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HENSELIAN DISCRETE VALUED FIELDS ADMITTING ONE-DIMENSIONAL LOCAL CLASS FIELD THEORY

I. D. Chipchakov

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Abstract. This paper gives a characterization of Henselian discrete valued fields whose finite abelian extensions are uniquely determined by their norm groups and related essentially in the same way as in the classical local class field theory. It determines the structure of the Brauer groups and character groups of Henselian discrete valued strictly primary quasilocal (or PQL-) fields, and thereby, describes the forms of the local reciprocity law for such fields. It shows that, in contrast to the special cases of local fields or strictly PQL-fields algebraic over a given global field, the norm groups of finite separable extensions of the considered fields are not necessarily equal to norm groups of finite Galois extensions with Galois groups of easily accessible structure.

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Introduction. The purpose of this paper is to characterize the fields pointed out in the title, as well as to shed light on the structure of their Brauer groups, and on some properties of their norm groups, along the lines drawn in [10] and [11]. The obtained results on this topic bear an accomplished character in the special case where the considered fields are strictly primarily quasilocal. In particular, they fully describe the forms of the local reciprocity law, possible in this case. They also clarify the specific nature both of the classical norm limitation theorem (cf. [20, Ch. 8, Theorem 6]), and of the description in [12] of the norm groups of finite extensions of strictly primarily fields, algebraic over a given global field E_0. This can serve as a basis for a substantial progress in the study of arbitrary fields admitting one-dimensional local class field theory, started in [9] and [10]. The research in this area could stimulate the efforts to answer the question of whether central division algebras of prime exponent p over a field E are similar to tensor products of cyclic division E-algebras of index p (see also [27, Sect. 4, Theorem 2]).

Let us note that a field E is called strictly primarily quasilocal, if it satisfies the following conditions with respect to every prime p, for which E does not equal its maximal p-extension $E(p)$ (in a fixed separable closure E_{sep}): the p-component $\mathrm{Br}(E)_p$ of $\mathrm{Br}(E)$ is nontrivial, and the relative Brauer group $\mathrm{Br}(L/E)$ of each cyclic extension L of E of degree p equals the maximal subgroup of $\mathrm{Br}(E)$ of exponent p. We say that E admits (one-dimensional) local class field theory, if the canonical mapping π of the set $\mathrm{Ab}(E)$ of finite abelian extensions of E in E_{sep} into the set $\mathrm{Nor}(E)$ of norm subgroups of the multiplicative group E^*, maps injectively field compositums into group intersections, and field intersections into inner group products, i.e. if $N(M_1.M_2) = N(M_1/E) \cap N(M_2/E)$ and $N((M_1 \cap M_2)/E) = N(M_1/E) \cdot N(M_2/E)$, $\forall M_1, M_2 \in \mathrm{Ab}(E)$. This occurs, if and only if E admits (one-dimensional) local p-class field theory (i.e. π acts in the same way on the set of finite abelian p-extensions of E in $E(p)$), for each prime p (see Lemma 1.7). It has been proved in [11] that strictly PQL-fields admit such a theory, and also, that they are subject to an exact analogue to the local reciprocity law, and a partial analogue to Hasse's symbol for local fields. It is not known whether or not fields with local class field theory are necessarily strictly PQL. The answer to this question would be affirmative, if the well-known open problem of whether central division algebras of prime exponent p are similar to tensor products of cyclic algebras of index p, has a positive solution.

Here is an overview of the present paper: For convenience of the reader, we state in Section 1 the main results of [10] and [11], used in the sequel. Sections 2 and 3 contain characterizations of some of the basic types of Henselian discrete valued fields admitting such a theory. In Section 4 we show that an abelian torsion group T is realizable as a Brauer group of a Henselian discrete valued strictly PQL-field if and only if T is divisible with a 2-component isomorphic to the quasicyclic 2-group $\mathbb{Z}(2^\infty)$. This is obtained by classifying, up-to an isomorphism, abstract abelian groups realizable as Brauer groups of fields of the considered type, whose groups of roots of unity are isomorphic to a given subgroup R of the quotient group $\mathbb{Q}/\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Z}$ being, as usual, the additive groups of rational numbers and of integer numbers, respectively. The obtained result enables one to describe the forms of the local reciprocity law (see Proposition 1.3), possible in the studied situation. It also sheds light on the structure of the character groups of Henselian discrete valued strictly PQL-fields. Section 5 shows when a Henselian discrete valued field (K, v) admitting such a theory, has the property that its finite separable extensions are subject to the norm limitation theorem. It gives a necessary condition that every finite separable extension R/K satisfies the equality $N(R/K) = N(\Phi(R)/K)$, for a suitably chosen finite abelian extension $\Phi(R)$ of K, and proves its sufficiency in the special case where the residue field of (K, v) is of zero characteristic. Generally, however, we show there that $N(R/K)$ does not necessarily equal the norm group of a finite Galois extension of K with a Galois group of sufficiently simple structure (see Theorems 5.8 and 5.9).

Throughout the paper, algebras are assumed to be associative with a unit, simple algebras are supposed to be finite-dimensional over their centres, Brauer groups of fields are viewed as additively presented, and Galois groups are regarded as profinite with respect to the Krull topology. For each algebra A, we consider only subalgebras of A containing its unit; we write A^* for the multiplicative group of A. For each abelian torsion group T, $T(P)$ denotes the set of those prime numbers p, for which the p-component T_p of T is nontrivial. We denote by $\overline{P}$ the set of prime numbers, and put $P(E) = \{p \in \overline{P} : E(p) \neq E\}$, for each field E. Also, we associate with E the presentation of $\overline{P}$ as a disjoint union $P_0(E) \cup P_1(E) \cup P_2(E)$, where $P_0(E)$ is the set of those prime numbers p, for which E contains a primitive p-th root of unity, or else, $p = \mathrm{char}(E)$, and $P_1(E) = \{p \in \overline{P} \setminus P_0(E) : E^* \neq E^{*p}\}$. As usual, a nontrivial valuation v of a field K into a linearly ordered abelian group, written additively, is called Henselian, if v is uniquely extendable to a valuation v_L on each algebraic extension L of K; when this occurs, we denote by $\widehat{L}$ the residue field, and by $v(L)$ the value group of (L, v_L). Our basic terminology concerning valuation theory, simple algebras, Brauer groups and abstract abelian groups is standard (such as can be found, for example, in [21, 33, 29, 27, 18]), as well as the one concerning profinite groups, Galois cohomology, field extensions and Galois theory (see [30, 23, 25]).

1. Preliminaries. This section contains a brief account of the main result of one-dimensional abstract local class field theory, established in [10] and [11]; a list of books containing an introduction to the classical theory can be found in the overview of Koch's survey on the subject in [24, Ch. 2]. First we present characterizations of the basic types of fields admitting such a theory.

Proposition 1.1. *Strictly PQL-fields admit local class field theory.*

Conversely, a field E admitting local class field theory is strictly PQL, provided that $\mathrm{Br}(E) \neq \{0\}$ *and every central division E-algebra of prime exponent p is similar to a tensor product of cyclic E-algebras of index p.*

Corollary 1.2. *In order that finite extensions of a field E admit local class field theory it is necessary and sufficient that these extensions are strictly PQL-fields.*

Our next result generalizes the classical local reciprocity law to the case of strictly PQL-fields, and so emphasizes the significance of their Brauer groups for one-dimensional local class field theory.

Proposition 1.3. *Let E be a strictly PQL-field, $P(E)$ the set of prime numbers p for which $E(p) \neq E$, and for each $p \in P(E)$, let ${}_p\mathrm{Br}(E)$ be the maximal subgroup of* $\mathrm{Br}(E)$ *of exponent p, I_p a basis and $d(p)$ the dimension of ${}_p\mathrm{Br}(E)$ as a vector space over the field with p elements, and $G(M/E)^{\mathrm{Br}(E)}$ the direct sum $\oplus_{p\in P(E)} G(M/E)_p^{d(p)}$, where $G(M/E)_p$ is the Sylow p-subgroup of $G(M/E)$, and $G(M/E)_p^{d(p)}$ is a direct sum of isomorphic copies of $G(M/E)_p$ indexed by I_p. Then the quotient group $E^*/N(M/E)$ is isomorphic to $G(M/E)^{\mathrm{Br}(E)}$.*

The following three statements shed light on the scope of the classical norm limitation theorem. Let us recall that this theorem states that the norm group of each finite extension R of K equals the norm group of the maximal abelian subextension of K in R.

Proposition 1.4. *Let E be a field, M/E a finite Galois extension, R an intermediate field of M/E, and R_{ab} the maximal abelian extension of E in R. Then $N(R/E) = N(R_{\mathrm{ab}}/E)$ in each of the following special cases:*

(i) *E is strictly primarily quasilocal and the Galois group $G(M/E)$ is nilpotent;*

(ii) *E is a quasilocal field, such that the natural homomorphism of* $\mathrm{Br}(E)$ *into* $\mathrm{Br}(F)$ *is surjective, for every finite extension of E.*

Note that condition (ii) of Proposition 1.4 is in force, in case E is a local field. This can be deduced from the fact that then $\mathrm{Br}(E)$ is isomorphic to the quotient group $\mathbb{Q}/\mathbb{Z}$ of the additive group $\mathbb{Q}$ of rational numbers by the subgroup $\mathbb{Z}$ of integer numbers. For a similar reason, Proposition 1.4 (ii) also applies to the more general case of a field with local class field theory in the sense of [28] (see also [3, Ch. 15]).

Proposition 1.5. *For each nonnilpotent finite group G, there exists a strictly PQL-field $E(G)$ and a Galois extension $M(G)$ of $E(G)$, for which the following is true:*

(i) *$E(G)$ is an algebraic extension of the field $\mathbb{Q}$ of rational numbers;*

(ii) *The Galois group $G(M(G)/E(G))$ is isomorphic to G, and $N(M(G)/E(G))$ is a proper subgroup of $N(M(G)_{\mathrm{ab}}/E(G))$, where $M(G)_{\mathrm{ab}}$ is the maximal abelian extension of $E(G)$ in $M(G)$.*

It has been proved in [12] that if E is a strictly PQL-extension of a global field E_0, and R/E is a finite extension, then $N(R/E) = N(\Phi(R)/E)$, for some finite abelian extension $\Phi(R)$ of E. Our next result shows, however, that strictly PQL-fields do not always preserve this property.

Proposition 1.6. *There exists a field E, for which the following assertions hold true:*

(i) *all finite extensions of E are strictly PQL-fields;*

(ii) *the absolute Galois group $G(K_{sep}/K) := G_K$ is not pronilpotent;*

(iii) *every finite extension R of K is subject to the following alternative:*

(α) *R is an intermediate field of a finite Galois extension $M(R)/K$ with a nilpotent Galois group;*

(β) *$N(R/K)$ does not equal the norm group of any abelian finite extension of K.*

We conclude these preliminaries with the statements of two basic lemmas (proved in [10]) that will often be used without an explicit reference.

Lemma 1.7. *Let E be a field, R a finite abelian extension of E, P the set of prime numbers dividing $[R : E]$, and R_p the maximal subextension of E in R of p-primary degreeq for each $p \in P$. Then R equals the compositum of the fields R_p: $p \in P$, $N(R/E) = \cap_{p\in P} N(R_p/E)$ and $E^*/N(R/E)$ is isomorphic to the direct product $\prod_{p\in P}(E^*/N(R_p/E))$.*

It is clear from Lemma 1.1 that a field E admits one-dimensional local class field theory if and only if it is a field with local p-class field theory, for every prime number p. As to the following one, it shows that the group $\mathrm{Br}(E)_p$ is necessarily nontrivial, if E admits local p-class field theory with respect to a prime number p for with $E(p) \neq E$.

Lemma 1.8. *Let E be a field, such that $\mathrm{Br}(E)_p = \{0\}$, for some prime number p. Then $\mathrm{Br}(U)_p = \{0\}$ and $N(U/E) = E^*$, for every finite extension U of E in $E(p)$.*

2. Characterization of Henselian discrete valued fields admitting one-dimensional local class field theory.

The main purpose of this Section is to prove the following theorem.

