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SYMMETRIC AND ASYMMETRIC GAPS IN SOME FIELDS OF FORMAL POWER SERIES

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Abstract. We consider non-archimedean real closed fields of cardinality $\aleph_1$ that have special type of symmetric gaps and compare these fields with well known η_1-fields (Hausdorff), semi-η_1-fields, and some super-real fields (Dales, Woodin). All these fields are realized as fields of formal power series. We describe all symmetric Dedekind and non-Dedekind gaps of semi-η_1-fields (in particular, for a nonstandard real line). We consider a construction of fields with symmetric gaps that are not semi-η_1. By this construction we give examples of fields with different asymmetric gaps.

1. Introduction. Throughout this paper we consider non-archimedean real closed totally ordered fields of cardinality $\aleph_1$. One of the directions for investigation of the totally ordered fields is gap (cut) theory. We will follow [6]. A pair (A, B) of non-empty subsets A, B of a field $(F, +, \cdot, <)$ is called a gap if

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$A < B$ (i. e., $\forall a \in A \forall b \in B : a < b$) and $A \cup B = F$. The set A is called a *short-shore* of a gap (A, B) in F if there exists $a_0 \in A$ such that for all $a \in A$, we have $a + (a - a_0) \in A$ (the "distance" between a_0 and every $a \in A$ is much less than "distance" between a_0 and B). If a shore is not short then it is called a *long shore*. If both A and B are long then (A, B) is called a *symmetric gap*. If one of the shores is long and the other one is short then (A, B) is called an *asymmetric gap*. Note that $(short, short)$-gap is impossibly (see [6] for details).

A subset $H \subset L$ is said to be cofinal (coinitial) in a totally ordered set L if $\forall l \in L \exists h \in H$ such that $l \leq h (l \geq h)$. $\min\{\mathrm{card}(H) |\ H$ is cofinal (coinitial) in $L\}$ is called a cofinality (coinitiality) of L and is denoted $\mathrm{cf}(L)$($coi(L)$). A gap (A, B) of F is said to have (α, β)*-type* if $\mathrm{cf}(A) = \alpha$ and $\mathrm{coi}(B) = \beta$ (see [1]).

Note that if (A, B) is a symmetric gap then $\mathrm{cf}(A) = \mathrm{coi}(B)$; the cardinal $\mathrm{cf}(A)$ is called *cofinality of* (A, B) and is denoted by $\mathrm{cf}(A, B)$ [6].

For x, y in a field $F \backslash \{0\}$, let $x \sim y$ if $\exists n \in \mathbb{N}$ such that $|x| \leq n|y|$ and $|y| \leq n|x|$. Let $\widehat{F}$ be the set of equivalence classes of F mod $\sim$. Let $\widehat{x} \in \widehat{F}$ such that $x \in \widehat{x}$. Define $x \ll y$ if $\forall n \in \mathbb{N}\ n|x| < |y|$ and $\widehat{x} < \widehat{y} \Leftrightarrow x \ll y$. Put $\widehat{x} \cdot \widehat{y} = \widehat{x \cdot y}$. So, we have $(\widehat{F}, \cdot, <)$ is a totally ordered group. $\widehat{F}$ is called *the group of archimedean classes of* F [6] or *the value group of* F [1]. If $x, y \in F \setminus \{0\}$ and $x + y \neq 0$ then $\widehat{x + y} = \max\{\widehat{x}, \widehat{y}\}$.

Hausdorff has introduced a notion η_1*-set*: a totally ordered set L is called an η_1-set if $\forall A, B \subseteq L$ such that $A < B$ with $|A \cup B| < \aleph_1$ there exists $t \in L$ with $A < t < B$.

There are two "isomorphism theorems" for real closed fields. A classical theorem of Erdös, Gilman and Henriksen [1] states that any two real closed fields that are η_1-sets of cardinality $\aleph_1$ are ordered isomorphic. This theorem is equivalent to CH [5]. Pestov introduced a notion of symmetric gap and proved the following isomorphism theorem:

Theorem 1.1 [6]. *Let F_1 and F_2 be really closed ordered fields such that* $\mathrm{card}(F_1) = \mathrm{card}(F_2) = \aleph_1$ *and cofinality of each symmetric gap in both fields is* $\aleph_1$. *Then F_1 and F_2 are isomorphic as ordered fields iff the groups of archimedean classes of both fields are order-isomorphic.*

In [4] we considered a class $\mathcal{K}$ of real closed fields to which Pestov's isomorphism theorem applies. A real closed field $F \in \mathcal{K}$ if

1) $\mathrm{card}(F) = \mathrm{card}(\widehat{F}) = \aleph_1$,

2) if (A, B) is a symmetric gap of F then $cf(A, B) = \aleph_1$.

