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INVARIANTS OF UNIPOTENT TRANSFORMATIONS ACTING ON NOETHERIAN RELATIVELY FREE ALGEBRAS

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ABSTRACT. The classical theorem of Weitzenböck states that the algebra of invariants $K[X]^g$ of a single unipotent transformation $g \in GL_m(K)$ acting on the polynomial algebra $K[X] = K[x_1, \dots, x_m]$ over a field K of characteristic 0 is finitely generated. This algebra coincides with the algebra of constants $K[X]^\delta$ of a linear locally nilpotent derivation δ of $K[X]$. Recently the author and C. K. Gupta have started the study of the algebra of invariants $F_m(\mathfrak{V})^g$ where $F_m(\mathfrak{V})$ is the relatively free algebra of rank m in a variety $\mathfrak{V}$ of associative algebras. They have shown that $F_m(\mathfrak{V})^g$ is not finitely generated if $\mathfrak{V}$ contains the algebra $UT_2(K)$ of 2×2 upper triangular matrices (and $g \neq 1$). The main result of the present paper is that the algebra $F_m(\mathfrak{V})^g$ is finitely generated if and only if the variety $\mathfrak{V}$ does not contain the algebra $UT_2(K)$. As a by-product of the proof we have established also the finite generation of the algebra of invariants T_{nm}^g where T_{nm} is the mixed trace algebra generated by m generic $n \times n$ matrices and the traces of their products.

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Introduction. Let K be any field of characteristic 0 and let $X = \{x_1, \dots, x_m\}$, where $m > 1$. Let $g \in GL_m = GL_m(K)$ be a unipotent linear operator acting on the vector space $KX = Kx_1 \oplus \cdots \oplus Kx_m$. By the classical theorem of Weitzenböck [16], the algebra of invariants

$$K[X]^g = \{f \in K[X] \mid f(g(x_1), \dots, g(x_m)) = f(x_1, \dots, x_m)\}$$

is finitely generated. A proof in modern language was given by Seshadri [12]. An elementary proof based on the ideas of [12] was presented by Tyc [14]. Since $g - 1$ is a nilpotent linear operator of KX, we may consider the linear locally nilpotent derivation

$$\delta = \log g = \sum_{i \geq 1} (-1)^{i-1} \frac{(g-1)^i}{i}$$

called a Weitzenböck derivation. (The K-linear operator δ acting on an algebra A is called a derivation if $\delta(uv) = \delta(u)v + u\delta(v)$ for all $u, v \in A$.) The algebra of invariants $\mathbb{C}[X]^g$ coincides with the algebra of constants $\mathbb{C}[X]^\delta$ $(= \ker(\delta)$). See the book by Nowicki [10] for a background on the properties of the algebras of constants of Weitzenböck derivations.

Looking for noncommutative generalizations of invariant theory, see e. g. the survey by Formanek [8], let $K\langle X\rangle = K\langle x_1, \dots, x_m\rangle$ be the free unitary associative algebra freely generated by X. The action of GL_m is extended diagonally on $K\langle X\rangle$ by the rule

$$h(x_{j_1} \cdots x_{j_n}) = h(x_{j_1}) \cdots h(x_{j_n}),\ h \in GL_m,\ x_{j_1}, \dots, x_{j_n} \in X.$$

For any PI-algebra R, let $T(R) \subset K\langle X\rangle$ be the T-ideal of all polynomial identities in m variables satisfied by R. The class $\mathfrak{V} = \mathrm{var}(R)$ of all algebras satisfying the identities of R is called the variety of algebras generated by R (or determined by the polynomial identities of R). The factor algebra $F_m(\mathfrak{V}) = K\langle X\rangle/T(R)$ is called the relatively free algebra of rank m in $\mathfrak{V}$. We shall use the same symbols x_j and X for the generators of $F_m(\mathfrak{V})$. Since $T(R)$ is GL_m-invariant, the action of GL_m on $K\langle X\rangle$ is inherited by $F_m(\mathfrak{V})$ and one can consider the algebra of invariants $F_m(\mathfrak{V})^G$ for any linear group G. As in the commutative case, if $g \in GL_m$ is unipotent, then $F_m(\mathfrak{V})^g$ coincides with the algebra $F_m(\mathfrak{V})^\delta$ of the constants of the derivation $\delta = \log g$.

