OPTIMIZATION OF THE INVESTMENT PORTFOLIO IN THE CONDITIONS OF UNCERTAINTY

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Introduction

Portfolio analysis exists, perhaps, as long, as people think about acceptance of rational decisions connected with use of the limited resources. However the occurrence moment of portfolio analysis can be dated precisely enough is having connected it with a publication of pioneer work of Harry Markovittz (Markovitz H. Portfolio Selection) in 1952. The model offered in this work, simple enough in essence, has allowed catching the basic features of the financial market, from the point of view of the investor, and has supplied the last with the tool for development of rational investment decisions.

The central problem in Markovitz theory is the portfolio choice that is a set of operations. Thus in estimation, both separate operations and their portfolios two major factors are considered: profitableness and risk of operations and their portfolios. The risk thus receives a quantitative estimation. The account of mutual correlation dependences between profitablenesses of operations appears the essential moment in the theory. This account allows making effective diversification of portfolio, leading to essential decrease in risk of a portfolio in comparison with risk of the operations included in it. At last, the quantitative characteristic of the basic investment characteristics allows defining and solving a problem of a choice of an optimum portfolio in the form of a problem of quadratic optimization.

However the worldwide market crises in 1997-1998 and in 2000-2001, which had yielded only to the American investors 10 billion dollar losses, have shown, that existing theories of optimization of share portfolios and forecasting of share indexes have exhausted itself, and essential revision of share management methods is necessary.

Thus, in the light of obvious insufficiency of available scientific methods for management of financial actives, the development of fundamentally new theory of management of the financial systems functioning in the conditions of essential uncertainty needed. The big assistance to this theory was rendered by the theory of the fuzzy sets which have been developed about half a century ago in fundamental works of Lofti Zadeh.

The purpose of the present work is research and the analysis of qualitatively new approach to management of the share portfolio, based on application of the theory of fuzzy sets, and also development of algorithms realizing the given approach and comparison of results of their application with the results received at use of classical probabilistic methods.

Problem statement

The purpose of the analysis and optimization of an investment portfolio is research in area of portfolio optimization, and also the comparative analysis of structure of the effective portfolios received at use of model Markovitz and fuzzy-set model of a share portfolio optimization.

Let us consider a share portfolio from *N* components and its expected behaviour at time interval $[0,T]$. Each of a portfolio component is characterized $i=1,\dots,N$ by the financial profitableness r_i .

The holder of a share portfolio – the private investor, the investment company, mutual fund – operates the investments, being guided by certain reasons. On the one hand, the investor tries to maximise the profitableness. On the other hand, it fixes maximum permissible risk of an inefficiency of the investments. We will assume the capital of the investor be equal 1. The problem of optimization of a share portfolio consists in a finding of a vector of share price distribution of papers in a portfolio $x=\{x_i\}\ i=\overline{1,N}$ of the investor maximising the income at the set risk level (obviously, that $\sum_{i-1}^{N} x_i = 1$).

Weaknesses of accurate model Markovitz

In process of practical application of model Markovitz its lacks were found out:

1. The hypothesis about normality profitableness distributions in practice does not prove to be true.

2. Stationarity of price processes also not always is in practice.

3. At last, the risk of actives is considered as a dispersion standard deviation of the prices of securities from expected value i.e. as decrease in profitableness of securities in relation to expected value, and profitableness increase in relation to an average are estimated absolutely the same.

Though for the proprietor of securities these events are absolutely not the same.

These weaknesses of Markovitz theory define necessity of use of essentially new approach of definition of an optimum investment portfolio.

Fuzzy sets method of portfolio optimization

Main principles and idea of a method

The risk of a portfolio is not its volatility, but possibility that expected profitableness of a portfolio will appear below some preestablished planned value.

- Correlation of assets in a portfolio is not considered and not accounted.
- Profitableness of each asset is not random fuzzy number. Similarly, restriction on extremely low level of profitableness can be both usual scalar and fuzzy number of any kind. Thus, we reduce two sources of the information (average profitableness and volatility of asset) in one (a settlement corridor of profitableness or the price) and by that unite two sources of uncertainty into one.
- Therefore optimize a portfolio in such statement may mean, in that specific case, the requirement to maximize expected profitableness of a portfolio in a point of time T at the fixed risk level of a portfolio
- Profitableness of a security on termination of ownership term is expected to be equal r and is in a settlement range. For i -th security:

$\overline{r_i}$ – expected profitableness of i -ой security;

r_{i1} – the lower border of profitableness of i -th security;

r_{i2} – the upper border of profitableness of i -th security.

