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INFORMATION MATRIX FOR BETA DISTRIBUTIONS

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Abstract. The Fisher information matrix for three generalized beta distributions are derived.

1. Introduction. Beta distributions are very versatile and a variety of uncertainties can be usefully modeled by them. Many of the finite range distributions encountered in practice can be easily transformed into the standard distribution. In reliability and life testing experiments, many times the data are modeled by finite range distributions, see for example [1].

The standard beta distribution is defined by the pdf

$$f(x) \quad = \quad \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$$

for $0 < x < 1$, $a > 0$ and $b > 0$. Three popular generalizations of this distribution are given by the pdfs:

$$(1) \qquad f(x) \quad = \quad \frac{1}{(d-c)B(a,b)} \left(\frac{x-c}{d-c}\right)^{a-1} \left(1 - \frac{x-c}{d-c}\right)^{b-1}$$

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for $c \leq x \leq d$,

$$f(x) = \frac{p x^{ap-1} \left(q^p - x^p\right)^{b-1}}{q^{p(a+b-1)} B(a, b)} \qquad (2)$$

for $0 \leq x \leq q$, and

$$f(x) = \frac{\lambda^a x^{a-1} (1-x)^{b-1}}{B(a, b) \left\{1 - (1-\lambda) x\right\}^{a+b}} \qquad (3)$$

for $0 \leq x \leq 1$, where $a > 0$, $b > 0$, $-\infty < c < d < \infty$, $p > 0$, $q > 0$ and $\lambda > 0$. We refer to (1) as the translated beta distribution. The generalizations (2) and (3) are due to McDonald [8] and Libby and Novick [7], respectively; and, so we refer to them as the McDonald's beta and Libby and Novick's beta, respectively. The aim of this note is to calculate the Fisher information matrix corresponding to each of the pdfs given by (1)–(3). For a given observation x, the Fisher information matrix is defined by

$$(I_{jk}) = \left\{ E\left(\frac{\partial \log L(\theta)}{\partial \theta_j} \frac{\partial \log L(\theta)}{\partial \theta_k} \right) \right\} \qquad (4)$$

for $j = 1, 2, \ldots, p$ and $k = 1, 2, \ldots, p$, where $L(\theta) = \log f(x)$ and $\theta = (\theta_1, \theta_2, \ldots, \theta_p)$ are the parameters of the pdf f. It has the meaning "information about the parameters θ contained in the observation x." The information matrix plays a significant role in statistical inference in connection with estimation, sufficiency and properties of variances of estimators. It is related to the covariance matrix of the estimate of θ (being its inverse under certain conditions). See Cox and Hinkley [2] for details.

The exact forms of the information matrix are derived in Sections 2, 3 and 4. Some technical results required for the derivations are noted in the Appendix (Section 5). The calculations use the beta function and the Gauss hypergeometric function defined by

$$B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt,$$

and

$${}_2F_1\left(a, b; c; x\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

respectively, where $(c)_k = c(c+1) \cdots (c+k-1)$ denotes the ascending factorial. The properties of these special functions can be found in Gradshteyn and Ryzhik [4].

2. Information matrix for translated beta. As implied by the name, the translated beta distribution given by (1) is a translated version of the standard beta distribution. One can see c and d as location parameters and $d-c$ as a scale parameter. This distribution has been applied as widely as the standard beta distribution – see Gupta and Nadarajah [5] for illustrations of some of the application areas.

If x is a single observation from (1) then the log-likelihood function can be written as

$$\log L(a,b,c,d) = (a-1)\log(x-c)+(b-1)\log(d-x)-\log B(a,b)-(a+b-1)\log(d-c).$$