Note that by the Theorem 1.1 every two fields from this class are isomorphic iff the groups of archimedean classes of the fields are isomorphic.

In section 2 we investigate asymmetric gaps of special fields from the class $\mathcal{K}$ and show that the class $\mathcal{K}$ is strictly wider then a class of all η_1- fields of cardinality $\aleph_1$. We consider also examples of fields from the class $\mathcal{K}$ that have an *asymmetric* $(\aleph_1, \aleph_1)$ gap (in particular, a nonstandard real line).

In section 3 we consider Dedekind and non-Dedekind symmetric gaps of fields with cofinality $\aleph_0$ from $\mathcal{K}$ (in particular, semi-η_1-fields); prove that the class $\mathcal{K}$ is wider then a class of all semi-η_1- fields of cardinality $\aleph_1$; using [1] we show that super real fields are in our class $\mathcal{K}$.

2. On asymmetric gaps in some fields of formal power series.

By definitions of symmetric gap and η_1-set, we evidently have the following

Proposition 2.1. *F is an η_1-field iff each gap (A, B) of F has only one of the following types $(\aleph_1, \aleph_1)$, $(\aleph_0, \aleph_1)$, $(\aleph_1, \aleph_0)$, $(1, \aleph_1)$, $(\aleph_1, 1)$ and $\mathrm{cf}(F) = \aleph_1$.*

In [4] it was shown that if F is a totally ordered real closed η_1-field with $\mathrm{card}(F) = \aleph_1$ then $F \in \mathcal{K}$.

Our aim here is to show that the class $\mathcal{K}$ is strictly wider then the class of all η_1- fields of cardinality $\aleph_1$. To this end we give examples of fields from the class $\mathcal{K}$ with $(\aleph_0, \aleph_0)$-asymmetric gaps. Note that any symmetric gap of $F \in \mathcal{K}$ has type $(\aleph_1, \aleph_1)$.

Denote by $\mathbb{R}[[G]]$ a field of formal power series $x = \sum\limits_{g \in G} r_g g$, where $r_g \in \mathbb{R}, \mathrm{supp}(x) = \{g \in G | r_g \neq 0\}$ is inversely well-ordered subset of a totally ordered group G (i. e., each subsets of $\mathrm{supp}(x)$ has a maximal element). The order in $\mathbb{R}[[G]]$ is as follows: $x > 0$ if $r_\gamma > 0$, where $\gamma = \max \mathrm{supp}(x)$. Let β be a regular cardinal with $\aleph_0 < \beta \leq \mathrm{card}(G)$. By $R[[G, \beta]]$ is denoted a subfield of $\mathbb{R}[[G]]$, which consists of such formal power series x that $\mathrm{card}(\mathrm{supp}(x)) < \beta$ (*the field of bounded formal power series*)[1, 2].

If G is divisible group then $R[[G, \beta]]$ are real-closed fields; if $\mathrm{card}(G) \geq \mathbf{c}$ then $\mathrm{card}(\mathbb{R}[[G, \aleph_1]]) = \mathrm{card}(G)$; if $\mathrm{card}(G) \geq \aleph_0$ then $2^{\mathrm{cf}(G)} \leq \mathrm{card}(\mathbb{R}[[G]]) \leq 2^{\mathrm{card}(G)}$ (see [1]). So, if $\aleph_1 = \mathbf{c} = \mathrm{card}(G)$, we have $\mathrm{card}(\mathbb{R}[[G, \aleph_1]]) = \mathrm{card}(G) = \aleph_1$.

We assume CH for the following description of $\mathcal{K}$ by means of fields of bounded formal power series[4, 3]: the class $\mathcal{K}$ coincides with a class of all fields of bounded formal power series $\mathbb{R}[[G, \aleph_1]]$, where G is a totally ordered divisible Abelian group and $\mathrm{card}(G) = \aleph_1$. The cofinality of a field $F \in \mathcal{K}$ and the cofinality of its group of archimedean classes are the same.