$r_i = \left(r_{1i}, \overline{r_i}, r_{2i}\right)$ – profitableness of i -th security, is triangular fuzzy number.

Then profitableness of a portfolio:

$$r = (r_{\min} = \sum_{i=1}^{N} x_i r_{1i}; \overline{r} = \sum_{i=1}^{N} x_i \overline{r_i}; r_{\max} = \sum_{i=1}^{N} x_i r_{2i}), \tag{9}$$

where x_i - weight of i -th asset in portfolio, and

$$\sum_{i=1}^{N} x_i = 1, \quad 0 \le x_i \le 1. \tag{10}$$

Critical level of profitableness of a portfolio at the moment of T may be fuzzy triangular type number $r^* = \left(r_1^*; \overline{r}^*; r_2^*\right)$.

Mathematical model of optimization of an investment portfolio by means of fuzzy sets

Let us consider a risk estimation of portfolio investments. On fig. 1 membership function r and criterion value r^* are shown

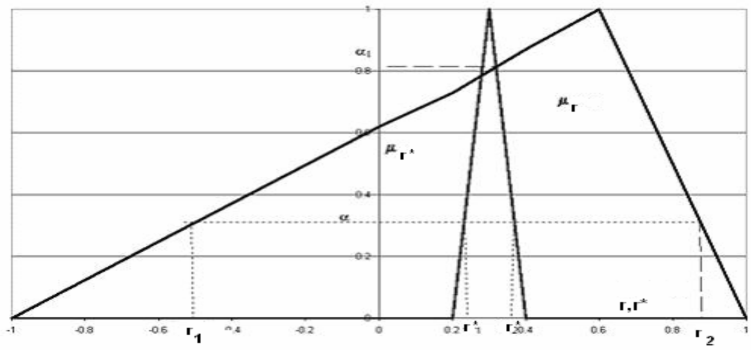


Fig. 1. Membership functions of *r* and *r**

Point with ordinate α_1 - the crossings point of two membership functions .Let us choose any level of membership α and define corresponding intervals $[r_1, r_2]$ and $[r_1^*, r_2^*]$. At $\alpha > \alpha_1$, $r_1 > r_2^*$, intervals are not crossed, the risk and inefficiencies level equals to zero. Level α_1 top border of risk zone. At $0 \le \alpha \le \alpha_1$ intervals are crossed.

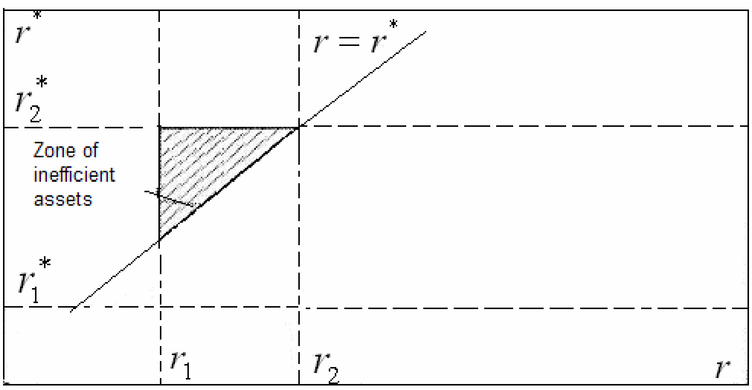


Fig. 2. Phase space (r, r^*)

$$S_\alpha = \begin{cases} 0, \ if \ \ r_1 \ge r_2^* \\ \dfrac{(r_2^* - r_1)^2}{2}, \ if \ \ r_2^* > r_1 \ge r_1^*; \ r_2 \ge r_2^* \\ \dfrac{(r_1^* - r_1) + (r_2^* - r_1)}{2} \cdot (r_2^* - r_1^*), \ \text{if } r_1 < r_1^*, \ r_2 > r_2^* \\ (r_2^* - r_1^*)(r_2 - r_1) - \dfrac{(r_2 - r_1^*)^2}{2}, \ if \ \ r_1 < r_1^* \le r_2; \ r_2 < r_2^* \\ (r_2^* - r_1^*)(r_2 - r_1), \ if \ \ r_2 \ge r_1^* \end{cases} \tag{11}$$