The first-order derivatives are:

$$\frac{\partial \log L}{\partial a} = \log(x-c) - \frac{\Gamma'(a)}{\Gamma(a)} + \frac{\Gamma'(a+b)}{\Gamma(a+b)} - \log(d-c),$$

$$\frac{\partial \log L}{\partial b} = \log(d-x) - \frac{\Gamma'(b)}{\Gamma(b)} + \frac{\Gamma'(a+b)}{\Gamma(a+b)} - \log(d-c),$$

$$\frac{\partial \log L}{\partial c} = \frac{1-a}{x-c} + \frac{a+b-1}{d-c},$$

and

$$\frac{\partial \log L}{\partial d} = \frac{b-1}{d-x} - \frac{a+b-1}{d-c}.$$

The second-order derivatives are:

$$\frac{\partial^2 \log L}{\partial a^2} = \frac{\Gamma(a+b)\Gamma''(a+b) - \left\{\Gamma'(a+b)\right\}^2}{\Gamma^2(a+b)} - \frac{\Gamma(a)\Gamma''(a) - \left\{\Gamma'(a)\right\}^2}{\Gamma^2(a)},$$

$$\frac{\partial^2 \log L}{\partial a \partial b} = \frac{\Gamma(a+b)\Gamma''(a+b) - \left\{\Gamma'(a+b)\right\}^2}{\Gamma^2(a+b)},$$

$$\frac{\partial^2 \log L}{\partial a \partial c} = \frac{1}{c-x} + \frac{1}{d-c},$$

$$\frac{\partial^2 \log L}{\partial a \partial d} = \frac{1}{c-d},$$

$$\frac{\partial^2 \log L}{\partial b^2} = \frac{\Gamma(a+b)\Gamma''(a+b) - \left\{\Gamma'(a+b)\right\}^2}{\Gamma^2(a+b)} - \frac{\Gamma(b)\Gamma''(b) - \left\{\Gamma'(b)\right\}^2}{\Gamma^2(b)},$$

$$\frac{\partial^2 \log L}{\partial b \partial c} = \frac{1}{d-c},$$

$$\frac{\partial^2 \log L}{\partial b \partial d} = \frac{1}{d-x} - \frac{1}{d-c},$$

$$\frac{\partial^2 \log L}{\partial c^2} = \frac{1-a}{(x-c)^2},$$

$$\frac{\partial^2 \log L}{\partial c \partial d} = \frac{1-a-b}{(d-c)^2},$$

and

$$\frac{\partial^2 \log L}{\partial d^2} = \frac{a+b-1}{(d-c)^2} + \frac{1-b}{(d-x)^2}.$$

Now, we can compute the elements of the Fisher information matrix. It is clear that

$$E\left(-\frac{\partial^2 \log L}{\partial a^2}\right) = \frac{\Gamma(a)\Gamma''(a) - \left\{\Gamma'(a)\right\}^2}{\Gamma^2(a)} - \frac{\Gamma(a+b)\Gamma''(a+b) - \left\{\Gamma'(a+b)\right\}^2}{\Gamma^2(a+b)},$$

$$E\left(-\frac{\partial^2 \log L}{\partial a \partial b}\right) = -\frac{\Gamma(a+b)\Gamma''(a+b) - \left\{\Gamma'(a+b)\right\}^2}{\Gamma^2(a+b)},$$

$$E\left(-\frac{\partial^2 \log L}{\partial a \partial d}\right) = \frac{1}{d-c},$$

$$E\left(-\frac{\partial^2 \log L}{\partial b^2}\right) = \frac{\Gamma(b)\Gamma''(b) - \left\{\Gamma'(b)\right\}^2}{\Gamma^2(b)} - \frac{\Gamma(a+b)\Gamma''(a+b) - \left\{\Gamma'(a+b)\right\}^2}{\Gamma^2(a+b)},$$

$$E\left(-\frac{\partial^2 \log L}{\partial b \partial c}\right) = \frac{1}{c-d},$$

and

$$E\left(-\frac{\partial^2 \log L}{\partial c \partial d}\right) = \frac{a+b-1}{(d-c)^2}.$$

By application of Lemma 1,

$$E\left(-\frac{\partial^2 \log L}{\partial a \partial c}\right) = \frac{b}{(a-1)(d-c)},$$

$$E\left(-\frac{\partial^2 \log L}{\partial b \partial d}\right) = \frac{a}{(c-d)(b-1)},$$

$$E\left(-\frac{\partial^2 \log L}{\partial c^2}\right) = \frac{(a+b-1)(a+b-2)}{(a-2)(d-c)^2},$$

and

$$E\left(-\frac{\partial^2 \log L}{\partial d^2}\right) = \frac{a(a+b-1)}{(b-2)(d-c)^2}$$

provided that $a > 2$ and $b > 2$.