Till the end of the paper we fix the integer $m > 1$, the variety $\mathfrak{V}$, the unipotent linear operator $g \in GL_m$ and the derivation $\delta = \log g$.

The author and C. K. Gupta [6] have started the study of the algebra of invariants $F_m(\mathfrak{V})^g$. They have shown that if $\mathfrak{V}$ contains the algebra $UT_2(K)$ of

2×2 upper triangular matrices and g is different from the identity of GL_m, then $F_m(\mathfrak{V})^g$ is not finitely generated for any $m > 1$. They have also established that, if $UT_2(K)$ does not belong to $\mathfrak{V}$, then, for $m = 2$, the algebra $F_2(\mathfrak{V})^g$ is finitely generated.

In the present paper we close the problem for which varieties $\mathfrak{V}$ and which m the algebra $F_m(\mathfrak{V})^g$ is finitely generated. Our main result is that this holds, and for all $m > 1$, if and only if the variety $\mathfrak{V}$ does not contain the algebra $UT_2(K)$.

It is natural to expect such a result by two reasons. First, it follows from the proofs of Seshadri [12] and of Tyc [14], see also the paper by Onoda [11], that the algebra $K[X]^g$ is isomorphic to the algebra of invariants of certain SL_2-action on the polynomial algebra in $m + 2$ variables. One can prove a similar fact for $F_m(\mathfrak{V})^g$ and $(K[y_1, y_2] \otimes_K F_m(\mathfrak{V}))^{SL_2}$. Second, the results of Vonessen [15], Domokos and the author [3] give that $F_m(\mathfrak{V})^G$ is finitely generated for all reductive G if and only if the finitely generated algebras in $\mathfrak{V}$ are one-side noetherian. For unitary algebras this means that $\mathfrak{V}$ does not contain $UT_2(K)$ or, equaivalently, $\mathfrak{V}$ satisfies the Engel identity $[x_2, x_1, \ldots, x_1] = 0$. In our proof we use the so called proper polynomial identities introduced by Specht [13], the fact that the Engel identity implies that the vector space of proper polynomials in $F_m(\mathfrak{V})$ is finite dimensional and hence $F_m(\mathfrak{V})$ has a series of ideals such that the factors are finitely generated $K[X]$-modules. As a by-product of the proof we have established also the finite generation of the algebra of invariants T_{nm}^g, where T_{nm} is the mixed trace algebra generated by m generic $n \times n$ matrices $x_1, \ldots, x_m$ and and the traces of their products $\mathrm{tr}(x_{i_1} \cdots x_{i_k})$, $k \geq 1$.

1. Preliminaries. We fix two finite dimensional vector spaces U and V, $\dim U = p$, $\dim V = q$, and representations of the infinite cyclic group $G = \langle g \rangle$:

$$\rho_U : G \to GL(U) = GL_p, \quad \rho_V : G \to GL(V) = GL_q,$$

where $\rho_U(g)$ and $\rho_V(g)$ are unipotent linear operators. Fixing bases $Y = \{y_1, \ldots, y_p\}$ and $Z = \{z_1, \ldots, z_q\}$ of U and V, respectively, we consider the free left $K[Y]$-module $M(Y, Z)$ with basis Z. Then g acts diagonally on $M(Y, Z)$ by the rule

$$g : \sum_{j=1}^{q} f_j(y_1, \ldots, y_p) z_j \to \sum_{j=1}^{q} f_j(g(y_1), \ldots, g(y_p)) g(z_j), \quad f_j \in K[Y],$$

where, by definition, $g(y_i) = \rho_U(g)(y_i)$ and $g(z_j) = \rho_V(g)(z_j)$. Let $M(Y, Z)^g$ be the set of fixed points in $M(Y, Z)$ under the action of g. Since $\rho_U(g)$ and $\rho_V(g)$

are unipotent operators, the operators $\delta_U = \log\rho_U(g)$ and $\delta_V = \log\rho_V(g)$ are well defined. Denote by δ the induced derivation of $K[Y]$. We extend δ to a derivation of $M(Y,Z)$, denoted also by δ, i. e. δ is the linear operator of $M(Y,Z)$ defined by

$$\delta : \sum_{j=1}^{q} f_j(Y)z_j \to \sum_{j=1}^{q} \delta(f_j(Y))z_j + \sum_{j=1}^{q} f_j(Y)\delta(z_j).$$