Let our class $\mathcal{K} = \mathcal{K}^0 \cup \mathcal{K}^1$, where $F \in \mathcal{K}^i$ if $\mathrm{cf}(F) = \aleph_i$ $(i \in \{0; 1\})$.

Proposition 2.2. *If $F \in \mathcal{K}^1$ then F has a symmetric gap.*

Proof. If $F \in \mathcal{K}^1$ then $F \cong \mathbb{R}[[\widehat{F}, \aleph_1]] \subset \mathbb{R}[[\widehat{F}]]$ and under CH, $\mathrm{card}(\mathbb{R}[[\widehat{F}, \aleph_1]]) = \aleph_1 < \mathrm{card}(\mathbb{R}[[\widehat{F}]]) = 2^{\aleph_1}$. Hence $\mathbb{R}[[\widehat{F}]] \setminus \mathbb{R}[[\widehat{F}, \aleph_1]] \neq \emptyset$ and F has a symmetric gap (see Proposition 2.1 from [3]). □

Lemma 2.1. *Let F be a real closed field. (A, B) is an (α, β) gap in $\widehat{F}$, where α, β are infinite regular cardinals. Then there exists an asymmetric (α, β) gap in F.*

Proof. Let $A_1 = \{x \in F | \ \exists\ g \in A, \hat{x} < g\}$, $B_1 = \{x \in F | \ \exists\ g \in B,\ g < \hat{x}\}$. If $x \in F$ then $\hat{x} \in \widehat{F} = A \bigcup B$. If $\hat{x} \in A$ then $\exists\ g \in A$ (there is no the last element in A_1 because of $\mathrm{cf}(A) = \alpha$ is infinite) such that $\hat{x} < g$. Thus $x \in A_1$. By the same argument, if $\hat{x} \in B$ then $\exists\ g \in B$ such that $g < \hat{x}$ and $x \in B_1$. It is obvious that $\mathrm{cf}(A) = \mathrm{cf}(A_1) = \alpha, \mathrm{coi}(B) = \mathrm{coi}(B_1) = \beta$. Hence (A_1, B_1) is a (α, β) gap in F. Let $x_0 \in A_1 \subset F$ and $x_0 < x,\ x \in A_1$. Consider $x + (x - x_0) = 2x - x_0$. We have $\widehat{2x - x_0} = \max\{\widehat{x}, \widehat{x_0}\} = \hat{x} \in A$. Therefore $x + (x - x_0) \in A_1$ and (A_1, B_1) is asymmetric. □

Now we remaind the construction of a group $(G(L, P), \cdot, <)$ with $\mathbb{R}[[G(L, P), \aleph_1]] \in \mathcal{K}$ [4]. Let L be a totally ordered set and $\mathrm{cf}(L) \geqslant \aleph_0$. Let P be a totally ordered infinite field and $\max\{|L|, |P|\} = \aleph_1$. The totally ordered Abelian divisible group $(G(L, P), \cdot, <)$ is as follows: $G(L, P) = \{(t_{i_1}^{r_{i_1}} t_{i_2}^{r_{i_2}} \cdots t_{i_n}^{r_{i_n}}) \mid t_{i_j} \in L, r_{i_j} \in P, j = \overline{1, n},\ n \in \mathbb{N}\}$. We suppose that $t_{i_1} > t_{i_2} > \cdots > t_{i_n}$ and for given element $t_1 \in L$, let $t_1^r = 1\ \forall r \in P$. Put $(t_{i_1}^{r_{i_1}}) \cdot (t_{i_1}^{q_{i_1}}) = (t_{i_1}^{r_{i_1} + q_{i_1}})$ and $(t_{i_1}^{r_{i_1}}) \cdot (t_{i_2}^{r_{i_2}}) = (t_{i_1}^{r_{i_1}} t_{i_2}^{r_{i_2}})$. For example, $(t_{i_1}^{1/2} t_{i_2}^{-1}) \cdot (t_{i_1} t_{i_2} t_{i_3}) = (t_{i_1}^{3/2} t_{i_3})$. Let $g_1 = (t_{i_1}^{r_{i_1}} \cdots t_{i_k}^{r_{i_k}})$ by definition, put $g_1 < g_2 \Leftrightarrow g_1 g_2^{-1} < 1$ and $g_1 < 1 \Leftrightarrow r_{i_1} < 0$. For example, we compare $(t_{i_1}^3 t_{i_2}^{-2} t_{i_3}^5)$ and $(t_{i_1}^3 t_{i_3})$. We have $(t_{i_1}^3 t_{i_2}^{-2} t_{i_3}^5) \cdot (t_{i_1}^{-3} t_{i_3}^{-1}) = t_{i_2}^{-2} t_{i_3}^4,\ t_{i_2} > t_{i_3},\ -2 < 0$ hence $(t_{i_1}^3 t_{i_2}^{-2} t_{i_3}^5) < (t_{i_1}^3 t_{i_3})$.

