceptional divisor (up to the action of ν).

Suppose we have such a curve (that is, a prime as above with $\delta \neq 0$). Then, applying the Weierstrass Preparation Theorem, we may write P as $(Z_1, Y_1 + H(X_1))$. We have the diagram

$$\begin{tikzcd}
K[[X,Y,Z]] \arrow[r, "\varphi"] \arrow[d, "\pi^M_{(1:0:0)}"'] & K[[X',Y',Z']] \arrow[d, "\pi^M_{(1:0:0)}"] \\
K[[X_1,Y_1,Z_1]] \arrow[r, "\psi"] & K[[X'_1,Y'_1,Z'_1]]
\end{tikzcd}$$

with changes of variables

$$\left\{ \begin{array}{lcl} \varphi(X) & = & X' \\ \varphi(Y) & = & Y' - X'H(X') \\ \varphi(Z) & = & Z' \end{array} \right. , \quad \left\{ \begin{array}{lcl} \psi(X_1) & = & X'_1 \\ \psi(Y_1) & = & Y'_1 - H(X'_1) \\ \psi(Z_1) & = & Z'_1 \end{array} \right.$$

So, looking at the right vertical arrow, we have found a quadratic transform on the direction $(1:0:0)$ which gives rise to the permitted curve (Z'_1, Y'_1). This clearly implies that (Z', Y') was permitted in $\mathcal{S}$. This proves the assertion.

Another way of seeing this is saying that, if there were no permitted curves which were transversal to the exceptional divisor, all permitted curves after the blowing–up must be tangent to it.

Let us prove now that, if $(Z_1, X_1 + Y_1^s v(Y_1))$ with $s > 1$, appears, so does (Z_1, X_1). Write

$$F^{(1)} = Z_1^n + \sum_{k=0}^{n-2} a_k^{(1)}(X_1, Y_1) Z_1^k,$$

where it must hold $a_k^{(1)} = (X_1 + Y_1^s v(Y_1))^{n-k} b_k^{(1)}(X_1, Y_1)$.

Fix then $k_0 \in \{0, \ldots, n-2\}$ and choose from all monomials in $b_{k_0}^{(1)}$ the minimal one for the lexicographic ordering, say $X_1^{i_0} Y_1^{j_0}$. Then all monomials appearing in $a_{k_0}^{(1)}$ have exponent in X_1 greater or equal than i_0 and, besides, the monomial $X_1^{i_0} Y_1^{j_0 + s(n-k_0)}$ actually appears in $a_{k_0}^{(1)}$, as it cannot be cancelled.

Now it is plain that $(i,j,k) \in N\left(F^{(1)}\right)$ if and only if $(i-j-k+n, j, k) \in N(F)$, so

$$i_0 \geq s\,(n-k_0) + k_0 - n \geq n - k_0.$$

Hence $(Z_1, X_1) \in \mathcal{E}\left(\mathcal{S}^{(1)}\right)$.

Let us prove then the existence, in this case, of the curve Q on $\mathcal{E}(\mathcal{S})$ announced in the theorem. As previously, we will consider $\alpha = 0$. Note $G(Y_1) = Y_1^2 v(Y_1)$. We will prove that there exists a power series $H(X,Y)$ verifying:

(1) $\operatorname{ord}(H) = \operatorname{ord}(G) = \lambda > 1$.

(2) H is regular on Y of order λ.

(3) There is a unit $u(X_1, Y_1)$ such that

$$\frac{1}{X_1^\lambda} H(X_1, X_1 Y_1) = u(X_1, Y_1)(X_1 + G(Y_1)).$$

This implies (quite straightforwardly) that $F \in Q^{(n)}$ and $(Z_1, X_1 + G(Y_1)) = \pi^M_{(1:0:0)}(Q)$, with $Q = (Z, H(X,Y))$. The second part is trivial and, for the first part it is enough proving

$$(X_1 + G(Y_1))^{n-k} \,|a_k^{(1)}(X_1, Y_1) \implies H(X,Y)^{n-k} | a_k(X,Y),$$

for $k = 0, \ldots, n-2$. Assume it is not so; then by the Weierstrass Preparation Theorem and being H regular with respect to Y, we can write

$$a_k(X,Y) = q(X,Y)H(X,Y)^{n-k} + \sum_{j=0}^{(n-k)\lambda - 1} \sigma_j(X) Y^j.$$