3. Information matrix for McDonald's beta. This distribution given by (2) is described from a probabilistic point of view in McDonald [8]. It has been used as a successful model in finance, reliability and queueing processes.

If x is a single observation from (2) then the log-likelihood function can be written as

$$\log L(a,b,p,q) = (ap-1)\log x + (b-1)\log\left\{1-\left(\frac{x}{q}\right)^p\right\} + \log p - ap\log q - \log B(a,b).$$

The first-order derivatives are:

$$\frac{\partial \log L}{\partial a} = p\log x - p\log q - \frac{\Gamma'(a)}{\Gamma(a} + \frac{\Gamma'(a+b)}{\Gamma(a+b)},$$

$$\frac{\partial \log L}{\partial b} = \log\left\{1-\left(\frac{x}{q}\right)^p\right\} - \frac{\Gamma'(b)}{\Gamma(b)} + \frac{\Gamma'(a+b)}{\Gamma(a+b)},$$

$$\frac{\partial \log L}{\partial p} = a\log x + (1-b)\frac{x^p \log(x/q)}{q^p - x^p} + \frac{1}{p} - a\log q,$$

and

$$\frac{\partial \log L}{\partial q} = \frac{(b-1)px^p}{q(q^p - x^p)} - \frac{ap}{q}.$$

The second-order derivatives are:

$$\frac{\partial^2 \log L}{\partial a^2} = \frac{\Gamma(a+b)\Gamma''(a+b) - \left\{\Gamma'(a+b)\right\}^2}{\Gamma^2(a+b)} - \frac{\Gamma(a)\Gamma''(a) - \left\{\Gamma'(a)\right\}^2}{\Gamma^2(a)},$$

$$\frac{\partial^2 \log L}{\partial a \partial b} = \frac{\Gamma(a+b)\Gamma''(a+b) - \left\{\Gamma'(a+b)\right\}^2}{\Gamma^2(a+b)},$$

$$\frac{\partial^2 \log L}{\partial a \partial p} = \log\left(\frac{x}{q}\right),$$

$$\frac{\partial^2 \log L}{\partial a \partial q} = -\frac{p}{q},$$

$$\frac{\partial^2 \log L}{\partial b^2} = \frac{\Gamma(a+b)\Gamma''(a+b) - \left\{\Gamma'(a+b)\right\}^2}{\Gamma^2(a+b)} - \frac{\Gamma(b)\Gamma''(b) - \left\{\Gamma'(b)\right\}^2}{\Gamma^2(b)},$$

$$\frac{\partial^2 \log L}{\partial b \partial p} = \frac{x^p \log(x/q)}{x^p - q^p},$$

$$\frac{\partial^2 \log L}{\partial b \partial q} = \frac{p x^p}{q\left(q^p - x^p\right)},$$

$$\frac{\partial^2 \log L}{\partial p^2} = \frac{(1-b) q^p x^p \left\{\log(x/q)\right\}^2}{\left(q^p - x^p\right)^2} - \frac{1}{p^2},$$

$$\frac{\partial^2 \log L}{\partial p \partial q} = \frac{(b-1)x^p}{q\left(q^p - x^p\right)} + \frac{(b-1)pq^{p-1}x^p \log(x/q)}{\left(q^p - x^p\right)^2} - \frac{a}{q},$$

and

$$\frac{\partial^2 \log L}{\partial q^2} = \frac{ap}{q^2} + (b-1)px^p\left(q^{-(2p+2)}x^p - 2pq^{-(2p+2)}x^p - q^{-(p+2)} - pq^{-(p+2)}\right).$$