Note that the group $G(L, P)$ is isomorphic to the subgroup of finite sums $(P[[L, \aleph_0]], +, <)$ of the group of formal power series $P[[L, \aleph_1]]$. We show here that the group $P[[L, \aleph_0]]$ has an $(\aleph_0, \aleph_0)$ gap and so it is not a η_1-set.

Theorem 2.1. *The group $G(L, P) \cong P[[L, \aleph_0]]$ has an $(\aleph_0, \aleph_0)$ gap.*

Proof. Since $\aleph_0 \leqslant \mathrm{cf}(L) \leqslant \aleph_1$ there exists a sequence $\{q_n\}_{n \in \mathbb{N}} \subset P[[L, \aleph_0]]$ such that $q_1 \gg q_2 \gg \cdots q_n \gg \cdots$ i.e. $\forall n \in N\ \forall i \in \mathbb{N}\ q_{i+1} \cdot n < q_i$. Let $\forall k \in \mathbb{N}$

$$a_k = q_1 + q_2 + \cdots + q_k; b_k = q_1 + q_2 + \cdots + q_{k-1} + \frac{2}{3} q_k.$$

$$A := \{g \in P[[L, \aleph_0]] | \ \exists n \in N \quad g < a_n\},\ B := \{g \in P[[L, \aleph_0]] | \ \exists n \in N \quad g > b_n\}.$$

Let us show that (A, B) is a gap in $P[[L, \aleph_0]]$. Suppose that there exists $c \in P[[L, \aleph_0]]$ such that $\forall k \in N \quad a_k < c < b_k$. Let

$$c = \gamma_1 h_1 + \gamma_2 h_2 + \cdots + \gamma_{k_0} h_{k_0}; h_i \in L, \gamma_i \in P.$$

We clime that $c = a_{k_0}$. Indeed for $k = 1$, we have

$$a_1 < c < b_1;\ q_1 < \gamma_1 h_1 + \gamma_2 h_2 + \cdots + \gamma_{k_0} h_{k_0} < \frac{2}{3} q_1 \ \Rightarrow\ h_1 = q_1.$$

For $k = 2$, we have

$$a_2 < c < b_2;\ q_1 + q_2 < \gamma_1 q_1 + \gamma_2 h_2 + \cdots + \gamma_{k_0} h_{k_0} < q_1 + \frac{2}{3} q_2 \ \Rightarrow$$

$$\Rightarrow\ q_2 < (\gamma_1 - 1) q_1 + \gamma_2 h_2 + \cdots + \gamma_{k_0} h_{k_0} < \frac{1}{2} q_2 \ \Rightarrow$$

$$\Rightarrow\ \gamma_1 = 1,\ h_2 = q_2 \ \Rightarrow\ c = q_1 + \gamma_2 q_2 + \gamma_3 h_3 + \cdots + \gamma_{k_0} h_{k_0}.$$

If for $k = n$
$c = q_1 + q_2 + \cdots + q_{n-1} + \gamma_n q_n + \gamma_{n+1} h_{n+1} + \cdots + \gamma_{k_0} h_{k_0}$ is true then for $k = n+1$, we will have

$$a_{n+1} < c < b_{n+1};\quad q_1 + q_2 + \cdots + q_{n+1} <$$

$$< q_1 + q_2 + \cdots + q_{n-1} + \gamma_n q_n + \gamma_{n+1} h_{n+1} + \cdots + \gamma_{k_0} h_{k_0} < q_1 + q_2 + \cdots + \frac{1}{2} q_{n+1} \Rightarrow$$

$$\Rightarrow q_{n+1} < (\gamma_n - 1) q_n + \gamma_{n+1} h_{n+1} + \cdots + \gamma_{k_0} h_{k_0} < \frac{1}{2} q_{n+1} \Rightarrow$$

$$\Rightarrow \gamma_n = 1, h_{n+1} = q_{n+1} \Rightarrow c = q_1 + q_2 + \cdots + q_n + \gamma_{n+1} q_{n+1} + \gamma_{n+2} h_{n+2} + \cdots + \gamma_{k_0} h_{k_0}.$$

