Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **30** (2004), 527–548

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

REGULAR AVERAGING AND REGULAR EXTENSION OPERATORS IN WEAKLY COMPACT SUBSETS OF HILBERT SPACES

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Communicated by S. Troyanski

Abstract. For weakly compact subsets of Hilbert spaces K, we study the existence of totally disconnected spaces L, such that $C(K)$ is isomorphic to $C(L)$.

We prove that the space $C(B_H)$ admits a Pełczyński decomposition and we provide a starshaped weakly compact K, subset of B_H with non-empty interior in the norm topology, and such that $C(K) \cong C(L)$ with L totally disconnected.

1. Introduction. A long standing problem concerning $C(K)$ spaces was if for every compact Hausdorff space K there exists a compact totally disconnected space L such that $C(K)$ is isomorphic to $C(L)$. In the positive direction

2000 *Mathematics Subject Classification:* Primary 46E15, 54C55; Secondary 28B20.

Key words: C(K) spaces, Weakly compact sets, Regular averaging operators, Regular extension operators.

*Research partially supported by EPEAEK program "Pythagoras"

the classical A. Milutin's theorem ([7]) settles this problem for the case of metrizable spaces K. We recall that A. Milutin has shown that if K is compact metric and uncountable then $C(K)$ is isomorphic to $C(\mathcal{C})$ where $\mathcal{C}$ denotes the Cantor set. Other approaches in this direction have been provided by A. Pełczyński in [8], S. Ditor in [3] and Y. Benyamini in [2].

In the non metrizable case, A. Pełczyński has shown in [8] that if K is either a compact Abelian group or a Cartesian product of metrizable compact spaces, then there exists a totally disconnected compact L such that $C(K)$ is isomorphic to $C(L)$.

On the other hand, recent achievements have settled the problem in the general case:

Indeed, assuming CH, P. Koszmider, [5], has shown that there is a compact space K, such that for any totally disconnected space L, $C(K)$ is not isomorphic to $C(L)$.

Recently, G. Plebanek [9], has obtained the same result without any additional axioms. Therefore in the general case the problem has negative solution.

Our main intention on the present paper is to study this problem for K belonging to the class of connected Eberlein compact sets.

The basic technique of attacking such a problem is due to Milutin and Pełczyński and goes as follows:

For a given compact K we are seeking for a totally disconnected space L such that $C(K)$ is a complemented subspace of $C(L)$ and $C(L)$ is a complemented subspace of $C(K)$. If in addition for either K or L we have that $C(K) \cong (C(K) \oplus C(K) \oplus \cdots)_0$ then Pełczyński's decomposition method (see [6, 8]) implies that $C(K) \cong C(L)$.

A sufficient condition for the complemented embedding of $C(K)$ into $C(L)$ is described by A. Pełczyński in [8] through a continuous onto map $\phi : L \to K$ that admits a regular averaging operator. That means that there exists a continuous $s : \mathcal{P}(K) \to \mathcal{P}(L)$ such that $\phi^* \circ s = \mathrm{id}_{\mathcal{P}(K)}$. Here $\mathcal{P}(K)$ and $\mathcal{P}(L)$ are the spaces of regular Borel probability measures endowed with the w^* topology and ϕ^* is the induced by ϕ map into these spaces. We refer to section 2 for further definitions. Also the complemented embedding of $C(L)$ into $C(K)$ follows from the existence of a continuous injection $\phi : L \to K$ which admits a regular extension operator. That means that there exists a continuous $s : \mathcal{P}(K) \to \mathcal{P}(L)$ such that $s \circ \phi^* = \mathrm{id}_{\mathcal{P}(L)}$.

Summing up all the above we may say that for a given compact K we are looking for a compact totally disconnected L such that, roughly speaking, $\mathcal{P}(K)$ is a retraction of $\mathcal{P}(L)$ and $\mathcal{P}(L)$ is a retraction of $\mathcal{P}(K)$. Thus this problem is of strong topological flavor.