Now, we can compute the elements of the Fisher information matrix. It is clear that

$$E\left(-\frac{\partial^2 \log L}{\partial a^2}\right) = \frac{\Gamma(a)\Gamma''(a) - \left\{\Gamma'(a)\right\}^2}{\Gamma^2(a)} - \frac{\Gamma(a+b)\Gamma''(a+b) - \left\{\Gamma'(a+b)\right\}^2}{\Gamma^2(a+b)},$$

$$E\left(-\frac{\partial^2 \log L}{\partial a \partial b}\right) = -\frac{\Gamma(a+b)\Gamma''(a+b) - \left\{\Gamma'(a+b)\right\}^2}{\Gamma^2(a+b)},$$

$$E\left(-\frac{\partial^2 \log L}{\partial a \partial q}\right) = \frac{p}{q},$$

and

$$E\left(-\frac{\partial^2 \log L}{\partial b^2}\right) = \frac{\Gamma(b)\Gamma''(b) - \left\{\Gamma'(b)\right\}^2}{\Gamma^2(b)} - \frac{\Gamma(a+b)\Gamma''(a+b) - \left\{\Gamma'(a+b)\right\}^2}{\Gamma^2(a+b)}.$$

By application of Lemma 2,

$$E\left(-\frac{\partial^2 \log L}{\partial p \partial q}\right) = \frac{ap}{(b-1)q}$$

and

$$E\left(-\frac{\partial^2 \log L}{\partial q^2}\right) = \frac{a(b-1)p(1-p)}{(a+b)q^2} + \frac{a(a+1)(b-1)p(p-1)}{(a+b)q^{p+2}} - \frac{ap}{q^2}.$$

By application of Lemma 3,

$$E\left(-\frac{\partial^2 \log L}{\partial a \partial p}\right) = \frac{1}{p}\left\{\frac{\Gamma'(a+b)}{\Gamma(a+b)} - \frac{\Gamma'(a)}{\Gamma(a)}\right\},$$

and

$$E\left(-\frac{\partial^2 \log L}{\partial b \partial p}\right) = \frac{a}{p(b-1)}\left\{\frac{\Gamma'(a+1)}{\Gamma(a+1)} - \frac{\Gamma'(a+b)}{\Gamma(a+b)}\right\}$$

provided that $b > 1$. By application of Lemma 4,

$$\begin{aligned} E\left(-\frac{\partial^2 \log L}{\partial p^2}\right) = \frac{1}{p^2} + \frac{a(b-1)}{p^2(a+b)}\Bigg[&\frac{\Gamma(a+1)\Gamma''(a+1) - \left\{\Gamma'(a+1)\right\}^2}{\Gamma^2(a+1)} \\ &-\frac{\Gamma(a+b+1)\Gamma''(a+b+1) - \left\{\Gamma'(a+b+1)\right\}^2}{\Gamma^2(a+b+1)} \\ &+\left\{\frac{\Gamma'(a+1)}{\Gamma(a+1)} - \frac{\Gamma'(a+b+1)}{\Gamma(a+b+1)}\right\}^2\Bigg]. \end{aligned}$$

By application of both Lemmas 2 and 3,

$$E\left(-\frac{\partial^2 \log L}{\partial p \partial q}\right) = -\frac{ap(a+b-1)}{q(b-2)}$$

provided that $b > 2$.

4. Information matric for Libby and Novick's beta. This beta distribution was first used by Libby and Novick [7] for utility function fitting and by Chen and Novick (1984) as a prior in some binomial sampling model. Two other applications are to the problem of Bayesian estimation of the ratio of two variances (Gelfand, [3]) and to model the proportion of time devoted to a specific work function in Bayesian work sampling (Pham-Gia, [9]).