It is easy to see that $\delta = \log g$ on $M(Y,Z)$ and $M(Y,Z)^g$ coincides with the kernel of δ, i. e. the set of constants $M(Y,Z)^\delta$. Changing the bases of U and V, we may assume that δ_U and δ_V have the form

$$\delta_U = \begin{pmatrix} J_{p_1} & 0 & \cdots & 0 & 0 \\ 0 & J_{p_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{p_{k-1}} & 0 \\ 0 & 0 & \cdots & 0 & J_{p_k} \end{pmatrix}, \quad \delta_V = \begin{pmatrix} J_{q_1} & 0 & \cdots & 0 & 0 \\ 0 & J_{q_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{q_{l-1}} & 0 \\ 0 & 0 & \cdots & 0 & J_{q_l} \end{pmatrix},$$

where J_r is the $(r+1)\times(r+1)$ Jordan cell

$$J_r = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{1}$$

with zero diagonal.

We denote by W_r the irreducible $(r+1)$-dimensional SL_2-module. It is isomorphic to the SL_2-module of the forms of degree r in two variables x,y. This is the unique structure of an SL_2-module on the $(r+1)$-dimensional vector space which agrees with the action of δ (and hence of g) as the Jordan cell (1): We can think of δ as the derivation of $K[x,y]$ defined by $\delta(x)=0$, $\delta(y)=x$. We fix the "canonical" basis of W_r

$$u^{(0)} = x^r, u^{(1)} = \frac{x^{r-1}y}{1!}, u^{(2)} = \frac{x^{r-2}y^2}{2!}, \ldots, u^{(r-1)} = \frac{xy^{r-1}}{(r-1)!}, u^{(r)} = \frac{y^r}{r!}. \tag{2}$$

We give U and V the structure of SL_2-modules

$$U = W_{p_1} \oplus \cdots \oplus W_{p_k}, \quad V = W_{q_1} \oplus \cdots \oplus W_{q_l}, \tag{3}$$

and extend it on $K[Y]$ and $M(Y,Z)$ via the diagonal action of SL_2. Again, this agrees with the action of g and δ. Then $K[U]$ and $M(Y,Z)$ are direct sums of irreducible SL_2-modules $U_{ri} \subset K[Y]$ and $W_{rj} \subset M(Y,Z)$ isomorphic to W_r, $i,j = 1,2,\ldots$, $r = 0,1,2,\ldots$, with canonical bases $\{u_{ri}^{(0)}, u_{ri}^{(1)}, \ldots, u_{ri}^{(r)}\}$ and $\{w_{rj}^{(0)}, w_{rj}^{(1)}, \ldots, w_{rj}^{(r)}\}$, respectively.

Lemma 1. *The elements $u \in K[Y]$ and $w \in M(Y,Z)$ belong to $K[Y]^\delta$ and $M(Y,Z)^\delta$, respectively, if and only if they have the form*

$$u = \sum_{r,i} \alpha_{ri} u_{ri}^{(0)}, \quad w = \sum_{r,j} \beta_{rj} w_{rj}^{(0)}, \quad \alpha_{ri}, \beta_{rj} \in K. \tag{4}$$

Proof. Almkvist, Dicks and Formanek [1] translated in the language of g-invariants results of De Concini, Eisenbud and Procesi [2] and proved that, in our notation, $g(u) = u$ and $g(w) = w$ if and only if u and w have the form (4). Since $g(u) = u$ if and only if $\delta(u) = 0$, and similarly for w, we obtain that (4) holds if and only if u and w are δ-constants. (The same fact is contained in the paper by Tyc [14] but in the language of representations of the Lie algebra $sl_2(K)$.) □

In each component W_r of U in (3), using the basis (2), we define a linear operator d by

$$d(u^{(k)}) = (k+1)(r-k)u^{(k+1)}, \quad k = 0,1,2,\ldots,r,$$

i. e., up to multiplicative constants, d acts by $u^{(0)} \to u^{(1)} \to u^{(2)} \to \cdots \to u^{(r)} \to 0$. We extend d to a derivation of $K[Y]$. As in the case of δ, again we can think of d as the derivation of $K[x,y]$ defined by $d(x) = y$, $d(y) = 0$.