Before passing to describe our result we should mention a result due to S. Ditor that, in a sense, solves a half of the above mentioned problem of retractions. S. Ditor in [3] has shown that for every compact K there exists a totally disconnected compact L, with the same as K topological weight, and a continuous surjection $\phi : L \to K$ which admits a regular averaging operator. This result and the main ideas of its proof play key role in our approach.

It is well known that the class of Eberlein compact spaces is the class which is nearest to the compact metrizable spaces. On the other hand it is far away from the class of products of metrizable spaces. So it seems very natural to consider this problem on this class. Furthermore, it is shown in [1] that if K is Eberlein compact, then there is a Ditor space L for K which is also an Eberlein compact. If it is moreover the ball of a Hilbert space endowed with its weak topology then L can be chosen to be embedable into K. Thus if there existed a regular extension operator for this embedding then the problem would be settled in this case. Unfortunately we are not able to prove this assertion.

In the rest of this paper, we will say that X admits a *p-Pełczyński decomposition* if X is isomorphic to $\left(\bigoplus_{n=1}^{\infty} X\right)_p$ for some $1 \le p < \infty$ or $p = 0$.

Let us explain now the results contained in this paper.

Section 2 is devoted to various definitions and notation.

In section 3 we briefly present some results which will be used in the next section.

In sections 4 and 5, we prove the following theorems:

Theorem 1.1. *Let K be the closed unit ball of an infinite dimensional Hilbert space with its weak topology. Then $C(K)$ admits a 0-Pełczyński decomposition.*

The proof requires the following steps: First we observe that denoting with I, $\mathcal{C}$ the unit interval and the Cantor set respectively, the spaces $C(I \times K)$ and $C(\mathcal{C} \times K)$ are isomorphic. Next, which is the main argument, we show that the space

$$C_{00}(I \times K) = \{f \in C(I \times K) : f(1,k) = f(-1,k) = 0, \text{ for all } k \in K\},$$

is isomorphic to the space

$$C_{00}(K) = \{f \in C(K) : f(x_1) = f(x_{-1}) = 0\}$$

for some $x_1, x_{-1} \in K$ and this yields the desired result.

Finally, using the fact that K is Eberlein compact, we get that $C_{00}(K)$ is isomorphic to $C(K)$.

It follows from our arguments that $C(K) \cong C(I \times K)$. Notice that it is not clear that $I \times K$ and K are homeomorphic.

Theorem 1.2. *Let H be a Hilbert space, $\epsilon > 0$. Then there exists a symmetric, starshaped weakly compact set K and a totally disconnected compact space L such that $B_H(1-\epsilon) \subset K \subset B_H(1)$ and $C(K) \cong C(L)$.*

This result is proved using an iteration method. Let us recall that if K_0 is the ball of a Hilbert space, then there exists a Ditor space L_1 for K_0 such that L_1 is embeddable in K_0, but it is not clear whether this embedding admits a regular extension operator. So in the next step, we take a subspace of K_0, let us call it K_1, containing the image of L_1, and such that the embedding in K_1 does admit a regular extension operator. The problem now is that L_1 is no longer a Ditor space for K_1. So we proceed by constructing L_2 which is a Ditor space for K_1 and finding a subspace K_2 of K_1, such that L_2 embeds in K_2 by a map that admits a regular extension operator. The procedure ends up by constructing two sequences of spaces K_n, and L_n, $n \in \mathbb{N}$, and at the end the desired spaces L and K are obtained as the inverse limits of K_n and L_n. The space L can be mapped onto K by a map that admits a regular averaging operator and at the same time it can be embedded in K by a map that admits a regular extension one. Moreover $C(K)$ admits a 0–Pełczyński decomposition.

2. Notation and definitions.

Let L, K be Hausdorff compact spaces and $\phi : L \to K$ a continuous map. We denote by $\phi^o : C(K) \to C(L)$ the map $\phi^o(f) = f\phi$. It is well known that ϕ^o is a bounded linear operator and moreover if ϕ is onto then ϕ^o is an isometric embedding; if ϕ is one to one then ϕ^o is onto.