If x is a single observation from (3) then the log-likelihood function can be written as

$$\log L(a, b, \lambda) = (a-1)\log x + (b-1)\log(1-x) - \\ -(a+b)\log\left\{1-(1-\lambda)x\right\} + a\log\lambda - \log B(a, b).$$

The first-order derivatives are:

$$\frac{\partial \log L}{\partial a} = \log x - \log\left\{1-(1-\lambda)x\right\} + \log\lambda - \frac{\Gamma'(a)}{\Gamma(a)} + \frac{\Gamma'(a+b)}{\Gamma(a+b)},$$

$$\frac{\partial \log L}{\partial b} = \log(1-x) - \log\left\{1-(1-\lambda)x\right\} - \frac{\Gamma'(b)}{\Gamma(b)} + \frac{\Gamma'(a+b)}{\Gamma(a+b)},$$

and

$$\frac{\partial \log L}{\partial \lambda} = \frac{a}{\lambda} - \frac{(a+b)x}{1-(1-\lambda)x}.$$

The second-order derivatives are:

$$\frac{\partial^2 \log L}{\partial a^2} = \frac{\Gamma(a+b)\Gamma''(a+b) - \left\{\Gamma'(a+b)\right\}^2}{\Gamma^2(a+b)} - \frac{\Gamma(a)\Gamma''(a) - \left\{\Gamma'(a)\right\}^2}{\Gamma^2(a)},$$

$$\frac{\partial^2 \log L}{\partial a \partial b} = \frac{\Gamma(a+b)\Gamma''(a+b) - \left\{\Gamma'(a+b)\right\}^2}{\Gamma^2(a+b)},$$

$$\frac{\partial^2 \log L}{\partial a \partial \lambda} = \frac{1}{\lambda} - \frac{x}{1-(1-\lambda)x},$$

$$\frac{\partial^2 \log L}{\partial b^2} = \frac{\Gamma(a+b)\Gamma''(a+b) - \left\{\Gamma'(a+b)\right\}^2}{\Gamma^2(a+b)} - \frac{\Gamma(b)\Gamma''(b) - \left\{\Gamma'(b)\right\}^2}{\Gamma^2(b)},$$

$$\frac{\partial^2 \log L}{\partial b \partial \lambda} = -\frac{x}{1-(1-\lambda)x},$$

and

$$\frac{\partial^2 \log L}{\partial \lambda^2} = \frac{(a+b)x^2}{\left\{1-(1-\lambda)x\right\}^2} - \frac{a}{\lambda^2}.$$

Now, we can compute the elements of the Fisher information matrix. It is clear that

$$E\left(-\frac{\partial^2 \log L}{\partial a^2}\right) = \frac{\Gamma(a)\Gamma''(a) - \left\{\Gamma'(a)\right\}^2}{\Gamma^2(a)} - \frac{\Gamma(a+b)\Gamma''(a+b) - \left\{\Gamma'(a+b)\right\}^2}{\Gamma^2(a+b)},$$

$$E\left(-\frac{\partial^2 \log L}{\partial a \partial b}\right) = -\frac{\Gamma(a+b)\Gamma''(a+b) - \left\{\Gamma'(a+b)\right\}^2}{\Gamma^2(a+b)},$$

and

$$E\left(-\frac{\partial^2 \log L}{\partial b^2}\right) = \frac{\Gamma(b)\Gamma''(b) - \left\{\Gamma'(b)\right\}^2}{\Gamma^2(b)} - \frac{\Gamma(a+b)\Gamma''(a+b) - \left\{\Gamma'(a+b)\right\}^2}{\Gamma^2(a+b)}.$$

By application of Lemma 5,

$$E\left(-\frac{\partial^2 \log L}{\partial a \partial \lambda}\right) = \frac{a\lambda^a}{a+b}\, {}_2F_1\left(a+b+1, a+1; a+b+1; 1-\lambda\right) - \frac{1}{\lambda},$$

$$E\left(-\frac{\partial^2 \log L}{\partial b \partial \lambda}\right) = \frac{a\lambda^a}{a+b}\, {}_2F_1\left(a+b+1, a+1; a+b+1; 1-\lambda\right),$$

and

$$E\left(-\frac{\partial^2 \log L}{\partial \lambda^2}\right) = \frac{a}{\lambda^2} - \frac{a(a+1)}{(a+b)(a+b+1)}\, {}_2F_1\left(a+b+2, a+2; a+b+2; 1-\lambda\right),$$

which, upon using special properties of the Gauss hypergeometric function, reduce to

$$E\left(-\frac{\partial^2 \log L}{\partial a \partial \lambda}\right) = -\frac{b}{(a+b)\lambda},$$

$$E\left(-\frac{\partial^2 \log L}{\partial b \partial \lambda}\right) = \frac{a}{(a+b)\lambda},$$

and

$$E\left(-\frac{\partial^2 \log L}{\partial \lambda^2}\right) = \frac{a}{\lambda^2}\left\{1 - \frac{a+1}{(a+b)(a+b+1)}\right\},$$

respectively.