So, by induction, $c = a_{k_0}$. It is a contradiction. □

Corollary 2.1. *Field* $\mathbb{R}[[G(L,P), \aleph_1]]$ *has an* $(\aleph_0, \aleph_0)$ *asymmetric gap.*

Proof. By Lemma 2.1, the gap (A, B) of $G(L, P)$ from the proof of Theorem 2.1 generates the gap $(\acute{A}, \acute{B})$ in the field $\mathbb{R}[[G(L,P), \aleph_1]]$, where

$$\acute{A} = \{x \in \mathbb{R}[[G(L,P), \aleph_1]] | \exists g \in A, x < 1 \cdot g\},$$

$$\acute{B} = \{x \in \mathbb{R}[[G(L,P), \aleph_1]] | \exists g \in B, 1 \cdot g < x\}$$

and this gap also has type $(\aleph_0, \aleph_0)$. □

Corollary 2.2. *Group* $G(L, P)$ *and field* $\mathbb{R}[[G(L,P), \aleph_1]]$ *are not* η_1*-sets.*

We consider also examples of fields from the class $\mathcal{K}$ that have an *asymmetric* $(\aleph_1,\aleph_1)$ gap.

Let S and T be two totally ordered sets. Then $S\odot T$ denotes the totally ordered set which is 'S followed by T': $\forall s\in S \forall t\in T\ s<t$ [1].

1) Let $L=\omega_1\odot\omega_1^*\odot\omega_1^{**}$, where ω_1 and ω_1^{**} are two copies of the ordinal ω_1; ω_1^* is the ordinal ω_1 with the inverse of the usual order. So the set $\omega_1\odot\omega_1^*$ give us the $(\aleph_1,\aleph_1)$ gap in L, which generates the $(\aleph_1,\aleph_1)$ asymmetric gap in the field $\mathbb{R}[[G(L,P),\aleph_1]]\in\mathcal{K}$.

2) Now we consider a non-standard real line $^*\mathbb{R}$, which is an ultrapower of $\mathbb{R}$ by an $\aleph_1$-good ultrafilter over $\mathbb{N}$. It is known [1] that $^*\mathbb{R}$ is η_1-field and it is order-isomorphic to the field $\mathbb{R}[[\mathbf{G},\aleph_1]]$ of bounded formal power series with $\mathbf{G}=\mathbb{R}[[\mathbf{Q},\aleph_1]]$ and $\mathbf{Q}$ is Sierpinski's set. $\mathbf{Q}$ consists of dyadic sequences $\alpha=(\alpha_\tau)_{\tau<\omega_1}$ with lexicographic order such that $\{\tau<\omega_1:\alpha_\tau=1\}$ is non-empty and has a largest member.

Describe a $(\aleph_1,\aleph_1)$ gap in $\mathbf{Q}$. Let $(a^\sigma)_{\sigma<\omega_1}$ be a sequence in $\mathbf{Q}$ such that
$$a^\sigma(\tau)=\begin{cases}0, & \tau>\sigma\vee\tau \text{ is "even"};\\ 1, & \tau \text{ is "odd"}\vee\tau\text{ is limit.}\end{cases}$$

So, we have $a^1=(100\dots0\dots)$, $a^2=(101000\dots0\dots)$, $a^3=(10101000\dots 0\dots),\dots,a^\omega=(10101000\dots|_\omega1000\dots)$, $a^{\omega+1}=(10101000\dots|_\omega101000\dots),\dots$. It is an increasing sequence.

Let $(b^\sigma)_{\sigma<\omega_1}$ in $\mathbf{Q}$ such that
$$b^\sigma(\tau)=\begin{cases}0, & \tau>2\sigma+2\ \vee\ (\tau<2\sigma+2 \text{ and } \tau \text{ is "even" });\\ 1, & \tau=2\sigma+2\ \vee(\tau<2\sigma+2 \text{ and } \tau \text{ is "odd"})\ \vee\ \tau \text{ is limit.}\end{cases}$$

That is $b^1=(1011000\dots0\dots)$, $b^2=(101011000\dots0\dots)$,
$b^3=(10101011000..0\dots),\dots,b^\omega=(10101000\dots|_\omega11000\dots)$,
$b^{\omega+1}=(10101000\dots|_\omega1011000\dots),\dots$. It is a decreasing sequence.