Lemma 2. (i) *The derivation d acts on each irreducible component U_{ri} of $K[Y]$ by*

$$d(u_{ri}^{(k)}) = (k+1)(r-k)u_{ri}^{(k+1)}, \quad k = 0,1,\ldots,r.$$

(ii) *If $f = f(Y) \in K[Y]$, then $\delta^{s+1}(f) = 0$ if and only if f belongs to the vector space*

$$K[Y]_s = \sum_{t=0}^{s} d^t(K[Y]^\delta). \tag{5}$$

Proof. Part (i) follows from the fact that the SL_2-action on U is the only action which agrees with the action of δ as well as with the action of d (as the derivations of $K[x,y]$ defined by $\delta(x) = 0$, $\delta(y) = x$ and $d(x) = y$, $d(y) = 0$, respectively), and the extension of this SL_2-action to $K[U]$ also agrees with the action of δ and d on $K[U]$.

(ii) Since the irreducible SL_2-submodules of $K[Y]$ are δ- and d-invariant, it is sufficient to prove the statement only for $f \in W_r \subset K[Y]$. Considering the basis (2) of W_r, we have that $\delta^{s+1}(f) = 0$ if and only if

$$f = \alpha_0 u^{(0)} + \alpha_1 u^{(1)} + \cdots + \alpha_s u^{(s)}, \quad \alpha_k \in K.$$

Since $W_r^\delta = Ku^{(0)}$ and $d^t(u^{(0)}) \in Ku^{(t)}$, we obtain that $W_r \cap K[Y]_s$ is spanned by $u^{(0)}, u^{(1)}, \dots, u^{(s)}$ and coincides with the kernel of δ^{s+1} in W_r. $\square$

In principle, the proof of the following proposition can be obtained following the main steps of the proof of Tyc [14] of the Weitzenböck theorem. The proof of the three main lemmas in [14] uses only the fact that the ideals of the algebra $K[Y]$ are finitely generated $K[Y]$-modules. Instead, we shall give a direct proof, using the idea of the proof of Lemma 3 in [14].

Proposition 3. *The set of constants $M(Y,Z)^\delta$ is a finitely generated $K[Y]^\delta$-module.*

Proof. Let N_i be the $K[Y]$-submodule of $M(Y,Z)$ generated by the basis elements z_j of $V = Kz_1 \oplus \cdots \oplus Kz_q$ corresponding to the i-th Jordan cell J_{q_i}. Since $M(Y,Z) = N_1 \oplus \cdots \oplus N_l$ and $M(Y,Z)^\delta = N_1^\delta \oplus \cdots \oplus N_l^\delta$, it is sufficient to show that each N_i^δ is a finitely generated $K[Y]^\delta$-module. Hence, without loss of generality we may assume that $q = r+1$ and $\delta(z_0) = 0$, $\delta(z_j) = z_{j-1}$, $j = 1, 2, \dots, r$. Let

$$f = f_0(Y)z_0 + f_1(Y)z_1 + \cdots + f_r(Y)z_r \in M(Y,Z)^\delta, \quad f_j(Y) \in K[Y]. \tag{6}$$

Then

$$\delta(f) = (\delta(f_0) + f_1)z_0 + (\delta(f_1) + f_2)z_1 + \cdots + (\delta(f_{r-1}) + f_r)z_{r-1} + \delta(f_r)z_r$$

and this implies that

$$\delta(f_j) = -f_{j+1}, \quad j = 0, 1, \dots, r-1,$$

$$\delta(f_r) = \delta^2(f_{r-1}) = \cdots = \delta^r(f_1) = \delta^{r+1}(f_0) = 0.$$

Hence, fixing any element $f_0(Y)$ from $K[Y]_r$, we determine all the coefficients $f_1, \dots, f_r$ from (6). By Lemma 2 it is sufficient to show that the $K[Y]^\delta$-module generated by $d^t(K[Y]^\delta)$ is finitely generated. By the theorem of Weitzenböck, $K[Y]^\delta$ is a finitely generated algebra. Let $\{h_1, \dots, h_n\}$ be a generating set of $K[Y]^\delta$. Then $d^t(K[Y]^\delta)$ is spanned by the elements $d^t(h_1^{a_1} \cdots h_n^{a_n})$. Since d is a derivation, $d^t(K[Y]^\delta)$ is spanned by elements of the form

$$h_1^{c_1} \cdots h_n^{c_n} \left(\prod d^{t_{i_1}}(h_1)\right) \cdots \left(\prod d^{t_{i_n}}(h_n)\right), \quad \sum t_{i_1} + \cdots + \sum t_{i_n} = t.$$