Now we apply $\pi^M_{(1:0:0)}$ and we obtain

$$\begin{aligned} X_1^a a_k^{(1)}(X_1, Y_1)^{n-k} &= \\ &= X_1^b q'(X_1, Y_1)(X_1 + G(Y_1))^{n-k} + \sum_{j=0}^{(n-k)\lambda-1} \left(\sigma_j(X_1) X_1^j\right) Y_1^j. \end{aligned}$$

Now, as $(X_1 + G(Y_1))^{n-k}$ divides $a_k^{(1)}(X_1, Y_1)$ it also must divide $X_1^a a_k^{(1)}(X_1, Y_1)$, hence the uniqueness of quotient and remainder in the Weierstrass Preparation Theorem imply

$$\sigma_j(X_1) X_1^j = 0, \text{ for all } j = 0, \ldots, (n-k)\lambda - 1,$$

and subsequently $H(X,Y)^{n-k} | a_k(X,Y)$.

So let us prove the existence of H and u. Write up $X_1 + G(Y_1)$ as

$$X_1 + G(Y_1) = X_1 + \sum_{i \geq \lambda} \alpha_i Y_1^i,$$

and the power series we look for as

$$H(X_1, Y_1) = \sum_{i+j=k\geq\lambda} \beta_{ij} X_1^i Y_1^j, \quad u(X_1, Y_1) = \sum_{i+j=k\geq 0} \gamma_{ij} X_1^i Y_1^j.$$

It must hold

$$\sum_{i+j=k\geq\lambda} \beta_{ij} X_1^{k-\lambda} Y_1^j = \left(\sum_{i+j=k} \gamma_{ij} X_1^i Y_1^j \right) \left(X_1 + \sum_{i\geq\lambda} \alpha_i Y_1^i \right),$$

which, for order 0, amounts to

$$\beta_{\lambda,0} = \gamma_{0,0} 0 = 0.$$

On the other hand, for order 1 we have

$$\beta_{\lambda+1,0} X_1 + \beta_{\lambda-1,1} Y_1 = \gamma_{0,0} X_1;$$

that is, $\beta_{\lambda-1,1} = 0$ and $\beta_{\lambda+1,0} = \gamma_{0,0}$, whose value can be taken to be 1.

As for order 2,

$$\beta_{\lambda+2,0} X_1^2 + \beta_{\lambda,1} X_1 Y_1 + \beta_{\lambda-2,2} Y_1^2 = \gamma_{0,0} \alpha_2 Y_1^2 + \gamma_{1,0} X_1^2 + \gamma_{0,1} X_1 Y_1,$$

which forces $\beta_{\lambda-2,2} = \alpha_2$ and let us freedom for fixing $\beta_{\lambda+2,0} = \gamma_{1,0}$, and $\beta_{\lambda,1} = \gamma_{0,1}$.

Observe then the following facts:

- Each β_{ij} appears only for order $i + 2j - \lambda$.
- In order k, all γ_{ij} with $i + j < k$ appear, but they never have relations among them, only those of the type

$$\beta_{ab} = \sum \gamma_{cd} \alpha_e.$$

Therefore it is clear that we can choose arbitrarily the γ_{ij}, and this choice determines the β_{ij}. Therefore both H and u exist.

It only remains proving that H can be chosen such that $H(0, Y)$ has order λ. But this is direct from the formula for order λ;

$$\beta_{0,\lambda} Y_1^\lambda = \gamma_{0,0} \alpha_\lambda Y_1^\lambda \neq 0,$$

so $\beta_{0,\lambda} \neq 0$.