We see that $\forall\sigma<\omega_1\ \forall\delta<\omega_1\ a^\sigma<b^\delta$. Between these sequences there is the only dyadic sequence $(10101010\dots1010\dots)$ of length ω_1. Both our sequences "converge" to $(101010\dots1010\dots)\notin\mathbf{Q}$. Therefore the sequences generate a gap in $\mathbf{Q}$. It is clearly, that the gap has type $(\aleph_1,\aleph_1)$. This gap generates the $(\aleph_1,\aleph_1)$ asymmetric gap in the field $\mathbb{R}[[\mathbf{G},\aleph_1]]$.

Remark 2.1. $^*\mathbb{R}$ has symmetric $(\aleph_1,\aleph_1)$ gaps [3] and it has asymmetric $(\aleph_1,\aleph_1)$ gaps as well.

3. Semi-η_1-super-real fields from the class $\mathcal{K}$. Dales and Woodin in [1] introduced a semi-η_1-field, which is generalization of η_1-field: a totally ordered field F is called a *semi-η_1-field* if for each strictly increasing sequence

$(s_n)_{n\in\mathbb{N}}$ and strictly decreasing sequence $(t_n)_{n\in\mathbb{N}}$ with $s_n < t_m$ $\forall m, n \in \mathbb{N}$ there exists $x \in F$ such that $s_n < x < t_m$ $\forall n, m \in \mathbb{N}$. It is easy to see that

Proposition 3.1. *F is a semi-η_1-field iff each gap (A, B) of F has only one of the following types* $(\aleph_1, \aleph_1)$, $(\aleph_0, \aleph_1)$, $(\aleph_1, \aleph_0)$, $(1, \aleph_1)$, $(\aleph_1, 1)$, $(1, \aleph_0)$, $(\aleph_0, 1)$. *Clearly, each η_1-field is a semi-η_1-field.*

By Proposition 3.1 and Corollary 2.1, we obtain the following

Proposition 3.2. $\mathbb{R}[[G(L, P), \aleph_1]]$ *(see section 2 of this paper) is not a semi-η_1-field.*

Note 3.1. *Each η_1-field of cardinality $\aleph_1$ belongs to $\mathcal{K}^1$.*

A gap (A, B) of a field F is called a *Dedekind gap* or a *fundamental gap* if $\forall \varepsilon \in F^+$ there exist $x \in A, y \in B$ such that $|y - x| < \varepsilon$ [1, 6]. It is easy to see by the definition that each Dedekind gap without the first and the last elements is symmetric.

Proposition 3.3 [3]. *Let F be a η_1-field with* $\mathrm{card}(F) = \aleph_1$. *Then*

(a) *there exist $2^{\aleph_1}$ symmetric Dedekind gaps;*

(b) *there exist $2^{\aleph_1}$ symmetric non-Dedekind gaps;*

(c) *if (A, B) is symmetric gap then* $\mathrm{cf}(A, B) = \aleph_1$.

Theorem 3.1. *Let $F \in \mathcal{K}^0$ and $\mathbb{R}[[\widehat{F}]] \setminus \mathbb{R}[[\widehat{F}, \aleph_1]] \neq \emptyset$. Then*

(a) *there is no symmetric Dedekind gap in F;*

(b) *F has $2^{\aleph_1}$ symmetric non-Dedekind gaps.*

P r o o f. (a). Since $\mathbb{R}[[\widehat{F}]] \setminus \mathbb{R}[[\widehat{F}, \aleph_1]] \neq \emptyset$, by Proposition 2.1. from [3], F has a symmetric gap. A symmetric gap (A, B) is Dedekind iff (see Proposition 2.2. from [3]) $\exists x_0 \in R[[\widehat{F}]] \setminus R[[\widehat{F}, \aleph_1]]$ $A < x_0 < B$ such that

$(*)$ $\mathrm{supp}(x_0)$ is inversely order-isomorphic to $\aleph_1$ and coinitial in $\widehat{F}$.

Since $\mathrm{cf}(F) = \aleph_0$ then $\mathrm{cf}(\widehat{F}) = \aleph_0$. Therefore if $x_0 \in R[[\widehat{F}]] \setminus R[[\widehat{F}, \aleph_1]]$ and $\mathrm{supp}(x_0)$ is coinitial in $\widehat{F}$ then $\mathrm{coi}(supp(x_0)) = \aleph_0$. Whence $supp(x_0)$ is not inversely order-isomorphic to $\aleph_1$. So by $(*)$, (A, B) is not Dedekind.