There is only a finite number of possibilities for $t_{i_1},\dots,t_{i_n}$, and we obtain that $d^t(K[Y]^\delta)$ generates a finitely generated $K[Y]^\delta$-module. $\square$

Corollary 4. *Let, in the notation of this section, U and V be polynomial GL_m-modules, let $g\in GL_m$ be a unipotent matrix and let $M(Y,Z)$ be equipped with the diagonal action of GL_m. Then, for every GL_m-submodule M_0 of $M(Y,Z)$, the natural homomorphism $M(Y,Z)\to M(Y,Z)/M_0$ induces an epimorphism $M(Y,Z)^g\to (M(Y,Z)/M_0)^g$, i. e. we can lift the g-invariants of $M(Y,Z)/M_0$ to g-invariants of $M(Y,Z)$.*

Proof. The lifting of the constants was established in [6] in the case of relatively free algebras and the same proof works in our situation. Since U and V are polynomial GL_m-modules, the module $M(Y,Z)$ is completely reducible. Hence $M(Y,Z)=M_0\oplus M'$ for some GL_m-submodule M' of $M(Y,Z)$ and $M(Y,Z)/M_0\cong M'$. If $w+M_0=\overline{w}\in(M(Y,Z)/M_0)^g$, then $w=w_0+w'$, $w_0\in M_0$, $w'\in M'$, and $g(w)=g(w_0)+g(w')$. Since $g(\overline{w})=\overline{w}$, we obtain that $g(w')=w'$ and the g-invariant $\overline{w}$ is lifted to the g-invariant w'. $\square$

Remark 5. The proof of Proposition 3 gives also an algorithm to find the generators of $M(Y,Z)^\delta$ in terms of the generators of $K[Y]^\delta$. The latter problem is solved by van den Essen [7] and his algorithm uses Gröbner bases techniques.

2. The main results. The following theorem is the main result of our paper. For $m=2$ it was established in [6] using the description of the g-invariants of $K\langle x,y\rangle$.

Theorem 6. *For any variety $\mathfrak{V}$ of associative algebras which does not contain the algebra $UT_2(K)$ of 2×2 upper triangular matrices, the algebra of invariants $F_m(\mathfrak{V})^g$ of any unipotent $g\in GL_m$ is finitely generated.*

Proof. We shall work with the linear locally nilpotent derivation $\delta=\log g$ instead with g.

It is well known that any variety $\mathfrak{V}$ which does not contain $UT_2(K)$ satisfies some Engel identity $[x_2,x_1,\dots,x_1]=0$. By the theorem of Zelmanov [17] any Lie algebra over a field of characteristic zero satisfying the Engel identity is nilpotent. Hence we may assume that $\mathfrak{V}$ satisfies the polynomial identity of Lie nilpotency $[x_1,\dots,x_{c+1}]=0$. (Actually, this follows from much easier and much earlier results on PI-algebras.)

Let us consider the vector space $B_m(\mathfrak{V})$ of so called proper polynomials in $F_m(\mathfrak{V})$. It is spanned by all products $[x_{i_1},\dots,x_{i_k}]\cdots[x_{j_1},\dots,x_{j_l}]$ of commutators of length ≥ 2. One of the main results of the paper by the author [4] states that if $\{f_1, f_2, \ldots\}$ is a basis of $B_m(\mathfrak{V})$, then $F_m(\mathfrak{V})$ has a basis