Theorem 2.1. *A Henselian discrete valued field (K, v) admits local class field theory if and only if $\widehat{K}$ is a nonreal perfect field, $P(K) = P(\widehat{K})$, and $\widehat{K}(p)/\widehat{K}$ is a $\mathbb{Z}_p$-extension, for each $p \in P(\widehat{K})$. When this occurs, $\mathrm{Br}(\widehat{K})_p = \{0\}$, for any $p \in P_0(K)$.*

P r o o f. It is well-known that $\mathrm{Br}(\widehat{K})_p = \{0\}$, whenever $\widehat{K}$ is a perfect field of characteristic p (cf. [2, Ch. VII, Theorem 22]), and also, in the special case where $\widehat{K}$ contains a primitive p-th root of unity and $G(\widehat{K}(p)/\widehat{K})$ is a free pro-p-group (cf. [36, Theorem 3.1] and [37, p. 725]). In view of Scharlau's generalization of Witt's theorem (cf. [34, (3.10)] and the references there), these observations, reduce our latter assertion to a consequence of the former one. The former statement of Theorem 2.1 is proved in several steps described by the following three lemmas. □

Lemma 2.2. *Let (K, v) be a Henselian valued field, such that $v(K) \neq pv(K)$, for some prime number p. Suppose also that K admits local p-class field theory. Then the following assertions are true:*

(i) *If $\widehat{K}(p) \neq \widehat{K}$, then $\widehat{K}(p)/\widehat{K}$ is a $\mathbb{Z}_p$-extension;*

(ii) *$\widehat{K}$ is perfect, provided that $\mathrm{char}(\widehat{K}) = p$ and there are distinct totally ramified extensions K_1 and K_2 of K in $K(p)$ of degree p.*

P r o o f. Suppose first that $\widetilde{K}(p) \neq \widehat{K}$, and fix extensions $\widetilde{L}_1$ and $\widetilde{L}_2$ of $\widehat{K}$ in $\widehat{K}(p)$ of degree p. Then the inertial lifts L_1 and L_2 in K_{sep} over K of $\widetilde{L}_1$ and $\widetilde{L}_2$, respectively, are cyclic extensions of K of degree p. Since $v(K) \neq pv(K)$ and $v(\alpha) \in pv(K)$, for each $\alpha N(L_1/K) \cdot N(L_2/K)$, the assumption that K admits local p-class field theory ensures that $L_1 = L_2$, which implies that $\widetilde{L}_1 = \widetilde{L}_2$ (cf. [21, p. 135]). Hence, by [9, Lemmas 3.3 and 3.6], $\widehat{K}(p)/\widehat{K}$ is a $\mathbb{Z}_p$-extension, unless $p = 2$

and $\widehat{K}$ is formally real. The latter case, however, is impossible, since then $\widehat{K}^{*2}$ contains the residue class $\hat{\beta}$, for each $\beta \in N(K(\sqrt{-1}/K)\cdot N(K(\sqrt{-\pi})/K)$ of value zero, where π is an element of K^*, such that $v(\pi) \notin 2v(K)$. The obtained results prove Lemma 2.2 (i), so we assume further that $p = \mathrm{char}(\widehat{K})$, and K_1, K_2 are distinct totally ramified extensions of K in $K(p)$ of degree p. The subnormality of proper subgroups of finite p-groups (cf. [25, Ch. I, Sect. 6]), together with Galois theory, ensures that K_1 and K_2 are cyclic over K, and because of the hypothesis on K, this means that $N(K_1/K) \cdot N(K_2/K) = K^*$. Hence, the assumption that $\widehat{K}$ is not perfect implies the existence of elements $\pi_1 \in K_1^*$ and $\pi_2 \in K_2^*$, such that $v'(\pi_1) = v'(\pi_2)$ and $v'(\pi_1) \notin v(K)$, where v' is the valuation of $K(p)$ extending v. Applying Ostrowski's theorem, one also obtains that $v(K_1) = v(K_2)$, and each element $\lambda_j \in K_j^*$ $(j = 1, 2)$ is presentable as a product $\lambda_j = \pi_j^{m(j)}\mu_j\rho_j$, for a suitably chosen nonnegative integer $m(j) < p$, and some elements $\mu_j \in K$ and $\rho_j \in K_j$ taken so that $v'(\rho_j) = 0$. These observations lead to the conclusion that $\widehat{K}^* = \cup_{i=0}^{p-1}\hat{a}^i\widehat{K}^{*p}$, where $a = N_K^{K_1}(\pi_1) \cdot N_K^{K_2}(\pi_2^{-1})$. At the same time, it is easily verified that if $\hat{a} \notin \widehat{K}^{*p}$, then $\hat{a}+1 \notin \cup_{i=0}^{p-1}\widehat{K}^{*p}$. The obtained contradiction proves Lemma 2.2 (ii). □

Lemma 2.3. *Let (K, v) be a Henselian discrete valued field with a residue field $\widehat{K}$ of characteristic $p \neq 0$. Then the following statements are true:*

(i) *If $\widehat{K}$ is infinite, then there exist infinitely many totally ramified extensions of K in $K(p)$ of degree p;*

(ii) *If K admits local p-class field theory, then $\widehat{K}(p)/\widehat{K}$ is a $\mathbb{Z}_p$-extension.*

P r o o f. It is well-known that if $\widehat{K}$ is finite, then $\widehat{K}(r)/\widehat{K}$ is a $\mathbb{Z}_r$-extension, for each prime r, so we assume further that $\widehat{K}$ is infinite. Suppose first that $\mathrm{char}(K) = p$, fix an element π of K^* so that $v(\pi) > 0$ and $v(\pi) \notin pv(K)$, and denote by V_π the vector subspace of K generated by the set $\Sigma_\pi = \{\pi^{-1-pn} : n \in \mathbb{N}\}$ over the prime subfield $\mathbb{F}_p$ of K. It is easily verified that Σ_π is a basis of V_π, and $v(\theta) \notin pv(K)$, for all $\theta \in (V_\pi \setminus \{0\})$. This implies that the polynomial $X^p - X - \theta$ has no root in K, and by the Artin-Schreier theorem (cf. [25, Ch. VIII, Sect. 6]), the set of subgroups of V_π of order p embeds in the set of extensions of K in $K(p)$ of degree p. The obtained result proves Lemma 2.3 (i) in the special case where $\mathrm{char}(K) = p$. Assume now that $\mathrm{char}(K) = 0$, ε is a primitive p-th root of unity in $\overline{K}$, $[K(\varepsilon) : K] = m$, R is the valuation ring of (K, v), μ is a generator of the maximal ideal M of R, and $\Sigma(B) = \{\sigma \in R : \hat{\sigma} \in B\}$ is a system of representatives of some basis B of $\widehat{K}$ as a vector space over the prime subfield of $\widehat{K}$. Observing that the polynomial $g_a(X) = (X+1)^p - (1+a\mu)$ is Eisensteinian with respect to μ, for each $a \in R^*$, one obtains from [5, Ch. I, Theorem 6.1] that the extension $K(\xi_a)/K$, where ξ_a is a p-th root of $1+a\mu$ in $\overline{K}$, is totally ramified of degree p. Regarding as we can K^*/K^{*p} as a vector space over the prime field of characteristic p, one also sees that the co-sets $\{(1+\sigma\mu)K^{*p} : \sigma \in \Sigma(B)\}$, are linearly independent in K^*/K^{*p}. In view of Kummer's theory, this proves Lemma 2.3 (i) in the special case where $\varepsilon \in K$. Henceforth, we assume that $\varepsilon \notin K$, denote by R' the valuation ring of the prolongation v' of v on $K(\varepsilon)$, and put $\delta = \mu$ or 1, depending on whether or not $v(p) \in pv(K)$. It is not difficult to see that $(p-1)v'(\varepsilon^i - 1) = v(p)$ and $v'(\varepsilon^{s^i} - 1 - s^i(\varepsilon - 1)) \geq v'((\varepsilon-1)^2)$, for each $i \in \{0, 1, \dots, p-1\}$. Note also that the extension L_α of $K(\varepsilon)$, obtained by adjoining the p-th roots in $\overline{K}$ of the element $1 + \alpha(\varepsilon - 1)\delta^{-1} := 1 + \tilde{\alpha}$, is totally ramified of degree p, for every $\alpha \in R'^*$. Suppose for a moment that $(1+\tilde{\alpha}) \in K(\varepsilon)^{*p}$. This means that $1 + \tilde{\alpha} = (1+\mu_1)^p$, for some element μ_1 of the maximal ideal M' of R', so it follows from Newton's binomial formula that $\tilde{\alpha} = \mu_1^p + p\mu_1\mu_1'$, with $\mu_1' \in R'$. The obtained result contradicts the fact that $v'(\tilde{\alpha}) \notin pv'(K(\varepsilon)$ and $v'(\tilde{\alpha}) \leq v'(\varepsilon - 1) < v(p)$, and thereby proves that $1+\tilde{\alpha} \notin K(\varepsilon)^{*p}$. Hence, the polynomial $g_\alpha(X) = (X+1)^p - (1+\tilde{\alpha})$ is irreducible over $K(\varepsilon)$. Taking now into consideration that the coefficients of $g_\alpha(X)$, except the leading one, lie in M', and the free term of $g_\alpha(X)$ equals $-\tilde{\alpha}$, one concludes that $L_\alpha/K(\varepsilon)$ is a totally ramified cyclic extension of degree p, as claimed. Our argument also shows that the additive group of the residue field of $(K(\varepsilon), v')$ is isomorhic to the quotient group $U'_\varepsilon/U_\varepsilon$, where $U'_\varepsilon = \{u' \in R' : v'(u'-1) \geq v'((\varepsilon-1)\delta^{-1})\}$ and $U_\varepsilon = \{u \in R' : v'(u-1) > v'((\varepsilon-1)\delta^{-1})\}$. Note finally that $K(\varepsilon)/K$ is a cyclic extension (cf. [25, Ch. VIII, Sect. 6]), and fix a K-automorphism φ of $K(\varepsilon)$ of order m, as well as integers s and l so that $\varphi(\varepsilon) = \varepsilon^s$ and $s \cdot l \equiv 1 (\mathrm{mod}\ p)$. Put $A(\varepsilon) = \{a \in K(\varepsilon)^* : \varphi(a)a^{-s} \in K(\varepsilon)^{*p}\}$, and $\eta(\rho) = \prod_{j=0}^{m-1}(1 + (\varphi^j(\varepsilon) - 1).\rho\delta^{-1})^{l^j}$, for every $\rho \in R'$. It is verified by direct calculations that $\eta(\rho) \in A(\varepsilon)$ and $\eta(\rho) - m\rho(\varepsilon-1)\delta^{-1} \in U_\varepsilon$, in case $\rho \in R^*$. Hence, by the already established properties of the elements of $U'_\varepsilon \setminus U_\varepsilon$, $\eta(\rho) \notin K(\varepsilon)^{*p}$, and by Albert's theorem (cf. [1, Ch. IX, Theorem 15]), one can then associate with the subgroup of $K(\varepsilon)^*/K(\varepsilon)^{*p}$, generated by the co-set of $\eta(\rho)$, a uniquely determined cyclic extension L_ρ of K of degree p, which turns out to be totally ramified over K. One also obtains from the inclusion $(K(\varepsilon)^{*p} \cap U'_\varepsilon) \subseteq U_\varepsilon$ that the co-sets $\sigma K(\varepsilon)^p : \sigma \in \Sigma(V)$, are linearly independent in $K(\varepsilon)/K(\varepsilon)^p$, regarded as a vector space over the field with p elements, which completes the proof of Lemma 2.3 (i).

Suppose now that K admits local p-class field theory. Then Lemma 2.2 (ii) and the former part of the present lemma ensure that $\widehat{K}$ is a perfect field, and since $K(p) \neq K$, one also concludes that $\mathrm{Br}(K)_p \neq \{0\}$. This, combined with the fact that $\mathrm{Br}(K)$ is isomorphic to the direct sum $\mathrm{Br}(\widehat{K}) \oplus \chi(G_{\widehat{K}})$, leads to the conclusion that the p-component of $C(G_{\widehat{K}})$, or equivalently, the character group of $G(\widehat{K}(p)/\widehat{K})$, is nontrivial, which reduces the latter assertion of Lemma 2.3 to a consequence of Lemma 2.2 (i). □

Lemma 2.4. *Let (K, v) be a Henselian valued field, such that $v(K) \neq pv(K)$, for some prime number p not equal to* $\mathrm{char}(\widehat{K})$. *Then the following statements are true:*

(i) *In order that every finite extension of K in $K(p)$ is inertial, it is necessary and sufficient that $\widehat{K}$ does not contain a primitive p-th root of unity; when this occurs, the Galois groups $G(K(p)/K)$ and $G(\widehat{K}(p)/\widehat{K})$ are canonically isomorphic;*

(ii) *K admits local p-class field theory, provided that $\widehat{K}(p)/\widehat{K}$ is a $\mathbb{Z}_p$-extension and some of the following two conditions is in force;*