For the results at points $(1:\alpha:0)$ it suffices considering the (commutative) diagram

$$\begin{tikzcd} K[[X,Y,Z]] \arrow[r, "\varphi"] \arrow[d, "\pi^M_{(1:\alpha:0)}"'] & K[[X',Y',Z']] \arrow[d, "\pi^M_{(1:0:0)}"] \\ K[[X_1,Y_1,Z_1]] \arrow[r, "\psi"] & K[[X_1',Y_1',Z_1']] \end{tikzcd}$$

with φ given by

$$\begin{cases} \varphi(X) &= X' \\ \varphi(Y) &= Y' - \alpha X' \\ \varphi(Z) &= Z' \end{cases}$$

Remark. This case (b.2), in geometrical terms, may be expressed as follows:

- Permitted curves transversal to the exceptional divisor cannot be created nor erased.
- Permitted curves tangent to the exceptional divisor are erased.
- Permitted curves tangent to the exceptional divisor can be created from desingularization of equimultiple (singular) curves. In this case, one of them must be the exceptional divisor itself.

$\boxed{\textbf{Case (b.1)}}$

Remark. In the conditions of (b.1), let $P = (\alpha : \beta : \gamma)$ a point in the tangent cone with multiplicity r. Then, the quadratic transform of $\mathcal{S}$ on $(\alpha : \beta : \gamma)$ has, at most, multiplicity r. This is straightforward, using, for instance, the Taylor expansion of F.

So we only need to be concerned about points of multiplicity n on the tangent cone. Changing the variables if needed we may consider that the point is $(0:1:0)$ and, subsequently, $\overline{F}$ does not depend on Y.

We will first prove that the quadratic transform cannot have permitted curves. Note that, in (b.2), we have showed that, if a new permitted curve appears, so does the exceptional divisor (and we did not use the fact that $\overline{F} = Z^n$ for proving this). But (Z,Y) cannot be a permitted curve, $\overline{F^{(1)}}$ having monomials in $K[X,Z]$ other than Z^n.

Now we explain why the quadratic transform does not erase permitted curves either. In fact if there is a permitted curve (only one is possible), we may take it to be (Z, X), after the customary change of variables. Then it is plain that, after a quadratic transform on the direction $(0 : 1 : 0)$, (Z_1, X_1) remains permitted, simply looking at the characterization given in the previous section.

This finishes the proof of the theorem.

REFERENCES

[1] S. S. Abhyankar. Resolution of Singularities of Embedded Algebraic Surfaces. Academic Press, New York, 1966.

[2] S. S. Abhyankar. Good points of a hypersurface. *Adv. Math.* **68** (1988), 87–256.

[3] B. Levi. Risoluzione delle singolarità puntualli delle superficie algebriche. *Atti Accad. Sci. Torino* **33** (1897), 66–86.

[4] S. B. Mulay. Equimultiplicity and hyperplanarity. *Proc. Amer. Math. Soc.* **89** (1983), 407–413.

[5] R. Narasimhan. Hyperplanarity of the equimultiple locus. *Proc. Amer. Math. Soc.* **87** (1983), 403–408.

[6] R. Narasimhan. Monomial equimultiple curves in positive characteristic. *Proc. Amer. Math. Soc.* **89** (1983), 402–406.

[7] R. Piedra. Estudio local de singularidades de superficies sobre cuerpos de característica arbitraria. Ph. D. Thesis, Univ. de Sevilla, 1978.

[8] T. Sánchez Giralda. Teoría de singularidades de superficies algebroides sumergidas. Ph. D. Thesis, Univ. Complutense, 1976.

[9] M. Spivakovski. A counterexample to the theorem of Beppo Levi in three dimensions. *Invent. Math.* **96** (1989), 181–183.

[10] J. M. Tornero. Aspectos locales de singularidades de superficies: Superficies de Puiseux. Ph. D. Thesis, Univ. de Sevilla, 2001.

[11] O. Zariski. Reduction of singularities of algebraic three dimensional varieties. *Ann. of Math.* **45** (1944), 472–542.

Departamento de Álgebra
Universidad de Sevilla.
Apdo 1160, 41080 Sevilla, Spain
e-mail: piedra@algebra.us.es
e-mail: tornero@algebra.us.es

Received March 22, 2004