Where S_α are shaded areas of flat figure. Since all realizations (r, r^*) at set membership level $\varphi(\alpha)$ equally possible, so the degree of inefficiencies risk $\varphi(\alpha)$ geometrical probability of event of hit of a point in (r, r^*) to the zone of inefficient distribution of the capital []:

$$\varphi(\alpha) = \frac{S_\alpha}{(r_2^* - r_1^*) \cdot (r_2 - r_1)}, \tag{12}$$

total value of risk level of portfolio inefficiency:

$$\beta = \int_0^{\alpha_1} \varphi(\alpha) \partial\alpha, \tag{13}$$

when the criterion of efficiency is defined accurately level, r^* limiting transition at $r_2^* \to r_1^* \to r^*$ gives:

$$\varphi(\alpha)=\begin{cases}0, & if\ \ r^*<r_1\\ \dfrac{(r^*-r_1)}{(r_2-r_1)}, & if\ \ r_1\le r^*\le r_2;\ \alpha\in[0;1].\\ 1, & if\ \ r^*>r_2\end{cases} \quad (14)$$

For risk estimation are necessary:

1. two values of inverse function $\mu_r^{-1}(\alpha_1)$: r^*, $\tilde{r}^*$.
2) two values of inverse function $\mu_r^{-1}(0)$. $r_{\min}$, $r_{\max}$

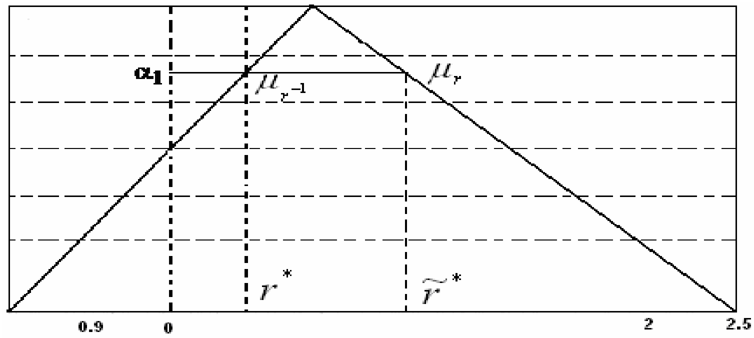


Fig. 3. An example of non-fuzzy efficiency criterion $\tilde{r}$

- The most expected value risk degree of a portfolio *β*:

$$\beta=\begin{cases}0, & npu\ \ r^*<r_{\min}\\ R\left(1+\dfrac{1-\alpha_1}{\alpha_1}\ln(1-a_1)\right), & if\ \ r_{\min}\le r^*\le\tilde{r}\\ 1-(1-R)\left(1+\dfrac{1-\alpha_1}{\alpha_1}\ln(1-a_1)\right), & if\ \ \tilde{r}\le r^*<r_{\max}\\ 1, & if\ \ r^*\ge r_{\max}\end{cases}. \quad (15)$$

where

$$R=\begin{cases}\dfrac{r^*-r_{\min}}{r_{\max}-r_{\min}}, & if\ \ r^*<r_{\max}\\ 1, & if\ \ r^*\ge r_{\max}\end{cases}. \quad (16)$$

$$\alpha=\begin{cases}0, & if\ \ r^*<r_{\min}\\ \dfrac{r^*-r_{\min}}{\tilde{r}-r_{\min}}, & if\ \ r_{\min}\le r^*<\tilde{r}\\ 1, & if\ \ r^*=\tilde{r}\\ \dfrac{r_{\max}-r^*}{r_{\max}-\tilde{r}}, & if\ \ \tilde{r}<r^*<r_{\max}\\ 0, & if\ \ r^*\ge r_{\max}\end{cases}. \quad (17)$$

Risk degree β accepts values from 0 to 1. Each investor, can define a piece of unacceptable values of risk, and also himself to execute the description of corresponding indistinct subsets, having set five membership functions $\mu^*(\beta)$.

Management model of a portfolio profitableness

To define structure of a portfolio which will provide the maximum profitableness at the set risk level, it is required to solve the following problem [1-4]:

$$\{x_{opt}\} = \{x\} \mid r \to \max, \quad \beta = const \tag{18}$$

where r и β are defined from (15)-(17), vector's components x satisfy (10).