5. Appendix. We need the following technical lemmas to calculate the elements of the Fisher information matrix.

Lemma 1. *For a random variable X with the pdf* (1),

$$E\left[\frac{1}{(X-c)^m (d-X)^n}\right] = \frac{B\left(a-m, b-n\right)}{(d-c)^{m+n} B\left(a, b\right)} \tag{5}$$

for $m < a$ and $n < b$.

Proof. The required expectation can be written as

$$\frac{B(a-m,b-n)\,I}{(d-c)^{m+n}B(a,b)},$$

where I denotes the integral

$$I = \int_c^d \frac{\left(\frac{x-c}{d-c}\right)^{a-m-1}\left(1-\frac{x-c}{d-c}\right)^{b-n-1}}{(d-c)B(a-m,b-n)}dx.$$

The integrand of I precisely takes the form of (1) and so $I = 1$. □

Lemma 2. *For a random variable X with the pdf (2),*

$$E\left[\frac{X^{p\alpha}}{(q^p - X^p)^\beta}\right] = \frac{q^{p(\alpha-\beta)}B(a+\alpha,b-\beta)}{B(a,b)} \quad (6)$$

for $a+\alpha > 0$ and $b > \beta$.

Proof. The required expectation can be written as

$$\frac{pI}{q^{p(a+\beta)}B(a,b)}, \quad (7)$$

where I denotes the integral

$$I = \int_0^q x^{p(a+\alpha)-1}\left\{1-\left(\frac{x}{q}\right)^p\right\}^{b-\beta-1} dx. \quad (8)$$

By setting $y = x/q$, (8) can be rewritten as

$$I = a^{p(a+\alpha)}\int_0^1 y^{p(a+\alpha)-1}\left\{1-y^p\right\}^{b-\beta-1} dy. \quad (9)$$

Using equation (3.251.1) in Gradshteyn and Ryzhik (2000), the integral on the right of (9) can be calculated as

$$\frac{1}{p}B(a+\alpha,b-\beta). \quad (10)$$

The result in (6) follows by combining (7), (9) and (10). □

Lemma 3. *For a random variable X with the pdf* (2),

$$E\left[\frac{X^{p\alpha}\log(X/q)}{(q^p - X^p)^\beta}\right] = \tag{11}$$
$$= \frac{q^{p(\alpha-\beta)}B(a+\alpha, b-\beta)}{B(a,b)}\left\{\frac{\Gamma'(a+\alpha)}{\Gamma(a+\alpha)} - \frac{\Gamma'(a+b+\alpha-\beta)}{\Gamma(a+b+\alpha-\beta)}\right\}$$

for $a+\alpha > 0$ and $b > \beta$.

P r o o f. The required expectation can be written as

$$\frac{pI}{q^{p(a+\beta)}B(a,b)}, \tag{12}$$

where I denotes the integral

$$I = \int_0^q x^{p(a+\alpha)-1}\left\{1 - \left(\frac{x}{q}\right)^p\right\}^{b-\beta-1}\log\left(\frac{x}{q}\right)dx. \tag{13}$$

By setting $y = x/q$, (13) can be rewritten as

$$I = a^{p(a+\alpha)}\int_0^1 y^{p(a+\alpha)-1}\{1-y^p\}^{b-\beta-1}\log y dy. \tag{14}$$