$$\{x_1^{p_1} \cdots x_m^{p_m} f_i \mid p_j \geq 0, i = 1, 2, \ldots\}.$$

Let $B_m^{(k)}(\mathfrak{V})$ be the homogeneous component of degree k of $B_m(\mathfrak{V})$. It follows from the proof of Theorem 5.5 in [4], that for any Lie nilpotent variety $\mathfrak{V}$, and for a fixed positive integer m, the vector space $B_m(\mathfrak{V})$ is finite dimensional. Hence $B_m^{(n)}(\mathfrak{V}) = 0$ for n sufficiently large, e. g. for $n > n_0$. Let I_k be the ideal of $F_m(\mathfrak{V})$ generated by $B_m^{(k+1)}(\mathfrak{V}) + B_m^{(k+2)}(\mathfrak{V}) + \cdots + B_m^{(n_0)}(\mathfrak{V})$. Since $wx_i = x_i w + [w, x_i]$, $w \in F_m(\mathfrak{V})$, applying Lemma 2.4 [4], we obtain that I_k/I_{k+1} is a free left $K[X]$-module with any basis of the vector space $B_m^{(k)}(\mathfrak{V})$ as a set of free generators. Since δ is a nilpotent linear operator of $U = KX = Kx_1 \oplus \cdots \oplus Kx_m$, it acts also as a nilpotent linear operator of $V_k = B_m^{(k)}(\mathfrak{V})$. Proposition 3 gives that $(I_k/I_{k+1})^\delta$ is a finitely generated $K[X]^\delta$-module. Clearly, $B_m^{(0)}(\mathfrak{V}) = K$, $B_m^{(1)}(\mathfrak{V}) = 0$, $B_m^{(2)}(\mathfrak{V})$ is spanned by the commutators $[x_{i_1}, x_{i_2}]$, etc. Hence $I_0/I_1 \cong K[X]$ and by the theorem of Weitzenböck $(I_0/I_1)^\delta$ is a finitely generated algebra. We fix a system of generators $\overline{f}_1, \ldots, \overline{f}_a$ of the algebra $(I_0/I_1)^\delta$ and finite sets of generators $\{\overline{f}_{k1}, \ldots, \overline{f}_{kb_k}\}$ of the $K[X]^\delta$-modules $(I_k/I_{k+1})^\delta$, $k = 2, 3, \ldots, n_0$. The vector space U is a GL_m-module and its GL_m-action makes V_k a polynomial GL_m-module. We apply Corollary 4 and lift all $\overline{f}_i$ and $\overline{f}_{kj}$ to some δ-constants $f_i, f_{kj} \in F_m(\mathfrak{V})^\delta$. The algebra S generated by $f_1, \ldots, f_a$ maps onto $(I_0/I_1)^\delta$ and hence $(I_k/I_{k+1})^\delta$ is a left S-module generated by $\overline{f}_{k1}, \ldots, \overline{f}_{kb_k}$. The condition $I_{n_0+1} = 0$ together with Corollary 4 gives that the f_i and f_{kj} generate $F_m(\mathfrak{V})^\delta$. $\square$

Together with the results of [6] Theorem 6 gives immediately:

Corollary 7. *For $m \geq 2$ and for any fixed unipotent operator $g \in GL_m$, $g \neq 1$, the algebra of g-invariants $F_m(\mathfrak{V})^g$ is finitely generated if and only if $\mathfrak{V}$ does not contain the algebra $UT_2(K)$.*

We refer to the books [9] and [5] for a background on the theory of matrix invariants. We fix an integer $n > 1$ and consider the generic $n \times n$ matrices $x_1, \ldots, x_m$. Let C_{nm} be the pure trace algebra, i. e. the algebra generated by the traces of products $\mathrm{tr}(x_{i_1} \cdots x_{i_k})$, $k = 1, 2, \ldots$, and let T_{nm} be the mixed trace algebra generated by $x_1, \ldots, x_m$ and C_{nm}. It is well known that C_{nm} is finitely generated. (For an explicit set of generators, the Nagata-Higman theorem states that the nil polynomial identity $x^n = 0$ implies the identity of nilpotency $x_1 \cdots x_d = 0$. If $d(n)$ is the minimal d with this property, one may take as generators $\mathrm{tr}(x_{i_1}\cdots x_{i_k})$ with $k \le d(n)$.) Also, T_{nm} is a finitely generated C_{nm}-module.