(b). Let (A, B) be a symmetric non-Dedekind gap. Then (see Proposition 2.1. from [3]) $\exists x_0 \in R[[\widehat{F}]] \setminus R[[\widehat{F}, \aleph_1]]$ and $\rceil(*)$holds. Let $x_0 = \sum_{g\in\widehat{F}} r_g g$. Put $r_g = x_0(g)$. Since $\mathrm{supp}(x_0) = \{g \in \widehat{F} | \; x_0(g) \neq 0\}$ is inversely well-ordered subset of $\widehat{F}$ and $\mathrm{card}(\mathrm{supp}(x_0)) = \aleph_1$, there exists $\Gamma \subset \widehat{F}$ with $\mathrm{card}(\Gamma) = \aleph_1$ and Γ is inversely order-isomorphic to $\aleph_1$. By $\rceil(*)$ and $\mathrm{coi}(\widehat{F}) = \aleph_0$, Γ is not coinitial in $\widehat{F}$.

Denote by S the set of all $x \in \mathbb{R}[[\widehat{F}]] \setminus \mathbb{R}[[\widehat{F}, \aleph_1]]$ such that $\operatorname{supp} x = \Gamma$. Each $x \in S$ generates a symmetric non-Dedekind gap in $\mathbb{R}[[\widehat{F}, \aleph_1]]$. Let $x_1, x_2 \in S$ and $x_1 < x_2$. Denote by (A_i, B_i) gaps in $\mathbb{R}[[\widehat{F}, \aleph_1]]$, which produced by x_i $(i = 1, 2)$. $A_i = \{x \in \mathbb{R}[[\widehat{F}]] \mid x < x_i\}$, $B_i = \{x \in \mathbb{R}[[\widehat{F}]] \mid x > x_i\}$. Prove that the gaps $(A_1, B_1), (A_2, B_2)$ are different.

There exists $g_0 = \max\{g \in \widehat{F} | x_1(g) \neq x_2(g)\}$. Define $x_3 \in \mathbb{R}[[\widehat{F}, \aleph_1]]$ such that $\operatorname{supp}(x_3) = \{g \in \Gamma \mid g \geq g_0\}$ and

1) $x_3(g) = x_1(g) = x_2(g)$ if $g > g_0$,

2) $x_3(g_0) = \frac{1}{2}(x_1(g_0) + x_2(g_0))$.

Since Γ is inversely order-isomorphic to $\aleph_1$, we have $\operatorname{card}(\operatorname{supp}(x_3)) = \aleph_0$. Hence $x_3 \in \mathbb{R}[[\widehat{F}, \aleph_1]]$. Evidently $x_1 < x_3 < x_2$. Therefore $x_3 \in B_1$ and $x_3 \in A_2$. So, $A_1 \neq A_2$. Thus x_1, x_2 produce different gaps in $\mathbb{R}[[\widehat{F}, \aleph_1]]$.

The cardinality of the set of all formal power series with support Γ equals $\mathbb{R}^{\operatorname{card}\Gamma} = 2^{\operatorname{card}\Gamma} = 2^{\aleph_1}$. So, the cardinality of the set of all symmetric non-Dedekind gaps is not less then $2^{\aleph_1}$. On the other hand, this is not greater then $2^{\aleph_1}$ since $\mathbb{R}[[\widehat{F}, \aleph_1]]$ has at most $2^{\aleph_1}$ gaps. □

Proposition 3.4. *Let F be a semi-η_1-field, which is not η_1-field with* $\operatorname{card}(F) = \aleph_1$. *Then*

a) $\operatorname{coi}\{y \in \widehat{F} : y > 1\} = \aleph_1$;

b) $F \in \mathcal{K}^0$.