Using equation (4.253.1) in [4], the integral on the right of (14) can be calculated as

$$\frac{1}{p^2}B(a+\alpha, b-\beta)\left\{\frac{\Gamma'(a+\alpha)}{\Gamma(a+\alpha)} - \frac{\Gamma'(a+b+\alpha-\beta)}{\Gamma(a+b+\alpha-\beta)}\right\}. \tag{15}$$

The result in (11) follows by combining (12), (14) and (15). $\square$

Lemmd 4. *For a random variable X with the pdf (2),*

$$E\left[\frac{X^{p\alpha}\{\log(X/q)\}^2}{(q^p - X^p)^\beta}\right] =$$

$$= \frac{q^{p(\alpha-\beta)}B(a+\alpha, b-\beta)}{p^2 B(a,b)}\left[\frac{\Gamma(a+\alpha)\Gamma''(a+\alpha) - \{\Gamma'(a+\alpha)\}^2}{\Gamma^2(a+\alpha)}\right.$$

$$-\frac{\Gamma(a+b+\alpha-\beta)\Gamma''(a+b+\alpha-\beta) - \{\Gamma'(a+b+\alpha-\beta)\}^2}{\Gamma^2(a+b+\alpha-\beta)}$$

$$\left. + \left\{\frac{\Gamma'(a+\alpha)}{\Gamma(a+\alpha)} - \frac{\Gamma'(a+b+\alpha-\beta)}{\Gamma(a+b+\alpha-\beta)}\right\}^2\right] \tag{16}$$

for $a+\alpha > 0$ and $b > \beta$.

Proof. The required expectation can be written as

$$\frac{pI}{q^{p(a+\beta)}B(a,b)}, \tag{17}$$

where I denotes the integral

$$I = \int_0^q x^{p(a+\alpha)-1}\left\{1 - \left(\frac{x}{q}\right)^p\right\}^{b-\beta-1}\left\{\log\left(\frac{x}{q}\right)\right\}^2 dx. \tag{18}$$

By setting $y = x/q$, (18) can be rewritten as

$$I = a^{p(a+\alpha)}\int_0^1 y^{p(a+\alpha)-1}\{1 - y^p\}^{b-\beta-1}(\log y)^2 dy. \tag{19}$$

Using equation (4.261.21) in Gradshteyn and Ryzhik (2000), the integral on the right of (19) can be calculated as

$$\frac{1}{p^3}B(a+\alpha, b-\beta)\left[\frac{\Gamma(a+\alpha)\Gamma''(a+\alpha) - \{\Gamma'(a+\alpha)\}^2}{\Gamma^2(a+\alpha)}\right.$$

$$-\frac{\Gamma(a+b+\alpha-\beta)\Gamma''(a+b+\alpha-\beta) - \{\Gamma'(a+b+\alpha-\beta)\}^2}{\Gamma^2(a+b+\alpha-\beta)}$$

$$\left. + \left\{\frac{\Gamma'(a+\alpha)}{\Gamma(a+\alpha)} - \frac{\Gamma'(a+b+\alpha-\beta)}{\Gamma(a+b+\alpha-\beta)}\right\}^2\right]. \tag{20}$$

The result in (16) follows by combining (17), (19) and (20). □

Lemma 5. *For a random variable X with the pdf* (3),

(21) $$E\left[\frac{X^m}{\{1-(1-\lambda)X\}^n}\right] = \frac{\lambda^a B\left(a+m,b\right)}{B\left(a,b\right)} \; {}_2F_1\left(a+b+n,a+m;a+b+m;1-\lambda\right)$$

for $a+m>0$ *and* $b>0$.

P r o o f. The required expectation can be written as

(22) $$\frac{\lambda^a I}{B(a,b)},$$

where I denotes the integral

(23) $$I = \int_0^1 \frac{x^{a+m-1}(1-x)^{b-1}}{\{1-(1-\lambda)x\}^{a+b+n}} dx.$$

Using equation (3.197.3) in Gradshteyn and Ryzhik (2000), (23) can be calculated as

(24) $$I = B\left(a+m,b\right) \; {}_2F_1\left(a+b+n,a+m;a+b+m;1-\lambda\right).$$

The result in (21) follows by combining (22) and (24). □

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