Theorem 8. *For any unipotent operator $g \in GL_m$, the algebra T_{nm}^g is finitely generated.*

Proof. Consider the vector space U of all formal traces $y_i = \mathrm{tr}(x_{i_1}\cdots x_{i_k})$, $i_j = 1,\dots,m$, $1 \le k \le d(n)$. Let Y be the set of all y_i. It has a natural structure of a GL_m-module and hence $\delta = \log g$ acts as a nilpotent linear operator on U. Also, consider a finite system of generators $f_1,\dots,f_a$ of the C_{nm}-module T_{nm}. We may assume that the f_j do not depend on the traces and fix some elements $h_j \in K\langle X\rangle$ such that $h_j \to f_j$ under the natural homomorphism $K\langle X\rangle \to T_{nm}$ extending the mapping $x_i \to x_i$, $i = 1,\dots,m$. Let V be the GL_m-submodule of $K\langle X\rangle$ generated by the h_j. Again, δ acts as a nilpotent linear operator on V. We fix a basis $Z = \{z_1,\dots,z_q\}$ of V. Consider the free $K[Y]$-module $M(Y,Z)$ with basis Z. Proposition 3 gives that $M(Y,Z)^\delta$ is a finitely generated $K[Y]^\delta$-module and the theorem of Weitzenböck implies that $K[Y]^\delta$ is a finitely generated algebra. Since the algebra C_{nm} and the C_{nm}-module T_{nm} are homomorphic images of the algebra $K[Y]$ and the $K[Y]$-module $M(Y,Z)$, Corollary 4 gives that $K[Y]^\delta$ and $M(Y,Z)^\delta$ map on C_{nm}^δ and T_{nm}^δ, respectively. Hence T_{nm}^δ is a finitely generated module of the finitely generated algebra C_{nm}^δ and, therefore, the algebra T_{nm}^δ is finitely generated. $\square$

REFERENCES

[1] G. Almkvist, W. Dicks, E. Formanek. Hilbert series of fixed free algebras and noncommutative classical invariant theory. *J. Algebra* **93** (1985), 189–214.

[2] C. De Concini, D. Eisenbud, C. Procesi. Young diagrams and determinantal varieties. *Invent. Math.* **56** (1980), 129–165.

[3] M. Domokos, V. Drensky. A Hilbert-Nagata theorem in noncommutative invariant theory. *Trans. Amer. Math. Soc.* **350** (1998), 2797–2811.

[4] V. Drensky. Codimensions of T-ideals and Hilbert series of relatively free algebras. *J. Algebra* **91** (1984), 1–17.

[5] V. Drensky, E. Formanek. Polynomial Identity Rings, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser, Basel-Boston, 2004 (to appear).

[6] V. Drensky, C. K. Gupta. Constants of Weitzenböck derivations and invariants of unipotent transformations acting on relatively free algebras. Preprint.

[7] A. van den Essen. An algorithm to compute the invariant ring of a G_a-action on an affine variety. *J. Symbolic Computation* **16** (1993), 551–555.

[8] E. Formanek. Noncommutative invariant theory. *Contemp. Math.* **43** (1985), 87–119.

[9] E. Formanek. The Polynomial Identities and Invariants of $n \times n$ Matrices. CBMS Regional Conf. Series in Math. **78**, Published for the Confer. Board of the Math. Sci. Washington DC, AMS, Providence RI, 1991.

[10] A. Nowicki. Polynomial Derivations and Their Rings of Constants. Uniwersytet Mikolaja Kopernika, Torun, 1994, http://www-users.mat.uni.torun.pl/~anow/ploder.html.

[11] N. Onoda. Linear actions of G_a on polynomial rings. Proceedings of the 25th Symposium on Ring Theory (Matsumoto, 1992), 11-16, Okayama Univ., Okayama, 1992.

[12] C. S. Seshadri. On a theorem of Weitzenböck in invariant theory. *J. Math. Kyoto Univ.* **1** (1962), 403–409.

[13] W. Specht. Gesetze in Ringen, I. *Math. Z.* **52** (1950), 557–589.

[14] A. Tyc. An elementary proof of the Weitzenböck theorem. *Colloq. Math.* **78** (1998), 123–132.

[15] N. Vonessen. Actions of Linearly Reductive Groups on Affine PI-Algebras. *Mem. Amer. Math. Soc.* **414**, 1989.

[16] R. Weitzenböck. Über die Invarianten von linearen Gruppen. *Acta Math.* **58** (1932), 231–293.

[17] E. I. Zelmanov. On Engel Lie algebras. Sibirsk. Mat. Zh. **29**, 5 (1988), 112-117 (in Russian); English translation: *Sib. Math. J.* **29** (1988), 777–781.

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