It is easy to see that (17) can be defined as follows

$$\alpha_1 = \begin{cases} 0, & if \quad r^* < r_{\min} \\ \dfrac{r^* - r_{\min}}{\tilde{r} - r_{\min}}, & if \quad r_{\min} \le r^* < \tilde{r} \\ \dfrac{r_{\max} - r^*}{r_{\max} - \tilde{r}}, & if \quad \tilde{r} \le r^* < r_{\max} \\ 0, & if \quad r^* \ge r_{\max} \end{cases} . \tag{19}$$

Having recollected also, that profitableness of a portfolio is:

$$r = (r_{\min} = \sum_{i=1}^{N} x_i r_{1i};\ \tilde{r} = \sum_{i=1}^{N} x_i \tilde{r}_i;\ r_{\min} = \sum x_i r_{2i}),$$

where $\left(r_{1i}, \overline{r_i}, r_{2i}\right)$ – profitableness of i th security, we receive the following problem of optimisation (20)-(22):

$$\tilde{r} = \sum_{i=1}^{N} x_i \tilde{r}_i \to \max, \tag{20}$$

$$\beta = const, \tag{21}$$

$$\sum_{i=1}^{N} x_i = 1, \quad x_i \ge 0, \quad i = \overline{1, N} \tag{22}$$

At a risk level variation β 3 cases are possible. We will consider in detail each of them.

1. $\beta = 0$

From (15) it is visible, that this case is possible when $r^* < \sum_{i=1}^{N} x_i r_{i1}$.

We receive the following problem of linear programming (23)-(25):

$$\tilde{r} = \sum_{i=1}^{N} x_i \tilde{r}_i \to \max, \tag{23}$$

$$\sum_{i=1}^{N} x_i r_{i1} > r^*, \tag{24}$$

$$\sum_{i=1}^{N} x_i = 1 \quad x_i \ge 0 \quad i = \overline{1, N}. \tag{25}$$

Found result of the problem decision (23)-(25) vector $x = \{x_i\}$ $i = \overline{1, N}$ is a required structure of an optimum portfolio for the given risk level.

2. $\beta = 1$

From (15) follows, that this case is possible when $r^* \ge \sum_{i=1}^{N} x_i r_{i2}$.
We receive the following problem of linear programming:

$$\tilde{r} = \sum_{i=1}^{N} x_i \tilde{r}_i \to \max, \quad (26)$$

$$\sum_{i=1}^{N} x_i r_{i2} \le r^*, \quad (27)$$

$$\sum_{i=1}^{N} x_i = 1 \quad x_i \ge 0 \quad i = \overline{1, N}. \quad (28)$$

Found result of the problem decision (26)-(28) vector $x = \{x_i\}$ $i = \overline{1, N}$ is a required structure of an optimum portfolio for the given risk level.

3. $0 < \beta < 1$

From (15) it is visible, that this case is possible when $\sum_{i=1}^{N} x_i r_{i1} \le r^* \le \sum_{i=1}^{N} x_i \tilde{r}_i$, or when $\sum_{i=1}^{N} x_i \tilde{r}_i \le r^* \le \sum_{i=1}^{N} x_i r_{i2}$.

a) Let $\sum_{i=1}^{N} x_i r_{i1} \le r^* \le \sum_{i=1}^{N} x_i \tilde{r}_i$. Then using (15) - (17) problem (20) - (22) is reduced to the following problem of nonlinear programming:

$$\tilde{r} = \sum_{i=1}^{N} x_i \tilde{r}_i \to \max, \quad (29)$$

$$\frac{1}{\sum_{i=1}^{N} x_i r_{i2} - \sum_{i=1}^{N} x_i r_{i1}} \left(\left(r^* - \sum_{i=1}^{N} x_i r_{i1} \right) + \left(\sum_{i=1}^{N} x_i \tilde{r}_i - r^* \right) \cdot \ln \left(\frac{\sum_{i=1}^{N} x_i \tilde{r}_i - r^*}{\sum_{i=1}^{N} x_i \tilde{r}_i - \sum_{i=1}^{N} x_i r_{i1}} \right) \right) = \beta, \quad (30)$$

$$\sum_{i=1}^{N} x_i r_{i1} \le r^*, \quad (31)$$

$$\sum_{i=1}^{N} x_i \tilde{r}_i > r^*, \quad (32)$$

$$\sum_{i=1}^{N} x_i = 1 \quad x_i \ge 0 \quad i = \overline{1, N}. \quad (33)$$

b) Let $\sum_{i=1}^{N} x_i \tilde{r}_i \le r^* \le \sum_{i=1}^{N} x_i r_{i2}$. Then the problem (20) - (22) is reduced to the following problem of nonlinear programming:

$$\tilde{r} = \sum_{i=1}^{N} x_i \tilde{r}_i \to \max, \quad (34)$$

$$\frac{1}{\sum_{i=1}^{N} x_i r_{i2} - \sum_{i=1}^{N} x_i r_{i1}} \left(\left(r^* - \sum_{i=1}^{N} x_i r_{i1} \right) - \left(r^* - \sum_{i=1}^{N} x_i \tilde{r}_i \right) \cdot \ln \left(\frac{r^* - \sum_{i=1}^{N} x_i \tilde{r}_i}{\sum_{i=1}^{N} x_i r_{i2} - \sum_{i=1}^{N} x_i r_{i1}} \right) \right) = \beta, \quad (35)$$

$$\sum_{i=1}^{N} x_i r_{i2} > r^*, \quad (36)$$

$$\sum_{i=1}^{N} x_i \tilde{r}_i \le r^*, \quad (37)$$

$$\sum_{i=1}^{N} x_i = 1 \quad x_i \ge 0 \quad i = \overline{1, N}. \quad (38)$$

The R-algorithm of minimisation of not differentiated functions is applied to the decision of problems (29) - (33) and (34) - (38) [11]. Let both problems: (29) - (33) and (34) - (38) soluable. Then to the structure of a required optimum portfolio will correspond a vector – $x = \{x_i\}\ i = \overline{1, N}$ the decision of that problems (29) - (33), (34) - (38)) the criterion function value of which will be more.

The analysis and comparison of the results received by Markovitz models and fuzzy-sets model

Let"s consider the share portfolio consisting of 5 components.

The portfolio which provide the maximum profitableness at the risk level 0,05 set by the user, includes only two companies shares: MosEnergo (48,5 %) and Tatnft (51,5 %).

Having set various restriction levels on σ (portfolio risk), we receive effective border portfolio set – dependence of the maximum profitableness on risk kind of $r_{max} = r_{max}(\sigma)$.

Under the program we will construct effective border by points of the user, or by 10 automatically generated points.

For the comparative analysis of investigated methods of a share portfolio optimisation real data on share prices of the companies RAO» EES (EERS2) and Gazprom (GASP), were taken from February, 2000 till May, 2006 [10.

In Markovitz model expected profitableness of the share is calculated as a mean $m = M\{r\}$ and risk of an asset is considered as a dispersion of the expected profitableness value $\sigma^2 = M\left[(m - r)^2\right]$ i.e. level of variability of expected incomes.

In the fuzzy-sets model proceeding from a situation at the share market:

- shares profitability of EERS2 is in a settlement corridor [-1.0: 3.9], the most expected value of profitableness is 2,1 %
- shares profitability of GASP is in a settlement corridor [-4.1: 5.7], the most expected value of profitableness of 4,8 %

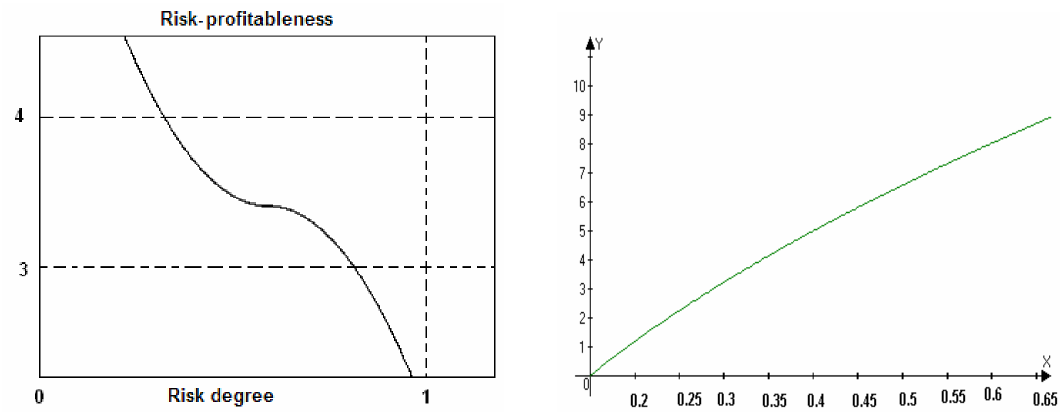
Let critical profitableness of a portfolio is 3,5 % i.e. portfolio investments which are bringing the income below 3.5 %, are considered as the inefficient.