(α) *$\widehat{K}$ does not contain a primitive p-th root of unity;*

(β) *The quotient group $v(K)/pv(K)$ is of order p.*

P r o o f. Statement (i) is contained in [8, Lemma 1.1]. It shows that if $\widehat{K}(p)/\widehat{K}$ is a $\mathbb{Z}_p$-extension and $\widehat{K}$ does not contain a primitive p-th root of unity, then K possesses a unique extension K_n in $K(p)$ of degree p^n, for any $\in \mathbb{N}$. Observing also that K_n is inertial over K, one obtains that the co-set $\pi N(K_n/K)$ is of order p^n in $K^*/N(K_n/K)$, whenever $\pi \in K^*$ and $v(\pi) \notin pv(K)$. The obtained result indicates that $K^*/N(K_n/K)$ are groups of exponent p^n, for all $n \in \mathbb{N}$, which completes the proof of Lemma 2.4 (ii) (α). Suppose now that $\widehat{K}$ contains a primitive p-th root of unity and $v(K)/pv(K)$ is of order p. Then it follows from the Henselian property of v, [8, Lemma 1.1] and Kummer's theory that $G(K(p)/K)$ is a pro-p-group of rank 2 and cohomological dimension 2. Hence, by Galois cohomology (cf. [36, Theorem 3.1] and [37, Lemma 7]), $G(K(p)/K)$ is a Demushkin group, and by [10, Proposition 5.1], this means that K admits local p-class field theory. Lemma 2.4 is proved. □

Corollary 2.5. *Let (K, v) be a Henselian discrete valued field with a residue field $\widehat{K}$. Then K is a strictly PQL-field if and only if it admits local class field theory and $\widehat{K}$ is p-quasilocal with respect to each prime $p \neq$* $\mathrm{char}(\widehat{K})$, *for which $\widehat{K}$ does not contain a primitive p-th root of unity.*

P r o o f. In view of Proposition 1.1, Theorem 2.1 and [7, Lemma 2.2], one may assume without loss of generality that $\widehat{K}(p)/\widehat{K}$ is a $\mathbb{Z}_p$-extension, for any

$p \in (P(\widehat{K}) \setminus P_0(\widehat{K}))$. Let L be the unique extension of K in $K(p)$ of degree p, ψ a K-automorphism of L of order p, π a generator of the maximal ideal of the valuation ring of (K, v), and D a central division K-algebra of exponent p. It follows from the generalized Witt theorem and Lemma 2.4 (i) that D is similar to the tensor product $S \otimes_K V$, where S is a central inertial division K-algebra of index p, uniquely determined by D, up-to a K-isomorphism, and V is the cyclic K-algebra $(L/K, \psi, \pi^j)$, for some nonnegative integer $j \leq p-1$. This indicates that L embeds in D as a K-subalgebra if and only if $\widehat{L}$ embeds in $\widehat{S}$ over $\widehat{K}$ (cf. [21, Sects. 2 and 3]). Since each central division $\widehat{K}$-algebra has an inertial lift over K (cf. [21, Sect. 2]), the obtained result proves our assertion. □

Note finally that the answer to the question of whether fields with local class field theory are strictly primarily quasilocal will be negative, if there exists a perfect field $\widetilde{K}$ with the following properties: (i) $\widehat{K}(p)/\widehat{K}$ is a $\mathbb{Z}_p$-extension whenever p is a prime number for which $\widehat{K}(p) \neq \widehat{K}$; (ii) $\widehat{K}(p) \neq \widehat{K}$, provided that $\widehat{K}$ contains a primitive p-th root of unity, or else, $p = \mathrm{char}(\widehat{K})$; (iii) $\widetilde{K}(\tilde{p}) \neq \widetilde{K}$ and there exists a central noncyclic division $\widehat{K}$-algebra of index $\tilde{p}$, for some prime $\tilde{p}$. Indeed, then one could take as a counter-example any Henselian discrete valued field with a residue field isomorphic to $\widetilde{K}$.

3. Henselian discrete valued fields whose finite extensions are strictly primarily quasilocal. In this section we characterize the fields pointed out in its title, and show which profinite groups are isomorphic to absolute Galois groups of such fields.

Proposition 3.1. *For a Henselian discrete valued field (K, v), the following conditions are equivalent:*

(i) *Every finite extension of K is a strictly PQL-field;*

(ii) *$\widehat{K}$ is perfect, the absolute Galois group $G_{\widehat{K}}$ is metabelian of cohomological p-dimension 1, for every $p \in \overline{P}$, and $P_0(\widetilde{L}) \subseteq P(\widetilde{L})$, for every finite extension $\widetilde{L}$ of $\widehat{K}$.*

P r o o f. This follows at once from Theorem 2.1 and the fact (cf. [7, Lemma 1.2]) that a profinite group G is metabelian of cohomological dimension ≤ 1 if and only if the Sylow pro-p-subgroups of G are isomorphic to $\mathbb{Z}_p$, whenever p is a prime number, for which $\mathrm{cd}_p(G) \neq 0$. □

In the rest of this section, we denote by G' the closure of the commutator subgroup of an arbitrary profinite group G, and by $r(E)$ the group of roots of unity in any field E.

Theorem 3.2. *Let G be a metabelian profinite group of cohomological p-dimension one, for each prime number p. Then there exists a Henselian discrete valued field (K, v), such that the absolute Galois group $G_{\widehat{K}}$ is isomorphic to G, and every finite extension of K is a strictly PQL-field. Moreover, if R is a subgroup of $\mathbb{Q}/\mathbb{Z}$, such that $2 \in P(R)$ and $\mathrm{cd}_p(G/G') = 1$, for each $p \in P(R)$, then K can be chosen so that $r(K) \cong R$.*

P r o o f. Let G be a nontrivial profinite group and $P(G)$ the set of those prime numbers p, for which the Sylow pro-p-subgroups of G are nontrivial. By [7, Lemma 1.2], G is metabelian of cohomological dimension $\mathrm{cd}(G) \leq 1$ if and only if its Sylow pro-p-subgroups are isomorphic to $\mathbb{Z}_p$, for each $p \in P(G)$; when this occurs, G is abelian if and only if $G \cong \prod_{p \in P(G)} \mathbb{Z}_p$. Note also that G is realizable as an absolute Galois group of a perfect field k_G, whenever $\mathrm{cd}(G) \leq 1$ (cf. [7, Remark 2.6]). Since perfect fields are realizable as residue fields of Henselian discrete valued fields (cf. [13, (5.7) – (5.10)]), these observations, combined with Proposition 3.1, prove the theorem in the special case where $G \cong \prod_{p \in \overline{P}} \mathbb{Z}_p$. In the rest of the proof, we assume that G is metabelian and nonabelian, $\mathrm{cd}(G) \leq 1$, and G_p is a Sylow pro-p-subgroup of G, for all $p \in P$. It is not difficult to see that then G has the following properties:

(3.1) (i) $P(G')$ does not contain the least element of $P(G)$; in particular, if $P(G) = \overline{P}$, then $2 \notin P(G')$;

(ii) For each $p \in P(G')$, G_p is a closed normal subgroup of G, as well as the centralizer $C(G_p) = \{z_p \in G : z_p g_p = g_p z_p$, for any $g_p \in G_p\}$; furthermore, the quotient group $G/C(G_p)$ is cyclic of order θ_p dividing $p-1$, and greater than 1;

(iii) $P(G') \cap P(G/G') = \phi$, and $P(G/G')$ contains a divisor q_p of $p-1$, $\forall p \in P(G')$.

We first show that if F/E is a Galois extension with $G(F/E) \cong G/G'$, then there exists an algebraically closed extension Σ of F and an intermediate field Σ_0 of Σ/E, such that E is algebraically closed in Σ_0, and one can find a Galois extension Σ_1 of Σ_0 in Σ so that $G(\Sigma_1/\Sigma_0) \cong G$, and the extension of Σ_0 in Σ_1 corresponding by Galois theory to the closure of the commutator subgroup of $G(\Sigma_1/\Sigma_0)$ is equal to the compositum $F\Sigma_0$. This is obtained as a consequence of the following two lemmas. □

Lemma 3.3. *Let E be a field, F a Galois extension of E in E_{sep} with $G(F/E) \cong \prod_{p \in P} \mathbb{Z}_p$, for some nonempty set P of prime numbers, and $\{F_i : i \in I\}$ a set of Galois extensions of E satisfying the following conditions:*

(i) $G(F_i/E)$ *is continuously isomorphic to the quotient group of* $G(F/E)$ *by some closed subgroup* H_i*, for each index* i*;*

(ii) E *is equal to the intersections* $F \cap \Omega$ *and* $F_i \cap \Omega_i$*,* $i \in I$*, where* Ω *is the compositum of the fields* F_j*,* $j \in I$*, and* Ω_i *is the compositum of* F *and the fields* F_j*,* $j \in (I \setminus \{j\})$*, for each* $i \in I$*.*

Assume also that φ *and* φ_i*,* $i \in I$*, are topological generators of* $G(F/E)$ *and* $G(F_i/E)$*,* $i \in I$*, respectively. Then the compositum* $F\Omega$ *possesses a subfield* W *with the following properties:*

(α) $F \cap W = E$ *and* $F\Omega/W$ *is an abelian extension with* $G(F\Omega/W)$ *canonically isomorphic to* $G(F/E)$*;*

(β) $F_i \cap W = E$*,* $i \in I$*, and for each* $i \in I$*, the unique* W*-automorphism of* F_iW *extending* φ_i *is induced by the unique* W*-automorphism of* $F\Omega$ *extending* φ*.*

P r o o f. For each $i \in I$, let L_i be the intermediate field of F/E corresponding by Galois theory to H_i, ψ_i the automorphism of L_i induced by φ, $\overline{\psi}_i$ the unique F_i-automorphism of L_iF_i extending ψ_i, $\overline{\varphi}_i$ the unique L_i-automorphism of L_iF_i extending φ_i, and W_i the intermediate field of L_iF_i, fixed by $\overline{\psi}_i\overline{\varphi}_i^{-1}$. Then one can take as W the compositum of the fields W_i: $i \in I$. □

Lemma 3.4. *Let* E *be a field,* P *a nonempty set of prime numbers,* F *a Galois extension of* E *in* $E_{\rm sep}$ *with* $G(F/E)$ *isomorphic to the topological group product* $\prod_{p \in P} \mathbb{Z}_p$*, and* $\{G_i : i \in I\}$ *a nonempty set of profinite groups, such that the quotient group of* G_i *by the closure* G_i' *of its commutator subgroup is isomorphic to a homomorphic image of* $G(F/E)$*, for each index* i*. Then there exist an extension* Φ' *of* F*, and a subfield* Φ *of* Φ'*, for which the following statements are true:*

(i) Φ'/Φ *is a Galois extension with* $G(\Phi'/\Phi)$ *isomorphic to* $G(F/E)$*;*

(ii) $E_{\rm sep} \cap \Phi = E$*, and for each* $i \in I$*, there exists a Galois extension* Λ_i *of* Φ *in* $\Phi'_{\rm sep}$*, such that* $G(\Lambda_i/\Phi) \cong G_i$ *and* Φ' *includes the intermediate field* Λ_i' *of* Λ_i/Φ *corresponding by Galois theory to* G_i'*.*

P r o o f. Let $S_i = \{U_{j(i)} : j(i) \in J_i\}$ be the set of open normal subgroups of G_i, $n_{j(i)}$ the order of the quotient group $G_i/U_{j(i)}$, for each $j(i) \in J_i$, and $E_{\rm sep}(Z)/E_{\rm sep}$ a purely transcendental field extension with a transcendency basis Z presentable as a disjoint union $\cup_{i \in I} Z_i$, where $Z_i = \{z_{j(i),\kappa} : \kappa = 1, \dots, n_{j(i)}$, $j(i) \in J_i\}$, for each $i \in I$. Observing that $G_i/U_{j(i)}$ embeds in the symmetric group ${\rm Sym}_{n_{j(I)}}$: $j(i) \in J_i$, and G_i is isomorphic to a closed subgroup of the topological group $\prod_{j(i) \in J_i}(G_i/U_{j(i)})$, one obtains that E has an extension T_i in $E(Z_i)$, such that $E(Z_i)$ is Galois over T_i with $G(E(Z_i)/T_i)$ isomorphic to G_i, for each $i \in I$ (this reproduces the proof of the main result of [38]). This implies that $F \cap E(Z) = E$, and $E(Z_i) \cap T_i(Z_{i'} : i' \in I, i' \neq i) = T_i$, for evary $i \in I$. In view of Galois theory, this means that if T is the compositum of the fields T_i, $i \in I$, then $F \cdot T$ and $E(Z_i) \cdot T$, $i \in I$, are Galois extensions of T with Galois groups canonically isomorphic to $G(F/E)$ and G_i, $i \in I$, respectively. Applying now Lemma 3.3, one completes the proof of Lemma 3.4. □

Suppose now that F/E is a Galois extension with $G(F/E) \cong G/G'$, Σ is an algebraically closed field including F as a subfield, Σ_0 and Σ_1 are intermediate fields of Σ/E, such that Σ_1/Σ_0 is a Galois extension with $G(\Sigma_1/\Sigma_0) \cong G$, E is separably closed in Σ_0, and $F\Sigma_0$ is the subfield of Σ_1, fixed by the elements of the commutator subgroup of $G(\Sigma_1/\Sigma_0)$. Also, let Σ_0' be the purely inseparable (algebraic) closure of Σ_0 in Σ. Considering Σ_0' and the compositum $\Sigma_1' := \Sigma_0'\Sigma_1$, instead of Σ_0 and Σ_1, respectively, one sees that Σ_0 can be chosen so as to be a perfect field. As $\mathrm{cd}(G) = 1$, G is a projective object in the category of profinite groups (cf. [19]), so it follows from Galois theory that there is an extension Σ_2 of Σ_0 in Σ, such that $\Sigma_1 \cap \Sigma_2 = \Sigma_0$ and $\Sigma_1\Sigma_2$ is an algebraic closure of Σ_0. Lemma 3.3 and the following lemma show how to construct E and F to ensure that Henselian discrete valued fields with residue fields isomorphic to Σ_2 satisfy the conditions of Proposition 3.1.