Proof. *a)* If $y \in \widehat{F}$ then the set y is an archimedean class and $\operatorname{cf}(y) = \aleph_0$. Consider a gap (A, B) in F with $A = \{x \in F | \, \widehat{x} \leq \widehat{1}\}, B = F \backslash A$. We have $\operatorname{cf}(A) = \operatorname{cf}(\widehat{1}) = \aleph_0$. Since F is a semi-η_1-field, $\operatorname{coi}(B) = \aleph_1$. Since there is no the first element in B, $\operatorname{coi}(B) = \operatorname{coi}(\widehat{B}) = \aleph_1$. Thus $\operatorname{coi}(\widehat{B}) = \operatorname{coi}\{y \in \widehat{F} : y > 1\} = \aleph_1$.

b) Let (A, B) be a symmetric gap in F. Put $\operatorname{cf}(A, B) = \alpha$ then (A, B) has type (α, α). Since F is a semi-η_1-field, $\alpha = \aleph_1$. By a), we have $\operatorname{card}(\widehat{F}) = \aleph_1$. Therefore $F \in \mathcal{K}$. Suppose that $\operatorname{cf}(F) = \aleph_1$ then (by Proposition 3.1. from [4]) there are no $(1, \aleph_0), (\aleph_0, 1)$ gaps in F and hence F is η_1-field. It is a contradiction. Thus $F \in \mathcal{K}^0$. □

Corollary 3.1. *Let F be a semi-η_1-field, which is not η_1-field with* $\operatorname{card}(F) = \aleph_1$. *Then*

(a) there is no symmetric Dedekind gap in F;

(b) there exist $2^{\aleph_1}$ symmetric non-Dedekind gaps;

(c) if (A, B) is symmetric gap then $\operatorname{cf}(A, B) = \aleph_1$.

Proof. *(a)* and *(b)* are consequences of Proposition 3.4 and Theorem 3.1. □

Remark 3.1. Let us compare our results with the following. [1, p.56, Corollary 2.35] Let F be a real-closed semi-η_1-field, which is a β_1-field. Then exactly one of the following occurs:

(I). $w(F) = \aleph_0$, and $F \cong \mathbb{R}$;

(II). $\mathrm{cf}(F) = \aleph_1$, and $F \cong \mathbf{R}$ ($\eta_\mathbf{1}$-field);

(III). $\mathrm{cf}(\widehat{F}) = (\aleph_0, 1)$, and then $F \cong \mathbb{R}[[\mathbb{R} \times \mathbf{G} \times \mathbb{R}, \aleph_\mathbf{1}]]$;

(IV). $\mathrm{cf}(\widehat{F}) = (\aleph_0, \aleph_0)$, and then $F \cong \mathbb{R}[[c_{00}(\mathbf{G}^\mathbf{N}), \aleph_\mathbf{1}]]$. (See [1] for details).

Under CH, F is β_1-field iff $\mathrm{card}(F) = \aleph_1$.

Note that at the present paper we describe all symmetric gaps of the fields (II)–(IV).

Now we consider question about existence a super-real field in our class $\mathcal{K}$.

Let X be a completely regular topological space and $C(X)$ be an algebra of continuous functions on X. Let P be a prime ideal in $C(X)$. $C(X)/P := A_P$ is a totally ordered commutative algebra. The quotient map from $C(X)$ onto A_P is denoted by π_P. Since $f = f^+ + f^-$ and $f^+ \cdot f^- = 0 \in P$, we have $a = \pi_P(f) \geq 0$ if $f \in f^+ + P, f^- \in P$.
The quotient fields of A_P is denoted by K_P and is called a super-real field (it is not equal to $\mathbb{R}$)[1]. It is known (see [1], p.96-98) that each of possibilities (II)–(IV) from the Remark 3.1 actually occurs in the class of super-real fields (ZFC+CH). Hence, we have

Corollary 3.2. *There are semi-η_1 $+ \beta_1$-super-real fields that belong to the class $\mathcal{K}$.*

Question. Is there a β_1-super-real field which is not semi-η_1 in the class $\mathcal{K}$?

REFERENCES

[1] H. J. Dales, H. Woodin. Super real fields. Clarenden Press. Oxford, 1996.

[2] L. Fuchs. Partially ordered algebraic systems. Pergamen Press, 1963.

[3] N. Yu. Galanova. Symmetric gaps of fields of bounded formal power series and nonstandard real line. *J. Algebra and Logic* **42**, *1* (2003), 26–36

[4] N. Yu. Galanova. An investigation of the fields of bounded formal power series by means theory of cuts. *Acta mathematica* **84** (2004) 2–3, Proc. of conf. “Algebra and its applications”.

[5] Ž. Perović. Isomorphism of real closed fields. *Publ. Inst. Math., Nouv. Sér.* **57** (71) (1995) 66–70.

[6] G. G. Pestov. To the gap theory of ordered fields *Siberian Math. J.* **42**, *6* (2001) 1350–1360 (in Russian).

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