A bounded operator $u : C(L) \to C(K)$ is called *regular* provided $u(f) \geq 0$ whenever $f \geq 0$ and $u(1_L) = 1_K$, where $1_L : L \ni \ell \mapsto 1 \in \mathbb{R}$ is the constant one function on L.

We say that ϕ admits a *regular averaging operator* if there exists a regular operator $u : C(L) \to C(K)$ such that $u\phi^o = \mathrm{id}_{C(K)}$ (where by id_X we denote the identity map on X). Also ϕ admits a *regular extension operator* if there exists a regular operator $u : C(L) \to C(K)$ such that $\phi^o u = \mathrm{id}_{C(L)}$.

We say that ϕ admits a *choice function,* if ϕ is onto and moreover there exists a continuous map $s : K \to L$ such that $s(k) \in \phi^{-1}(k)$ or equivalently $\phi s = \mathrm{id}_K$.

We denote by $\mathcal{M}(K)$ the regular measures on K. It is well known by Riesz's representation Theorem that $\mathcal{M}(K)$ can be identified with $C(K)^*$, i.e. the dual of $C(K)$. Also, by $\mathcal{P}(K)$ is denoted the regular probability measures on K. Unless otherwise stated $\mathcal{P}(K)$ will be endowed with the weak-* topology.

By δ_k we denote the Dirac measure on $k \in K$.

By ϕ^* is denoted the map $(\phi^o)^*|\mathcal{P}(L) : \mathcal{P}(L) \to \mathcal{P}(K)$, i.e. the restriction of the dual map of ϕ^o to $\mathcal{P}(L)$.

If $x \in A^\Gamma$ and $B \subset \Gamma$, by $x|B$ is denoted the unique element of A^B taking the same values on B as x, i.e.

$$x|B(b) = x(b) \text{ for every } b \in B.$$

3. Some preliminaries. Next we remind the definition and some notation about trees. A tree of height ω is a partially ordered set by a relation $\prec$, such that for every $t \in T$, the set $\{s \in T : s \prec t\}$ is linearly ordered and finite.

If $t \in T$, then the set of immediate successors of t will be denoted by S_t. We say that T is finitely branching, if S_t is finite for every $t \in T$. By $\mathcal{B}(T)$ we denote the set of branches of T, namely all maximal linearly ordered subsets of T. $\mathcal{B}(T)$ is naturally topologized by the sets $V_t = \{b \in \mathcal{B}(T) : t \in b\}$.

It can be easily shown that T is finitely branching if and only if $\mathcal{B}(T)$ is compact. In the sequel we assume that every T has a unique minimal element, denoted by $r(T)$.

Finally for $t \in T$, we denote by $|t|$, the cardinality of the set $\{s \in T : s \prec t \text{ and } s \neq t\}$ and for $b \in \mathcal{B}(T)$ (respectively $t \in T$) we denote by $b|n$, (resp. $t|n$) the unique $s \in b$, (resp. $s \prec t$) such that $|s| = n$.

We will make use of the following two theorems. Both are contained in [1]. The second one is due to A. Pełczyński and is also contained in [8].

Theorem 3.1. *Let T be a finitely branching tree and K a normal topological space. We assume that to each $t \in T$ an open subset U_t of K has been assigned, such that*

(1) $U_{r(T)} = K$.