Lemma 3.5. *Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$, P_0 and P be subsets of the set $\overline{P}$ of prime numbers, such that $P_0 \subseteq P$ and $2 \in P_0$, and ε_p be a primitive p-th root of unity in $\overline{\mathbb{Q}}$, for each $p \in \overline{P}$. Suppose that R_p is a nontrivial subgroup of $\mathbb{Z}(p^\infty)$, for every $p \in P$, and for each $p \in (\overline{P} \setminus P_0)$, fix an integer $\gamma_p \geq 2$ dividing $p - 1$ and not divisible by any prime $\tilde{p} \notin P$. Assume also that R_2' is a subgroup of $\mathbb{Z}(2^\infty)$ of order ≥ 8, in case R_2 is of order 2, and put $R_2' = R_2$, otherwise. Then $\overline{\mathbb{Q}}$ possesses a subfield F satisfying the following conditions:*

(i) *$P(F) = P$, $P_0(F) = P_0$, and G_F is isomorphic to the topological group product $\prod_{p\in P} \mathbb{Z}_p$;*

(ii) *The p-component of $r(F(\varepsilon_p))$ is isomorphic to R_p, $\forall p \in P$, and the 2-component of $F(\sqrt{-1})$ is isomorphic to R_2'; in particular, $R(F)$ is isomorphic to the direct sum $\oplus_{p\in P_0} R_p$;*

(iii) *The degree $[F(\varepsilon_p) : F]$ is equal to γ_p, for every $p \in (\overline{P} \setminus P_0)$.*

P r o o f. Fix a primitive root of unity $\eta_s \in \overline{\mathbb{Q}}$ of degree 2^s, for each $s \in \mathbb{N}$, put $\sqrt{-1} = \eta_2$, and denote by F_0 the extension of $\mathbb{Q}$, defined as follows:

(3.2) (i) $F_0 = \mathbb{Q}$, provided that R_2 is of order greater than 2;

(ii) $F_0 = \mathbb{Q}(\eta_\sigma - \eta_\sigma^{-1})$, in case R_2 is of order 2, and R_2' is of order 2^σ, for some integer $\sigma \geq 3$;

(iii) $F_0 = \mathbb{Q}(\eta, (\eta_s + \eta_s^{-1}) : s \in \mathbb{N})$, where η is a root in $\overline{\mathbb{Q}}$ of the polynomial $f(X) = X^4 + 6X^2 + 8X + 9$, in case R_2 is of order 2, and R_2' is infinite.

It is easily verified that $f(X)$ is irreducible over $\mathbb{Q}$ with a Galois group isomorphic to the alternating group Alt_4; in particular, this implies that $\mathbb{Q}(\eta)$ does not contain as a subfield any quadratic extension of $\mathbb{Q}$. Since $\mathbb{Q}(\eta_\sigma + \eta_\sigma^{-1})/\mathbb{Q}$ is a cyclic extension of degree $2^{\sigma-2}$, for each $\sigma \geq 2$, it becomes clear from Galois theory that $f(X)$ preserves the noted properties, regarded as a polynomial over the field $\mathbb{Q}(\eta_s + \eta_s^{-1}) : s \in \mathbb{N} := F_0'$. Our argument also shows that $\sqrt{-1} \notin F_0$ in the case singled out by (3.2) (iii). We prove that $\mathrm{Br}(F_0)_2 = \{0\}$ under the same hypothesis. Let p be a prime number and $\overline{\mathbb{Q}}_p$ an algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers. Identifying F_0' with its isomorphic copy in $\overline{\mathbb{Q}}_p$, and taking into account that $F_0'(\sqrt{-1}) = \mathbb{Q}(\eta_s : s \in \mathbb{N})$, one obtains from Galois theory that $F_0'\mathbb{Q}_p/\mathbb{Q}_p$ is an infinite abelian 2-extension. Note also that $f(X)$ has no roots in the field of real numbers (its values are positive in each of the intervals $(-\infty, -2]$, $[-2, -9/8]$ and $[-9/8, +\infty)$), so it follows from the Artin-Schreier theory (cf. [25, Ch. XI, Sect. 3]) that $\mathbb{Q}(\eta)$ is a nonreal field. Summing up these results, and applying the Brauer-Hasse-Noether-Albert theorem as in [30, Ch. II, 4.4] or [17, Sect. 2] (see also [12, Proposition 1.2]), one obtains that $\mathrm{Br}(F_0')_2 = \{0\}$, as well as the triviality of $\mathrm{Br}(F_0)_2$ in case (3.2) (iii). This means that $G(F_0(2)/F_0)$ is a free pro-2-group (cf. [36, Theorem 3.1] and [37, p. 725]), which leads to the following conclusion:

(3.3) The character group of $G(F_0/F_0)$ is nontrivial and divisible; equivalently, each cyclic extension of F_0 in $F_0(2)$ is a subfield of a $\mathbb{Z}_2$-extension of F_0 in $F_0(2)$.

Let $\Pi = \{p \in P : R_p \neq \mathbb{Z}(p^\infty)\}$, $p^{n(p)}$ be the order of $\mathbb{Z}(p^\infty)$, for each $p \in \Pi$, $\Gamma_{\overline{p}}$ be the unique $\mathbb{Z}_{\overline{p}}$-extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$, in case $\overline{p} \in \overline{P}$, and $a_{p'}$ an element of $\mathbb{Q}^* \setminus \mathbb{Q}^{*p'}$, for every $p' \in (P_0 \setminus \Pi)$. Denote by F_1 the compositum of the fields F_0, $\mathbb{Q}(\varepsilon_p) : p \in P_0$, $\Phi_\pi : \pi \in \Pi$, $\Gamma_{\pi'}$: $\pi' \in (\overline{P} \setminus \Pi)$, and $\Lambda_{p'} : p' \in (\overline{P} \setminus P_0)$, where Φ_π is the extension of $\mathbb{Q}$ in Γ_π of degree $\pi^{(n(\pi)-1)}$, for each $\pi \in \Pi$, and $\Lambda_{p'}$ is the extension of $\mathbb{Q}$ in $\mathbb{Q}(\varepsilon_{p'})$ of degree $(p'-1)/\gamma_{p'}$, for each $p' \in (\overline{P} \setminus P_0)$. It is clear from the definition of F_1 and Galois theory that every intermediate field of F_1/F_0 is abelian over F_0; using (3.3), one also sees that if condition (3.2) (iii) is in force, then $\sqrt{-1} \notin F_1$ and F_1 admits a $\mathbb{Z}_2$-extension in $\overline{\mathbb{Q}}$ containing $\sqrt{-1}$. At the same time, our choice of a_p ensures that the polynomial $X^p - a_p$ is irreducible over $\mathbb{Q}$, and its Galois group is nonabelian of order $p(p-1)$. Since finite extensions of $\mathbb{Q}$ in F_0 are of 2-primary degrees, these observations show that $X^p - a_p$ remains irreducible over F_1, for each $p \in (P_0 \setminus \Pi)$. Note also that there exists at least one

$\mathbb{Z}_{\pi'}$-extension of F_1 in $\overline{\mathbb{Q}}$, for each $\pi' \in (\overline{P} \setminus \Pi)$. In this case, we have $\Gamma_{\pi'} \subset F_1$, which ensures that the character group $C(F_1(\pi')/F_1)$ of $G(F_1(\pi')/F_1)$ is divisible [16, Proposition 2]. Therefore, it is sufficient to establish that $F_1(\pi') \neq F_1$, for an arbitrary $\pi' \in (\overline{P} \setminus \Pi)$. If $\pi' \in P_0$, this is contained in the fact that $X^{\pi'} - a_{\pi'}$ is irreducible over F_1, so we assume further that $\pi' \notin P_0$. Let $\Phi_{\pi'}$ be the extension of $\mathbb{Q}$ in $\Gamma_{\pi'}$ of degree π'. Then the π'-adic valuation of $\mathbb{Q}$ has a unique prolongation on $\Phi_{\pi'}$, which means that $\Phi_{\pi'} \otimes \mathbb{Q}_{\pi'}$ is a field (cf. [5, Ch. II, Theorem 10.2]). Observing also that this field does not contain a primitive root of unity of degree π' (cf. [20, Ch. 8, Theorem 1]), one obtains from the Shafarevich theorem [31] and the Grünwald-Wang theorem [35] (cf. also [3, Ch. 10, Theorem 5]) that $\overline{\mathbb{Q}}$ contains as a subfield an extension $\Phi'_{\pi'}$ of $\Phi_{\pi'}$ with the following properties:

(3.4) (i) $\Phi'_{\pi'}/\Phi_{\pi}$ is abelian of degree $\pi'^{(\pi'-1)}$ with a Galois group of exponent π';

(ii) $\Phi'_{\pi'} \cap \Gamma_{\pi'} = \Phi_{\pi'}$;

(iii) For each finite extension $U_{\pi'}$ of $\mathbb{Q}$ in $\Gamma_{\pi'}$, the valuation of the compositum $U_{\pi'} \cdot \Phi'_{\pi'}$ extending the π'-adic valuation of $\mathbb{Q}$ is unique and totally ramified; equivalently, π' is totally ramified in (the maximal order of) $U_{\pi'} \cdot \Phi'_{\pi'}$.

On the other hand, it follows from the law of decomposition of rational prime numbers in cyclotomic extensions of $\mathbb{Q}$ (cf. [26, Ch. IV, Sect. 1] and [4, Ch. V, Sect. 2]) that π' is tamely ramified in each finite extension $K_{\pi'}$ of Q in the compositum $F_1^{[\pi']}$ of all fields, except $\Gamma_{\pi'}$, noted in the definition of F_1. This implies that the prime ideal of the maximal order of $U_{\pi'}$, lying above π', is tamely ramified in $U_{\pi'} \cdot K_{\pi'}$. It is therefore clear from (3.4) that $U_{\pi'} \cdot K_{\pi'} \cap U_{\pi'} \cdot \Phi_{\pi'} = U_{\pi'}$. Since F_1 is a union of fields of type $U_{\pi'} \cdot K_{\pi'}$, these observations indicate that $\Phi_{\pi'} \not\subset F_1$, and more precisely, that $\Phi_{\pi'}.F_1$ is an abelian extension of F_1 of degree $\pi'^{(\pi'-1)}$. Thus the existence of a $\mathbb{Z}_{\pi'}$-extension of F_1 is proved. Let now Δ be the compositum of the fields Δ_p, $p \in P$, defined as follows:

(i) $\Delta_p = \Gamma_p \cdot F_1$, for every $p \in \Pi$, except in case $p = 2$ and (3.2) (iii) is in force;

(ii) Δ_p is the extension of F_1 generated by the roots in $\overline{\mathbb{Q}}$ of the polynomials $X^{p^n} - a_p$, $n \in \mathbb{N}$, for every $p \in (P_0 \setminus (\Pi \cup \{2\})$;

(iii) Δ_p is an arbitrary fixed $\mathbb{Z}_p$-extension of F_1 in $\overline{\mathbb{Q}}$, for each $p \in (\overline{P} \setminus (P_0 \cup \Pi))$, as well as in the special case where $p = 2$, R_2 is of order 2 and R'_2 is infinite; in this case, we assume that $\sqrt{-1} \in \Delta_2$.