Expected profitableness of the optimum portfolios received by means of Markovitz model, is higher, than profitableness of optimum portfolios, received by means of fuzzy-set model because in Markovitz model the calculation of expected share profitableness is based on indicators for the last periods and the situation in the share market at the moment of decision-making by the investor is not considered. As profitableness of shares EERS2 and GASP till July, 2006 was much more higher than at the present, Markovitz model gives unfairly high estimation.

In the fuzzy-set model profitableness of each asset is a fuzzy number. Its expected value is calculated not from statistical data, but by condition of the market at the moment of decision making by the investor. Thus, in the considered case, expected profitableness of a portfolio is not too high.

The structures of an optimum portfolio which we get as a result of use of both methods, for the same risk levels are quite different too. To find out the reason of this we consider following dependences (fig. 4) [9].

Dependence of expected profitableness from risk degree of the portfolio received



a) by fuzzy-set method. b) by Markovitz model

Fig. 4

Dependences of expected profitableness on degree of risk of the portfolio, received by the specified methods, are practically opposite. The reason of such result is the various understanding of a portfolio risk level.

In the fuzzy-set method the risk is recognised as a situation when expected profitableness of a portfolio drops below the set critical level, with decrease of expected profitableness increases risk of the income from portfolio investments will appear be less than critical value [9].

In Markovitz model the risk is considered as the degree of expected income variability of a portfolio, both in smaller, and in the big party that contradicts common sense. The various understanding of portfolio risk level is also the reason of distinctions of risk degree dependences on a share of this or that share in a portfolio, received by different methods.

In share EERS2, with the growth of low profitable securities in a portfolio, even in spite of the fact that the settlement corridor for EERS2 is narrower, rather than a settlement corridor for GASP, expected profitableness of a portfolio in general falls and the risk of an inefficiency portfolio selection grows.

Level of variability of expected incomes for shares EERS2 proceeding from data 2000-2006 is much lower, than for shares GASP. Therefore in Markovitz model which consider it as risk of portfolio investments, with the increase of ratio of share EERS2 the risk of a portfolio decreases.

From the point of view of the fuzzy-set approach, the more is the ratio of GASP shares in a portfolio, the less is the risk of that efficiency of share investments will appear below the critical level making in our case of 3.5 %.

From the point of view of Markovitz model, average mean deviation from average value for GASP shares is great enough, therefore with growth of their share the risk of a portfolio increases. It leads to that often share of highly profitable assets in the share portfolio received by means of Markovitz model is unfairly small.

According to Markovitz model, thanks to correlation between assets it is possible to receive a portfolio with a risk level less than volatility the least risk security.

In research we consider: having sink 96 % of the capital in EERS2 shares and 4 % in GASP shares, the investor receives portfolios with expected profitableness of 2.4 % and degree of risk 0.19. However investments with expected profitableness of 2.4 % in our fuzzy-set model are considered as the inefficient. If to set critical value of expected portfolio profitableness equal to 2.4% the risk of inefficient investments will decrease, too.

Conclusions

In this work the research in the field of portfolio management was carried out. Markovitz model, as one of most widely applied in the given area and rather recently arisen fuzzy-set approach to portfolio optimisation have been considered. As a result of research the mathematical model based on the fuzzy-set approach for a finding of structure of the optimum investment portfolio has been received, devoided of the majority of lacks of classical probabilistic models. On the basis of the theory of fuzzy sets the algorithm of optimisation of a share portfolio has been developed. Also the software in programming language C ++ has been developed.

In the course of research and the comparative analysis of Markovitz model and fuzzy-set model for finding the optimal share portfolio structure the following has been revealed:

1. Structures of an optimum portfolio and the indicators of its expected profitableness received by means of Markovitz model and fuzzy-set method principally differ.
2. With reduction of volume of initial data sample according to profitability of assets Markovitz model gives more reasonable results. However, sample that is too small should not be used because it cannot fully represent parameters under
3. Because deviation of expected profitability to the upper bound, as also to the lower bound, is considered in Markovitz model as a risk, dependencies of expected profitability on the risk level of portfolio computed using Markovitz model and fuzzy set method are completely opposite.
4. Due to the mentioned earlier reason, often the fraction of profitable assets in portfolio as computed by Markovitz model is unreasonably low.

Thus, lacks of Markovitz model have been visually proved. Especially is not justified the use of Markovitz model to the share markets of such countries as Russia and Ukraine where economy is very unstable.

Differences in profitableness of the optimum portfolios received using triangular, Gaussian and bell-shaped membership function, in the received results are small enough, especially between the models using Gaussian and bell-shaped membership function.

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