Using again the general properties of cyclotomic extensions of $\mathbb{Q}$, one obtains that the fields Δ and $F_1(\varepsilon_p)$, $p \in (\overline{P} \setminus P_0)$, satisfy the conditions of Lemma 3.4 (with respect to the ground field F_1). Also, it becomes clear that

$[F_1(\varepsilon_p) : F_1] = \gamma_p$, for every $p \in (\overline{P} \setminus P_0)$, the group of roots of unity of p-primary degrees in $F_1(\varepsilon_p)$ is isomorphic to R_p, for every $p \in P$, and the 2-component of $r(F(\sqrt{-1}))$ is isomorphic to R_2'. Set $\Delta' = \Delta(\varepsilon_p)$, $p \in (\overline{P} \setminus P_0)$, and take an intermediate field Δ_0 of Δ'/F_1 so that $\Delta_0 \cap \Delta = F_1$, $\Delta_0 \cdot \Delta = \Delta'$, and $[\Delta_0(\varepsilon_p) : \Delta_0] = \gamma_p$, $p \in (\overline{P} \setminus P_0)$. It follows from the choice of Δ_0 that Δ'/Δ_0 is a Galois extension with $G(\Delta'/\Delta_0) \cong G(\Delta/F_1) \cong \prod_{p \in P} \mathbb{Z}_p$. Hence, by Galois theory and the projectivity of $\prod_{p \in P} \mathbb{Z}_p$, $\overline{\mathbb{Q}}$ possesses a subfield F, such that $\Delta' \cdot F = \overline{\mathbb{Q}}$ and $\Delta' \cap F = \Delta_0$. Also, it is not difficult to see that F has the properties required by Lemma 3.5. □

We are now in a position to prove Theorem 3.2. Lemma 3.5 and the observations preceeding Lemma 3.3 imply the existence of a Henselian discrete valued field (K, v), such that $\mathrm{char}(\widehat{K}) = 0$, $r(\widehat{K}) \cong R$, $G_{\widehat{K}} \cong G$, and $[\widehat{K}(\varepsilon_p) : \widehat{K}] = \theta_p$: $p \in P(G')$, where θ_p is defined as in (3.1), for every $p \in P(G')$. This implies the inclusion $P_0(\widetilde{L}) \subseteq P(\widetilde{L})$, which means that finite extensions of K are strictly PQL-fields.

4. Brauer groups and character groups.

The purpose of this section is to determine the structure of the Brauer groups and character groups of Henselian discrete valued strictly PQL-fields as abstract abelian torsion groups. The obtained result describes the possible forms, for such fields, of the general local reciprocity law, stated by Proposition 1.3. Before presenting it we fix the notation $\mathbb{Z}_P$ for the direct sum $\oplus_{p \in P} \mathbb{Z}(p^\infty)$, taken over a nonvacuous set P of prime numbers.

Proposition 4.1. *Let (K, v) be a Henselian discrete valued strictly PQL-field, and $r(K)$ the group of all roots of unity in K. Then $\chi(\widehat{K}) \cong \mathbb{Z}_{P(\widehat{K})}$ and $\mathrm{Br}(K)$ is a divisible group with $\mathrm{Br}(K)_r$ isomorphic to $\mathbb{Z}(r^\infty)$, for every $r \in P(\widehat{K})$. In addition, if $\mathrm{char}(\widehat{K}) = 0$, then $\chi(K)$ is isomorphic to the direct sum $\chi(\widehat{K}) \oplus r(K)$.*

Proof. By the generalized Witt theorem, $\mathrm{Br}(K)$ is isomorphic to the direct sum $\mathrm{Br}(\widehat{K}) \oplus \chi(\widehat{K})$, so our statement concerning $\mathrm{Br}(K)$ and $\chi(\widehat{K})$ reduces to a consequence of Theorem 2.1 and Corollary 2.5. In view of [8, Lemma 1.1], the assertion about $\chi(K)$ will be proved, if we show that $\chi(K)_p \cong \mathbb{Z}(p^\infty) \oplus r(\widehat{K})_p$, for an arbitrary $p \in P_0(\widehat{K})$. In this case, by [6, (1.2)] and [8, Lemma 1.1], $G(K(p)/K)$ is a pro-p-group of rank 2 and cohomological dimension 2. As noted in the proof of Lemma 2.4, this means that $G(K(p)/K)$ is a Demushkin group. This enables one to deduce the statement about $\chi(K)_p$ from [37, Lemma 7 and Theorem 2] (concerning the special case of $p = 2$, see also [14, Theorem 4.7]). □

Theorem 4.2. *Let T be a nontrivial abelian torsion group, and $P(T)$ the set of those prime numbers p, for which $T_p \neq \{0\}$. Then T is isomorphic to the Brauer group of a suitably chosen Henselian dicsrete valued strictly PQL-field if and only if $T_2 \cong \mathbb{Z}(2^\infty)$. Moreover, the following assertions hold true:*

(i) *T is realizable as a Brauer group of a Henselian discrete valued strictly PQL-field with a residue field of characteristic $q > 0$, provided that $P(T) = \overline{P}$, and $T_p \cong \mathbb{Z}(p^\infty)$, for each prime divisor p of $q - 1$;*

(ii) *If R is a subgroup of $\mathbb{Q}/\mathbb{Z}$, such that $R_2 \neq \{0\}$, then the following conditions are equivalent:*

(α) *T is realizable as a Brauer group of a Henselian discrete valued strictly PQL-field (K_T, v_T), whose residue field has a group of roots of unity isomorphic to R;*

(β) *T_p is isomorphic to $\mathbb{Z}(p^\infty)$ whenever p is a prime number for which $R_p \neq \{0\}$.*

Proof. In view of Proposition 4.1, it suffices to prove the implication $(\beta)\to(\alpha)$ of Theorem 4.2 (ii). By Lemma 3.4, there exists a perfect field E, such that $r(E) \cong R$ and $G_E \cong \prod_{p\in P(T)} \mathbb{Z}_p$, which implies that if $T_p \cong \mathbb{Z}(p^\infty)$: $p \in P(T)$, then one can take as K any Henselian discrete valued field (K, v) with a residue field isomorphic to E (see also [30, Chapter II, Proposition 6]). Suppose now that the set $\widetilde{P}(T)$ of those prime numbers p, for which T_p does not embed in $\mathbb{Z}(p^\infty)$ is nonempty. This means that T can be presented as a direct sum of $\mathbb{Z}_{P(T)}$ by an isomorphic copy $\widetilde{T}$ of the quotient group $T/\mathbb{Z}_{P(T)}$. Our hypothesis on T indicates that $\widetilde{T}$ is divisible and $P(\widetilde{T}) = \widetilde{P}(T)$; we denote by $t(p)$ the dimension of $\widetilde{T}_p$ as a vector space over the field with p elements, and by Γ_p the additive subgroup of $\mathbb{Q}$ of rational numbers presentable as fractions of integers with denominators not divisible by p, for each $p \in \widetilde{P}(T)$. Consider now the direct sum $\Gamma = \oplus_{p\in\widetilde{P}(T)}\Gamma_p^{t(p)}$, where Γ_p is itself a direct sum of isomorphic copies of Γ_p, indexed by a set of cardinality $t(p)$, for every $p \in \widetilde{T}(P)$. It is easily seen that Γ can be provided with a structure of an ordered abelian group, which means that there exists a Henselian valued equicharacteristic field $(\widetilde{K}, \tilde{v})$, whose residue field and value group are isomorphic to E and Γ, respectively (cf. [15, Ch. 4, Sect. 7]). Note also that if $\mathrm{char}(E) = q > 0$, then $\widetilde{K}$ can be chosen so that its finite extensions in $\widetilde{K}(q)$ are inertial. This follows from the fact that the inertial lifts over $\widetilde{K}$ of the finite extensions of E in $E(q)$ are inertial, and the compositum I_q of their $\widetilde{K}$-isomorphic copies in $\widetilde{K}_{\mathrm{sep}}$ is a $\mathbb{Z}_q$-extension of $\widetilde{K}$; hence, by Galois theory and the projectivity of $\mathbb{Z}_q$ as an object in the category of profinite groups, there is an extension M_q of $\widetilde{K}$ in $\widetilde{K}(q)$, such that $I_q.M_q = \widetilde{K}(q)$, and $I_q \cap M_q = \widetilde{K}$. Considering M_q with its valuation extending $\tilde{v}$, instead of $(\widetilde{K}, \tilde{v})$, one obtains the required special choice of $\widetilde{K}$. We show that $P(\widetilde{K}) = \widetilde{P}(T)$, $\widetilde{K}(p)/\widetilde{K}$ is a $\mathbb{Z}_p$-extension, for all $p \in \widetilde{P}(T)$, and $\mathrm{Br}(\widetilde{K}) \cong T$. Indeed, condition (β) guarantees that $P_0(E) \cap \widetilde{P}(T) = \emptyset$, so our former two assertions can be deduced from [8, Lemma 1.1], Lemma 2.3 (i), the Henselian property of $\tilde{v}$, and the assumptions on E and Γ. Also, it follows from the definition of Γ that $t(p)$ equals the dimension of the quotient group $\Gamma/p\Gamma$ as a vector space over the field with p elements, for each $p \in \widetilde{P}(T)$, and $\Gamma = p\Gamma$, in case p is a prime number not lying in $\widetilde{P}(T)$. Taking now into account that E and $\widetilde{K}$ are perfect fields and $\mathrm{cd}(G_E) = 1$, one obtains consecutively that $\mathrm{Br}(E) = \{0\}$, $N(E_1/E) = E^*$, for every finite extension E_1 of E (cf. [30, Ch. II, Proposition 6]), and each central division $\widetilde{K}$-algebra $\widetilde{\Delta}$ is nicely semi-ramified in the sense of [21, Sect. 4]. Furthermore, it becomes clear that $\widetilde{\Delta}$ contains as a maximal subfield a $\widetilde{K}$-isomorphic copy of the unique extension of $\widetilde{K}$ in $\widetilde{K}_{\mathrm{sep}}$ of degree $\mathrm{ind}(\widetilde{\Delta})$. Since this extension is inertial over $\widetilde{K}$, our argument proves that $\mathrm{Br}(\widetilde{K})$ is isomorphic to $\widetilde{T}$, as claimed. Applying now Proposition 4.1, one concludes that every Henselian discrete valued field (K, v), such that $\widehat{K} \cong \widetilde{K}$ is strictly PQL, with $\mathrm{Br}(K) \cong T$ and $r(K) \cong R$, which completes the proof of Theorem 4.2. □

Remark 4.3. Let T be a divisible abelian torsion group, such that $T_2 \cong \mathbb{Z}(2^\infty)$, and let R be a subgroup of $\mathbb{Q}/\mathbb{Z}$, such that $R_2 \neq \{0\}$. It would be of interest to know for which pairs (T, R), there is a Henselian discrete valued strictly PQL-field (K, v) with $\mathrm{Br}(K) \cong T$ and $\mathrm{r}(K) \cong R$. The answer to this question would follow from Theorem 4.2, if it is true that $\mathrm{Br}(E)_p = \{0\}$ whenever E is a field and p is a prime number, for which $E(p) = E$.

5. Norm groups of Henselian discrete valued strictly PQL-fields. In this Section our consideration go along the lines drawn by Propositions 1.4 –1.6. We begin with a characterization of those Henselian discrete valued fields admitting local field theory whose finite separable extensions are subject to the norm limitation theorem.

Proposition 5.1. *For a Henselian discrete valued field (K, v) admitting local class field theory, the following conditions are equivalent:*

(i) $N(R/K) = N(R_{\mathrm{ab}}/K)$*, for each finite extension R of K in K_{sep};*

(ii) *Every finite extension of the residue field $\widehat{K}$ of (K, v) is cyclic;*

(iii) *For each finite extension L of K, the natural homomorphism of* $\mathrm{Br}(K)$ *into* $\mathrm{Br}(L)$ *is surjective.*

Proof. The implication (i)→(ii) follows at once from the fact that if U/K is an inertial finite extension of degree n, then $v(\lambda) \in nv(K)$, $\forall \lambda \in N(U/K)$. Suppose that condition (ii) is in force and fix a finite extension L of K. It is easily obtained that the absolute Galois groups $G_{\widehat{L}}$ and $G_{\widehat{K}}$ are isomorphic, whence $\chi(\widehat{L}) \cong \chi(\widehat{K}) \cong \mathbb{Z}_{P(\widehat{K})}$. Note also that $\mathrm{Br}(\widehat{K})$ and $\mathrm{Br}(\widehat{L})$ are trivial, since $\widehat{K}$ is perfect and $\mathrm{cd}(G_{\widehat{K}}) = 1$; this follows at once from Theorem 2.1 and [30, Ch. II, Proposition 6]. Hence, by the generalized Witt theorem $\mathrm{Br}(L) \cong \mathrm{Br}(K) \cong \mathbb{Z}_{P(\widehat{K})}$. Taking also into account that the relative Brauer group $\mathrm{Br}(L/K)$ is of finite exponent dividing $[L : K]$ (cf. [29, Sect. 13.4]), one proves the implication (ii)→(iii). The assertion that (iii) implies (i) is valid under the considerably more general hypotheses of Proposition 1.4 (ii), so Proposition 5.1 is proved. □

Corollary 5.2. *For a Henselian discrete valued field (K, v), the following conditions are equivalent:*

(i) *The residue field $\widehat{K}$ is quasifinite, i.e. perfect with $G_{\widehat{K}}$ isomorphic to the topological group product $\prod_{p \in \overline{P}} \mathbb{Z}_p$;*

(ii) *Every finite R extension of K is a strictly PQL-field, such that $N(R/K) = N(R_{\mathrm{ab}}/K)$.*

Proof. This follows at once from Propositions 3.1 and 5.1. □

Next we characterize those Henselian discrete valued fields (K, v) with residue fields of zero characteristic, and with the property that the norm group of each finite extension R/K equals $N(\Phi(R)/K)$, for some finite abelian extension $\Phi(R)$ of K.

Proposition 5.3. *Let (K, v) be a Henselian discrete valued field admitting local class field theory, and such that* $\mathrm{char}(\widehat{K}) = 0$. *Then the following conditions are equivalent:*

(i) *For each finite extension R of K in K_{sep}, there exists a finite abelian extension $\Phi(R)$ of K in K_{sep}, such that $N(\Phi(R)/K) = N(R/K)$;*

(ii) *The cohomological p-dimension $\mathrm{cd}_p(G_{\widehat{K}})$ is equal to zero, for each prime p not lying in $P(\widehat{K})$, $P_1(\widehat{K}) = \phi$, and $N(\widetilde{L}/\widehat{K}) = \widehat{K}^*$, for every finite extension $\widetilde{L}$ of $\widehat{K}$.*

When this occurs, the quotient group $K^*/N(R/K)$ is isomorphic to the Galois group $G(\Phi(R)/K)$.

P r o o f. Let π be a generator of the maximal ideal of the valuation ring of (K, v). We first prove the validity of (ii) without any restriction on $\mathrm{char}(\widehat{K})$, and under the weaker hypothesis that condition (i) is in force, for each finite Galois extension R of K. Our argument relies on the following well-known properties of $N(R/K)$: $N(R/K) \subseteq \{\alpha \in K^*\colon\ v(\alpha) \in [\widehat{R} : \widehat{K}]v(K)\}$; $\hat{u} \in \widehat{K}^{*e}$, provided that $u \in N(R/K)$, $v(u) = 0$ and $e = [R : K]/[\widehat{R} : \widehat{K}]$; $N(R/K)$ contains at least one element ρ of value $v(\rho)$ generating $[\widehat{R} : \widehat{K}]v(K)$. To begin with, our hypothesis ensures that $[R : K]$ is not divisible by any prime number not lying in $P(\widehat{K})$, which is equivalent to the assertion that $\mathrm{cd}_p(G_{\widehat{K}}) = 0$, $p \in (\overline{P} \setminus P)$. We prove that $P_1(\widehat{K}) = \phi$ by assuming the opposite. This means that K contains an element a, such that $v(a) = 0$ and $\hat{a} \notin \widehat{K}^{*p}$, for some $p \in (\overline{P} \setminus P_0)$. Put $[K(\varepsilon_p) : K] = m$, where ε_p is a primitive p-th root of unity in K_{sep}, and denote by M_π the root field in K_{sep} of the binomial $X^p - \pi$ over K. It is easily verified that $[M_\pi : K] = pm$ and $[\widehat{M_\pi} : \widehat{K}] = m$, so our starting observations indicate that $[\widehat{M_\pi} : \widehat{K}] = m$. This, combined with the fact that $p \notin P_0(\widehat{K})$, enables one to deduce from [8, Lemma 1.1] that p does not divide $[\Phi(M_\pi) : K]$. Our conclusion, however, contradicts the fact that the order in $K^*/N(M/K)$ of the co-set of a is divisible by p, which proves the assertion that $P_1(\widehat{K}) = \phi$. Let now $\widetilde{L}$ be a finite Galois extension of $\widehat{K}$, and L the inertial lift of $\widetilde{L}$ in K_{sep} over K. It follows from the Henselian property of v and the equality $N(L/K) = N(\Phi(L)/K)$ that $N(\widetilde{L}/\widehat{K}) = N(\widehat{\Phi}(L)/\widehat{K})$. Therefore, it suffices to establish that $N(\widetilde{L}/\widehat{K}) = \widehat{K}^*$ in the special case where $\widetilde{L}$ is abelian over $\widehat{K}$. Furthermore, Lemma 1.7 indicates that one may assume in addition that $\widetilde{L}/\widehat{K}$ is a p-extension, for some prime p. If $p \in P_0(\widehat{K})$, this follows at once from Theorem 2.1 and Lemma 1.8, and if $p \notin P_0(\widehat{K})$, it is an immediate consequence of the equality $\widehat{K}^* = \widehat{K}^{*p}$.

Suppose now that condition (ii) is in force, fix a finite tamely ramified extension R of K in K_{sep}, denote by R_0 the maximal inertial extension of K in R, and by $(U(R)$, $U(R_0)$ and $U(K)$ the multiplicative groups of the valuation rings of (R, v_R), (R_0, v_{R_0}) and (K, v), respectively. Also, let $[R_0 : K] = f$, $[R : R_0] = e$ and let n be the least positive integer dividing $[R : K]$ and not divisible by any prime number out of $P_0(\widehat{K})$. It is clear from the Henselian property of v, v_{R_0} and v_R that $N(R/K) = U(K)^{*e}\langle(N_K^{R_0}(\beta)\pi)^f\rangle$, for a suitably chosen element $\beta \in U(R_0)$. As $P_1(\widehat{K}) = \phi$, this implies the inclusion $U(K)^n\langle\pi^{[R:K]}\rangle \subseteq N(R/K)$. It is not difficult to see that $U(K)/U(K)^n$ is a cyclic group of order n_0, where n_0 is the greatest positive integer dividing n, for which $\widehat{K}$ contains a primitive n_0-th root of unity. This implies that the extension L_π of K in K_{sep} obtained by adjoining an n_0-th root of π is abelian of degree n_0, and such that $N(L_\pi/K) = U(K)^n\langle(-1)^{(n-1)/2}\pi\rangle$. At the same time, it follows from Theorem

2.1, condition (ii) and Galois theory that K has an abelian inertial extension R' in $K_{\rm sep}$ of degree $[R' : K] = [R : K]$. This indicates that $R'L_{\pi}/K$ is abelian and $N(R'L_{\pi}/K) = U(K)^n\langle\pi^{[R:K]}\rangle$. Taking finally into account that K admits local class field theory and $K^*/N((R'L_{\pi})/K) \cong G((R'L_{\pi})/K)$, one concludes that each subgroup of K^* including $N((R'L_{\pi})/K)$ is a norm group of a suitably chosen abelian extension of K in $R'L_{\pi}$, and so completes the proof of Proposition 5.3. □

Propositions 5.1 and 5.3 and Corollary 2.5 indicate that the study of norm groups of Henselian discrete valued strictly PQL-fields usually should not restrict to the special case of finite abelian extensions. Our main objective in the rest of this paper is to show that the research in this direction does not reduce to considering the special case of finite Galois extensions whose Galois groups belong to some of the most actively studied group formations. We refer the reader to [32], for a systematic presentation of this part of finite group theory, introducing here only two definitions needed for the further discussion.

Definition 5.4. *Let χ be a class of finite groups including all finite abelian groups. We say that χ is a regular group formation, if it satisfies the following conditions:*

(c) *If $H \in \chi$, then every isomorphic copy of H lies in χ;*

(cc) *χ is closed under the formation of subgroups and quotient groups, and contains $G/(A \cap B)$ whenever G is a group with normal subgroups A and B, such G/A and G/B lie in χ.*

Definition 5.5. *A pair (χ_1, χ_2) of regular group formations is called admissible, if $\chi_1 \subset \chi_2$, and the complement $\chi_2 \setminus \chi_1$ contains a group G satisfying the following condition:*

(α) *the commutator subgroup $[G, G]$ of G is its unique minimal normal subgroup, and the quotient group $G/[G, G]$ is cyclic.*

We say that (χ_1, χ_2) is P-admissible, for some set P of prime numbers, if (χ_1, χ_2) is admissible, $2 \in P$, and the group G can be chosen so that the following condition is fulfilled:

(β) *P includes the set of prime divisors of the index $|G : [G, G]|$, whereas the order of $[G, G]$ is divisible by at least one prime number not lying in P.*

Remark 5.6. (i) It is easily obtained from Galois theory that if χ is a regular group formation and L/K is a field extension presentable as a compositum of finite Galois extensions of K with Galois groups from χ, then L/K is a Galois extension and $G(M/K) \in \chi$, for every finite Galois subextension M of K in L.

(ii) Examples of regular group formations are provided by those consisting of all finite groups with the property, say $\widetilde{P}$, from the following list: nilpotent; supersolvable; solvable. Every pair of formations consecutively taken from the list, is admissible. A pair (χ_1, χ_2) of regular group formations, which are closed with respect to taking group extensions, is admissible if and only if χ_1 is a proper subclass of χ_2; when this occurs, $\chi_2 \setminus \chi_1$ contains at least one nonabelian simple group.

(iii) Every finite group G satisfying condition (α) with respect to a given admissible pair is necessarily nonnilpotent; moreover, if G is solvable, then $[G, G]$ is an elementary abelian p-group, for some prime p. This implies that the formation of finite groups with nilpotent commutator subgroups does not form an admissible pair with the class of solvable finite groups.

The following lemma describes some properties of the finite groups singled out in Definition 5.5, and used in the proof of the main result of this section.

Lemma 5.7. *Assume that G is a nonnilpotent finite group whose commutator subgroup $[G, G]$ is its unique minimal normal proper subgroup. Suppose also that the quotient group $G/[G, G]$ is cyclic. Then the centralizer $C_G([G, G])$ of $[G, G]$ in G is trivial, if G is nonsolvable, and equals $[G, G]$, otherwise. In the latter case, the order of $[G, G]$ is relatively prime to the index of $[G, G]$ in G.*

P r o o f. It is clear from the normality of $[G, G]$ that $C_G([G, G])$ is also a normal subgroup of G, so the former assumption of the lemma ensures that if $C_G([G, G]) \neq G$, then $[G, G] \subseteq C_G([G, G])$, whence $[G, G]$ is abelian. Furthermore, it becomes clear that $[G, G]$ is an elementary abelian p-group, for some prime number p. This implies that G possesses a normal Sylow p-subgroup G_p. Since the centre $Z(G_p)$ of G_p is a nontrivial characteristic subgroup of G_p, it is also normal in G and includes $[G, G]$, so it follows from the cyclicity of $G/[G, G]$ that G_p is abelian. We prove that p does not divide the index $|G : [G, G]|$ by assuming the opposite. This means that $[G, G] \neq G_p$, and because of the assumption that $[G, G]$ is the only minimal normal subgroup of G, and of the fact that $G_p/[G, G]$ is a cyclic group, leads to the conclusion that G_p is cyclic. Using again the cyclicity of $G/[G, G]$, one obtains consecutively that $[G, G]$ and G_p are subgroups of the centre of G. By the SchurZassenhaus theorem (cf. [22, Ch. 7, Sect. 2]), the normality of G_p in G implies the existence of a subgroup H_p of G isomorphic to G/G_p, so the established property of G_p indicates that there is a group isomorphism $G \cong G_p \times (G/G_p)$. This amounts to saying that H_p is normal in G, and since $H_p \cap [G, G] = \{1\}$, yields $H_p = \{1\}$ and $G = G_p$, which contradicts the assumption that G is nonnilpotent. Thus the assertion that p does not divide

$|G : [G, G]|$ is proved. Considering now $C_G([G, G])$ instead of G, and arguing in the same manner, one obtains that if $[G, G] \neq C_G([G, G])$, then $[G, G]$ is a direct summand in $C_G([G, G])$, and therefore, $C_G([G, G])$ has a nontrivial characteristic subgroup H_p of order not divisible by p. Hence, H_p must be normal in G, in contradiction with the assumption that $[G, G]$ is the unique minimal normal subgroup of G. The obtained result completes the proof of Lemma 5.7. □

The main results of this section are presented by the following two theorems, which are proved in the following section.

Theorem 5.8. *Assume that χ is a regular group formation, such that the set $\mathrm{Ad}(\chi)$ of those regular group formations χ', for which (χ, χ') is an admissible pair, is nonempty. Then there exists a Henselian discrete valued strictly PQL-field (K, v), for which the following is true:*

(i) *If M/K is a Galois extension with $G(M/K) \in \chi$, then $G(\widehat{M}/\widetilde{K})$ is cyclic and $N(M/K) = N(M_{\mathrm{ab}}/K)$;*

(ii) *For each $\chi_1 \in \mathrm{Ad}(\chi)$, there exists an inertial Galois extension $L(\chi_1)$ of K, such that $G(L(\chi_1)/K) \in (\chi_1 \setminus \chi)$; in addition $N(L(\chi_1)/K)$ does not equal the norm group of the maximal Galois extension of K in $L(\chi_1)$ with a Galois group from χ;*

(iii) *For each finite extension R of K in K_{sep}, there is an abelian finite extension $\Phi(R)$, such that $N(R/K) = N(\Phi(R)/K)$.*

Theorem 5.9. *Assume that χ is a regular group formation, and P is a set of prime numbers, such that $2 \in P$ and the set $\mathrm{Ad}_P(\chi)$ of those regular group classes χ', for which (χ, χ') is a P-admissible pair, is nonempty. Then there exists a Henselian discrete value strictly PQL-field (K, v), for which the following is true:*

(i) *$P(\widehat{K}) = P_0(\widehat{K}) = P$ and every Galois extension M of K with $G(M/K) \in \chi$ has the properties required by Theorem 5.8 (i);*

(ii) *For each $\chi_1 \in \mathrm{Ad}_P(\chi)$, there is an inertial Galois extension $L(\chi_1)$ of K, such that $G(L(\chi_1)/K) \in (\chi_1 \setminus \chi)$; also, $N(L(\chi_1)/K)$ does not equal the norm group of any finite Galois extension of K with a Galois group belonging to χ.*

6. Proofs of Theorems 5.8 and 5.9.

The existence of $\widehat{K}$ will be established on the basis of the following two lemmas.

Lemma 6.1. *Each field E has an extension Σ with the following two properties:*

(i) *E is algebraically closed in Σ;*

(ii) $N(\Sigma'/\Sigma) = \Sigma^*$, *for every finite extension Σ' of Σ in $\Sigma_{\rm sep}$.*

P r o o f. Denote by $F_{\rm ns}(E)$ the set of finite Galois extensions of E in $E_{\rm sep}$, and for each $M \in F_{\rm ns}(E)$, put $N_{M/E}(T) = \prod_{j=1}^{n(M)} (\sum_{i=1}^{n(M)} \varphi_j(\mu_i)T_i)$, where $\{\mu_i : i = 1, \dots, n(M)\}$, is a basis of M as a vector space over E, $\{\varphi_j : j = 1, \dots, n(M)\} = G(M/E)$, and $T = (T_1, \dots, T_{n(M)})$ is a set of algebraically independent elements over E. It is well-known that $N_{M/E}(T) \in E[T]$. Note also that the polynomial $N_{M/E}^{[a]}(T) = N_{M/E}(T) - a$ is irreducible over any algebraically closed extension U of E, $\forall a \in E^*$. This follows from the fact that $N_{M/E}(T) \in U(T)^{*p}$, for any prime p whence the polynomial $aT_0^{n(M)} - N_{M/E}(T) \in U(T)[T_0]$ is irreducible over $U(T)$ (cf. [25, Ch. VIII, Sect. 10]). We show that there exists an extension Σ_1 of E, such that E is algebraically closed in Σ_1, and the equation $N_{M/E}(X_1, \dots, X_{n(M)}) = a$ is solvable over Σ_1, for each $a \in E^*$. Consider the polynomial ring $\Sigma_0 = E[X_{\lambda,\nu} : \lambda \in (F_{\rm ns}(E) \times E^*), \nu = 1, \dots, n(\lambda)]$, where $n(\lambda) = n(M)$, in case $\lambda = (M, b)$, for some $b \in E^*$, and denote by I the ideal of Σ_0, generated by the set $\{N_{M/E}^{[a]}(X_\lambda) : \lambda = (M, a),\ X_\lambda = (X_{\lambda,1}, \dots, X_{\lambda,n(M)})\}$. It is easily verified that $I \neq \Sigma_0$; one also sees that the quotient ring $\Sigma_0/J := \Sigma_1$, where J is a maximal ideal of Σ_0 including I, is a field with the claimed properties. Now Lemma 6.1 can be proved by constructing Σ as a union of fields $\{\Sigma_n : n \in \mathbb{N}\}$, such that $\Sigma \subset \Sigma_{n+1}$, Σ_n is algebraically closed in Σ_{n+1}, and the norm equation associated as above with any finite Galois extension of Σ_n, and each element of Σ_n^*, is solvable over Σ_{n+1}, $\forall n \in \mathbb{N}$. □

Lemma 6.2. *Let E be a field, s a positive integer, L and $F_1, \dots, F_s$ Galois extensions of E in $E_{\rm sep}$, Φ and Φ_j: $j = 1, \dots, s$, the compositums of the fields F_i: $i = 1, \dots, s$ and $F_{j'}$, $j' \neq j$, for each $j \in \{1, \dots, s\}$, respectively. Assume also that χ is a regular group formation, $G(L/E)$ is a cyclic group, the Galois groups $G(F_i/E)$, $i = 1, \dots, s$, are pairwise nonisomorphic (in case $s \geq 2$), and for each index j, $G(F_j/E)$ satisfies the conditions of Lemma 5.7, $G(F_j/E) \notin \chi$, L includes as a subfield the extension L_j of E in F_j corresponding by Galois theory to $[G(F_j/E), G(F_j/E)]$, and F_j is not a subfield of $L\Phi_j$. Then the following assertions hold true:*

(i) *$(L\Phi)/E$ is a Galois extension, such that $G((L\Phi/L)$ is equal to the commutator subgroup of $G((L\Phi/E)$, and is isomorphic to the direct product*

$$\prod_{i=1}^{s} [G(F_j/E), G(F_j/E)];$$

(ii) $G((L\Phi)/(L\Phi_j))$ *is a minimal normal subgroup of* $G(L\Phi/E)$*, isomorphic to* $[G(F_j/E), G(F_j/E)]$*, for each index* j*;*

(iii) *If* U *is an intermediate field of* $L\Phi/E$*, such that* $U \cap L = E$ *and the Galois group of its normal closure over* E *lies in* χ*, then* $U = E$*.*

Proof. Clearly, $L\Phi$, $L\Phi_i$ and LF_i, $i = 1, \dots, s$, are Galois extensions of E. We first show that $G(F_j/E)$ has the following properties, for each index j:

(6.1) (i) The commutator subgroup of $G((LF_j)/E)$ is isomorphic to $[G(F_j/E), G(F_j/E)]$ and is equal to $G((LF_j)/L)$;

(ii) $G((LF_j)/L)$ is a minimal normal subgroup of $G((LF_j)/E)$;

(iii) A subgroup H_j of $G((LF_j)/E)$ is included in $G((LF_j)/F_j)$ if and only if it is normal and satisfies the condition $H_j \cap G((LF_j)/L) = \{1\}$.

Let L'_j be the maximal abelian extension of E in LF_j. Then it follows from Galois theory and the assumptions of Lemma 5.7 that $L \subseteq L'_j$ and $L'_j \cap F_j = L_j$. Furthermore, it becomes clear that $[LF_j : L_j] = [L : L_j] \cdot [F_j : L_j] = [L'_j : L_j] \cdot [F_j : L_j]$ (cf. [25, Ch. VIII, Theorem 5]), which proves the equality $L'_j = L$ as well as the fact that $G(F_j/L_j)$ is canonically isomorphic to $G((LF_j)/L)$. The obtained result is equivalent to (6.1) (i). The proof of (6.1) (ii) relies on the fact that $G((LF_j)/F_j)$ is a normal subgroup of $G((LF_j)/E)$, whose intersection with $G((LF_j)/L)$ is trivial. This implies that if U_j is a nontrivial normal subgroup of $G((LF_j)/E)$, included in $G((LF_j)/L)$, then the quotient group $U_jG((LF_j)/F_j)/G((LF_j)/F_j)$ is isomorphic to U_j, and is included in the commutator subgroup of $G((LF_j)/E)/G((LF_j)/F_j)$. Since, by Galois theory, $G((LF_j)/E)/G((LF_j)/F_j)$ is isomorphic to $G(F_j/E)$, this leads to the conclusion that $U_j = G((LF_j)/L)$, as required by (6.1) (ii). We turn to the proof of (6.1) (iii). It is clear from Galois theory and the cyclicity of L/E that $G((LF_j)/F_j)$ is a cyclic group. This ensures that the subgroups of $G((LF_j)/F_j)$ are characteristic, and also, normal in $G((LF_j)/E)$, because of the same property of $G((LF_j)/F_j)$. Let now H_j be a normal subgroup of $G((LF_j)/E)$, such that $H_j \cap G((LF_j)/L) = \{1\}$. Then H_j is a subgroup in the centralizer, say C_j, of $G((LF_j)/L)$ in $G((LF_j)/E)$. The inclusion $H_j \subseteq G((LF_j)/F_j)$ will be established in the process of proving the following statement:

(6.2) $G((LF_j)/F_j)$ is equal to C_j, if $G(F_j/E)$ is nonsolvable, and to the centre Z_j of $G((LF_j)/E)$, otherwise.

Note first that $C_j \subseteq G((LF_j)/L_j)$. Suppose for a moment that this is not the case. Observing that $C_j/G((LF_j)/F_j)$ is a subgroup of the centralizer of $G((LF_j)/L_j)/G((LF_j)/F_j)$ in $G((LF_j)/E)/G((LF_j)/F_j)$, taking into consideration that $G((LF_j)/L_j)$ equals the inner product $G((LF_j)/F_j)G((LF_j)/L)$, and arguing as in the proof of (6.2) (ii), one concludes that $[G(F_j/E), G(F_j/E)]$ does not include its centralizer in $G(F_j/E)$, in contradiction with Lemma 5.7. The obtained inclusion indicates that if $C_j \neq G((LF_j)/F_j)$, then $C_j \cap G((LF_j)/L) \neq \{1\}$, so it follows from (6.1) (ii) that $G((LF_j)/L) \subseteq C_j$. This proves (6.2) in the special case where $G(F_j/E)$ is nonsolvable. In the rest of the proof of (6.2) we assume that $G(F_j/E)$ is solvable. By (6.1) (i) and (ii), this means that $G((LF_j/L)$ is abelian, and by the Burnside-Wielandt theorem (cf. [22, Ch. 6, Sect. 2]), and the fact that $G((LF_j)/E)$ is nonnilpotent, there exists a maximal subgroup N_j of $G((LF_j)/E)$ not including $G((LF_j)/L)$. In view of the commutativity of $G((LF_j)/L)$, $N_j \cap G((LF_j)/L)$ is a normal subgroup of $G((LF_j)/E)$, so it becomes clear from (6.1) (ii) that $N_j \cap G((LF_j)/L) = \{1\}$. Therefore, N_j is isomorphic to $G((LF_j)/E)/G((LF_j)/L)$, i.e. to the cyclic group $G(L/E)$. Using the normality of $G((LF_j)/L_j)/L_j)$ in $G((LF_j)/E)$, one also obtains without difficulty that $(N_j \cap G((LF_j)/L_j)) := N'_j$ is a subgroup of Z_j of order equal to the orders of $G((LF_j)/F_j)$ and $G(L/L_j)$. This result indicates that $G((LF_j)/L_j) = G((LF_j)/L)N'_j$. At the same time, it follows from Lemma 5.7 and the fact that $G(F_j/E) \cong G((LF_j)/E)/G((LF_j)/F_j)$ that $Z_j \cap G((LF_j)/L) = \{1\}$. These observations and the remarks preceeding the statement of (6.2) enable one to deduce that $H_j \subseteq Z_j = G((LF_j)/F_j)$, and so to complete the proofs of (6.1) (iii) and (6.2). Our further considerations also rely on the following consequence of (6.2) and the conditions of Lemma 6.2 concerning $G(F_1/E), \dots, G(F_s/E)$:

(6.3) The groups $G((LF_i)/E)$, $i = 1, \dots, s$, are pairwise nonisomorphic, in case $s \geq 2$.

Our objective now is to prove Lemma 6.2 (i) and (ii). The corresponding assertions in the special case of $s = 1$ are contained in (6.1) (i) and (ii), so we assume that $s \geq 2$. Proceeding by induction on s, one sees that it is sufficient to establish the validity of our assertions, assuming additionally that they are true when s is replaced by $s - 1$ and $L\Phi$ by $L\Phi_i$, for each positive integer $i \leq s$. It is clear from Galois theory, statement (6.1) (ii) and the assumptions of the lemma that $LF_j \cap L\Phi_j = L$, for each index j. This is equivalent to the assertion that $G((L\Phi)/L) \cong \prod_{i=1}^{s} G((LF_i)/L)$, and by (6.1) (ii), to the former part of Lemma 6.2 (i). As L is abelian over E, $G((L\Phi)/L)$ includes the commutator subgroup of $G((L\Phi/E)$, so we have to show that $G((L\Phi/L)$ is a subgroup of $[G((L\Phi)/E), G((L\Phi)/E)]$. Suppose first that $G(F_i/E)$, $i = 1, \dots, s$, are nonsolvable groups. By (6.1) (i) and (ii), then $G((LF_i)/L)$, $i = 1, \dots, s$, are nonsolvable and characteristically simple, which implies that they coincide with their commutator subgroups. Let now $G(F_j/E)$ be a solvable group, for some index j, and let D_j be the extension of E in LF_j corresponding to the maximal subgroup N_j of

$G((LF_j)/E)$, considered in the proof of (6.2). The choice of N_j and Galois theory indicate that $D_j \cap L = E$, which combined with the fact that $LF_j \cap L\Phi_j = L$, yields $D_j \cap L\Phi_j = E$. Since $D_j L\Phi_j = L\Phi$, this means that the Galois groups $G((L\Phi_j)/E)$ and $G((L\Phi)/D_j)$ are canonically isomorphic. Hence, by Galois theory and the inductive hypothesis, the compositum $LD_j = LF_j$ is the maximal abelian extension of D_j in $L\Phi$. This implies that LF_j includes the maximal abelian extension of E in $L\Phi$, so it follows from (6.1) (i) that this extension is equal to L, which completes the proof of Lemma 6.2 (i). Returning to the equalities $LF_i \cap L\Phi_i = L$, $i = 1, \dots, s$, one obtains from Galois theory that $G(LF_i/E)$ is canonically isomorphic to the subgroup $G((L\Phi)/(L\Phi_i))$ of $G((L\Phi)/E)$, for each index i. Also, it becomes clear that the product of these subgroups is direct and equal to $G((L\Phi)/L)$. Taking now into account that the quotient group of $G((L\Phi)/E)$ by the product of the groups $G((L\Phi)/(L\Phi_{i'}))$, $i' \neq i$, is isomorphic to $G((LF_i)/E)$, for each i, and arguing as in the proof of (6.1) (ii), one completes the proof of Lemma 6.2 (ii).

It remains for us to prove Lemma 6.2 (iii). Denote by $\widetilde{U}$ the normal closure of U over E, and by H and $\widetilde{H}$ the subgroups of $G((L\Phi)/E)$ corresponding to U and $\widetilde{U}$, respectively. Suppose first that $U \neq E$ and $s = 1$. Then $\widetilde{H}$ does not include $G((LF_1)/L)$, which implies that $\widetilde{H} \cap G((LF_1)/L) = \{1\}$. Therefore, by (6.1) (iii) and Galois theory, $\widetilde{U}$ includes F_1, which leads to the conclusion that $G(F_1/E)$ is a homomorphic image of $G(\widetilde{U}/E)$. This, however, is impossible, since $G(\widetilde{U}/E) \in \chi$ whereas $G(F_1/E) \notin \chi$. The obtained contradiction proves Lemma 6.2 (iii) in the special case of $s = 1$. Henceforth, we assume that $s \geq 2$ and $U \neq E$. Proceeding by induction on s, one sees that it is sufficient to obtain a contradiction under the hypothesis that $U \cap (L\Phi_i) = E$: $i = 1, \dots, s$. In view of Galois theory and Lemma 6.1 (ii), then we have $\widetilde{H} \cap G((L\Phi)/(L\Phi_i)) = \{1\}$, for each i, which means that $\widetilde{U}$ is a subgroup of the centralizer, say C, of $G((L\Phi)/L)$. The crucial step towards the proof of Lemma 6.2 (iii) is to show that $\widetilde{H} \cap C = \{1\}$. This is obtained as a special case of the following statement:

(6.4) If Ω is a subgroup of C, normal in $G((L\Phi)/E)$ and such that $\Omega \cap G((L\Phi)/(L\Phi_i)) = \{1\}$, $i = 1, \dots, s$, then $\Omega \cap G((L\Phi)/L) = \{1\}$.

Proceeding by induction on s, one sees that it is sufficient to prove (6.4), on the additional assumption that its analogue is true, for $s - 1$ and $L\Phi_i$, for each i. Denote, for each $\theta \in G((L\Phi)/L)$, by $\theta_1, \dots, \theta_3$ the elements of $G((L\Phi)/(L\Phi_1))$,..., $G((L\Phi)/(L\Phi_s))$, respectively, taken so that $\prod_{i=1}^{s} \theta_i = \theta$. Suppose that $\Omega \neq \{1\}$, and put $\Omega_i = \langle\theta_i\rangle : \theta \in (\Omega \cap H)\}$, for each i. Note first that $\Omega_i \neq \{1\}$: $i = 1, \dots, s$. Indeed, it is easily seen that if $\Omega_j = \{1\}$, for some index j, then $\Omega G((L\Phi)/(L\Phi_j))/G((L\Phi)/(L\Phi_j))$ is a subgroup of the centralizer of $[G((L\Phi)/E)/G((L\Phi)/(L\Phi_j)), G((L\Phi)/E)/G((L\Phi)/(L\Phi_j)))$ in $G((L\Phi)/E)/G(L\Phi)/(L\Phi_j))$, such that $\Omega G((L\Phi)/(L\Phi_j))/G((L\Phi)/(L\Phi_j)) \cap G((L\Phi)/(L\Phi_{j'})G((L\Phi)/(L\Phi_j))/G((L\Phi)/(L\Phi_j)) = \{1\}$: $j' \neq j$, and the intersection of $\Omega G((L\Phi)/(L\Phi_j))/G((L\Phi)/(L\Phi_j))$ and $[G((L\Phi)/E)/G((L\Phi)/(L\Phi_j)), G((L\Phi)/E)/G((L\Phi)/(L\Phi_j)))$ is nontrivial. Since $G((L\Phi)/E)/G((L\Phi)/(L\Phi_j))$ is isomorphic to $G((L\Phi_j)/E)$, and $\Omega G((L\Phi)/(L\Phi_j))/G((L\Phi)/(L\Phi_j))$ is normal in $G((L\Phi)/E)/G((L\Phi)/(\Phi_j))$ our conclusion contradicts the inductive hypothesis and so proves the nontriviality of $\Omega_1, \dots, \Omega_s$. Note also that the groups $G((L\Phi)/(L\Phi_i))$, $i = 1, \dots, s$, are abelian. Suppose that $G((L\Phi)/(L\Phi_j))$ is nonabelian, for some j. Then $G((L\Phi)/(L\Phi_j))$ must equal its commutator subgroup, and possess a trivial centre, which necessitates that $\Omega_j = \{1\}$. The obtained contradiction proves the commutativity of $\Omega_1, \dots, \Omega_s$. Arguing in a similar way, one deduces from the inductive hypothesis that $\Omega_1, \dots, \Omega_s$ are elementary abelian p-groups, for some prime p. We show that one can assume without loss of generality that $[L : E]$ is not divisible by p. Let L_p be the extension of E in L corresponding to the Sylow p-subgroup of $G(L/E)$. It follows from Galois theory and the concluding assertion of Lemma 5.7 that $L\Phi = (L_p\Phi) \cdot L$ and $L_p\Phi \cap L = L_p$, whence $G((L\Phi)/E) \cong G((L_p\Phi)/E) \times G(L/L_p)$. This allows one to identify the commutator subgroups of $G((L\Phi)/E)$ and $G((L_p\Phi)/E)$, and thereby, to reduce our further considerations to the special case in which $L = L_p$, i.e. p does not divide $[L : E]$. Let now γ be an element of $G((L\Phi)/E)$ of order $[L : E]$, θ a nontrivial element of $\Omega \cap G((L\Phi)/L)$, such that the number of its nontrivial p-components is minimal, and μ, ν distinct indices for which $\theta_\mu \neq 1$ and $\theta_\nu \neq 1$. Lemma 6.2 (i), (ii) and the choice of θ indicate that $G((L\Phi)/(L\Phi_\mu))$ and $G((L\Phi)/(L\Phi_\nu))$ are of one and the same order, say k, the system $B_\kappa = \{\gamma^u \theta_\kappa \gamma^{-u} : u = 0, \dots, k-1\}$ is a basis of $G((L\Phi)/(L\Phi_\kappa))$ as a vector space over the field with p elements ($\kappa = \mu, \nu$), and the linear operators of $G((L\Phi)/(L\Phi_\mu))$ and $G((L\Phi)/(L\Phi_\nu))$, induced by γ by conjugation, have one and the same matrix with respect to B_μ and B_ν, respectively. This implies that the subgroups $G((L\Phi)/(L\Phi_\mu))\langle\gamma\rangle$ and $G((L\Phi)/(L\Phi_\nu))\langle\gamma\rangle$ are isomorphic. On the other hand, it is not difficult to see that the intersection $G((L\Phi)/(L\Phi_i))\langle\gamma\rangle$ and the product of the groups $G((L\Phi)/(L\Phi_{i'}))$: $i' \neq i$, is trivial, for each index i. This enables one to deduce that $G((L\Phi)/(L\Phi_i))\langle\gamma\rangle$ is isomorphic to $G((LF_i)/E)$, $\forall i$. Thus our argument leads to the conclusion that $G((LF_\mu)/E)$ and $G((LF_\nu)/E)$ are isomorphic. This, however, contradicts (6.3), and thereby, proves (6.4).

It is now easy to complete the proof of Lemma 6.2 (iii). Proceeding by induction on s, one obtains from (6.4) that the intermediate field of $(L\Phi)/E$ corresponding to Ω includes as a subfield the compositum of the fields F_i: $i =$

$1, \dots, s$. This, applied to the special case of $\Omega = \widetilde{H}$, leads to the conclusion that $G((F_i/E)$ is realizable as a quotient group of $G(\widetilde{U}/E)$, and therefore, must be an elements of χ, $\forall i$. The obtained contradiction is due to the assumption that $U \neq E$, so Lemma 6.2 (iii) is proved. □

Remark 6.3. (i) With assumptions and notations being as in Lemma 6.2, it is easily obtained from (6.4) and Galois theory that every nontrivial subgroup of $G((L\Phi)/L)$ which is normal in $G((L\Phi)/E)$ equals the product of some of the groups $G((L\Phi)/(L\Phi_i))$, $i = 1, \dots, s$.

(ii) Proceeding by induction on s, and applying Galois theory, (6.1) (i), (6.2) and the preceeding observation, one proves that the assumption of Lemma 6.2 that F_j is not a subfield of $L\Phi_j$, for any $j \geq s$, is superfluous.

Let now χ be a regular group formation, and $(P', A(\chi))$ some of the pairs $(\overline{P}, \mathrm{Ad}(\chi))$ or $(P, \mathrm{Ad}_P(\chi))$, where $\mathrm{Ad}(\chi)$ and $(P, \mathrm{Ad}_P(\chi))$ are defined as in Theorems 5.8 and 5.9, respectively. Fix a set $B(\chi) = \{G_i : i \in I\}$ of pairwise nonisomorphic finite groups satisfying the conditions of Lemma 5.7 so that, for each $\chi' \in \mathrm{Ad}(\chi)$, there is an index $i' \in I$ such that $G_{i'} \in (\chi' \setminus \chi)$, and take the fields E, F, Φ, Φ', Λ_i and Λ'_i, $i \in I$, in accordance with Lemma 3.4, assuming additionally that $\mathrm{char}(E) = 0$, $P_0(E) = P'$ and $P_0(\Phi') = \overline{P}$. Fixing an algebraically closed extension Θ of Φ' of sufficiently large transcendency degree, denote by $\widetilde{\Theta}$ the union of the fields V_n and Y_n, $n \in \mathbb{N}$, defined inductively in Θ, as follows:

(6.5) (i) V_n is an extension of Y_{n-1}, such that Y_{n-1} is algebraically closed in V_n and $N(V'_n/V_n) = V_n^*$, for each finite extension V'_n of V_n $(Y_0 := \Phi)$;

(ii) Y_n is an extension of V_n, such that $Y_n \cap (\Phi' V_n) = V_n$ and $Y_n \Phi'$ equals the compositum of all finite Galois extensions of V_n in Θ with Galois groups lying in χ.

It follows from the definition of $\widetilde{\Theta}$, Lemma 6.1 (iii) and Galois theory that the compositum of finite Galois extensions of $\widetilde{\Theta}$ with Galois groups in χ is equal to $\Phi'\widetilde{\Theta}$. One also sees that $\Lambda_i\widetilde{\Theta}$ is a Galois extension of $\widetilde{\Theta}$ with $G((\Lambda_i\widetilde{\Theta})/\widetilde{\Theta}) \cong G_i$, $\forall i \in I$, and $\Phi' \cap \widetilde{\Theta} = \Phi$. It is therefore clear that $P_0(\widetilde{\Theta}) = P'$ and each Henselian discrete valued field (K, v) with a residue field isomorphic to $\widetilde{\Theta}$ is strictly PQL. Observing now that if M/K is a finite Galois extension with $G(M/K) \in \chi$, then $G(M/K)$ is nilpotent, one obtains from Proposition 1.4 (i) that $G(M/K) = G(M_{\mathrm{ab}}/K)$. Also, it becomes clear that $N(M_i/K) \neq N(M_{i,\mathrm{ab}}/K)$: $i \in I$, where M_i is the inertial lift of Λ_i in Θ over K, $\forall i \in I$. Note finally that if $P' = P$, then $N(M_i/K) \neq N(M/K)$, since $K^* : N(M_i/K)|$ is divisible by at least one prime p_i not lying in P'. Summing up the obtained results, one concludes that K has the properties required by Theorem 5.8 or 5.9 depending on whether $A(\chi) = \mathrm{Ad}(\chi)$ or $A(\chi) = \mathrm{Ad}_P(\chi)$.

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Institute of Mathematics and Informtics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: `chipchak@math.bas.bg`

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