(2) *For every $t \in T$, $U_t = \cup\{U_s : s \in S_t\}$.*

Then there exists a continuous function $p : K \to (\mathcal{P}(\mathcal{B}(T)), w^*)$ such that for every $k \in K$, $p(k)$ is supported by

$$\{x \in \mathcal{B}(T) : k \in \cap_{s \in x} U_s\}.$$

Theorem 3.2. *Let $\{L_i\}_{i\in I}$, $\{K_i\}_{i\in I}$ be families of Hausdorff compact topological spaces and for each $i \in I$, $\phi_i : L_i \to K_i$ continuous maps. Set*

$$\phi : L = \prod_{i\in I} L_i \ni (\ell_i)_{i\in I} \mapsto (\phi_i(\ell_i))_{i\in I} \in \prod_{i\in I} K_i = K$$

Then

(1) *If each ϕ_i is onto and admits a regular averaging operator, then also ϕ is onto and admits a regular averaging operator.*

(2) *If each ϕ_i is one to one and admits a regular extension operator, then also ϕ is one to one and admits a regular extension operator.*

Let us make some observations regarding to the function ϕ^* introduced in the previous section. Recall that for a continuous map $\phi : L \to K$ we have denoted by $\phi^* : \mathcal{P}(L) \to \mathcal{P}(K)$ the induced affine map. Actually ϕ^* could be also defined as $\phi^*(p)(A) = p(\phi^{-1}(A))$, for every $p \in \mathcal{P}(L)$ and $A \subset K$ Borel measurable set. Therefore we easily obtain that if $p \in \mathcal{P}(L)$, then $\operatorname{supp} \phi^*(p) \subset \phi(\operatorname{supp} p)$. These observations lead to the proof of the next lemma which in the sequel we will repeatedly use.

Lemma 3.3. *Let $\phi : L \to K$ a continuous map with L and K Hausdorff compact spaces. Assume that M is a closed subset of K. Then $(\phi^*)^{-1}(\mathcal{P}(M)) = \mathcal{P}(\phi^{-1}(M))$.*

The following two Propositions are also contained in [1]. Proposition 3.4 is due to R. Haydon in [4]. We include their proves here for the convenience of the reader.

Proposition 3.4. *Let L, K be Hausdorff compact spaces and $\phi : L \to K$ a continuous map.*

(1) *Assume that ϕ is onto. Then the following are equivalent:*

  (a) *ϕ admits a regular averaging operator.*

  (b) *ϕ^* admits a choice function, i.e. there exists an one to one continuous map $s : \mathcal{P}(K) \to \mathcal{P}(L)$ such that $\phi^* s = \mathrm{id}_{\mathcal{P}(K)}$.*

  (c) *There exists a continuous map $v : K \to \mathcal{P}(L)$ such that $\phi^* v(k) = \delta_k$ for all $k \in K$.*

  (d) *There exists a continuous map $u : K \to \mathcal{P}(L)$ such that $u(k)$ is supported by $\phi^{-1}(k)$.*

(2) *Assume that ϕ is one to one. Then the following are equivalent:*

  (a) *ϕ admits a regular extension operator.*

  (b) *ϕ^* is a choice function for some continuous $s : \mathcal{P}(K) \to \mathcal{P}(L)$, i.e. $s\phi^* = \mathrm{id}_{\mathcal{P}(L)}$.*

(c) *There exists a continuous map $v : K \to \mathcal{P}(L)$ such that $v\phi(\ell) = \delta_\ell$ for all $\ell \in L$.*

P r o o f. 1. Lemma 3.3 yields that condition 1c is equivalent to 1d. For the remaining conditions:

1a $\Rightarrow$ 1b. Let u be the regular averaging operator for ϕ. Then $u\phi^o = \mathrm{id}_{C(K)}$ and consequently $(\phi^o)^* u^* = \mathrm{id}_{\mathcal{M}(K)}$. Therefore it suffices to show that for all $p \in \mathcal{P}(K)$, $u^*(p) \in \mathcal{P}(L)$, since then the map $s = u^*|\mathcal{P}(K)$ would be the choice function for ϕ^*. Observe that for $f \in C(L)$, $f \geq 0$, $u^*(p)(f) = p(u(f)) \geq 0$ and hence $u^*(p)$ is a positive measure. Moreover $u^*(p)(1_L) = p(u(1_L)) = p(1_K) = 1$ and therefore $u^*(p)$ is a probability measure.

1b $\Rightarrow$ 1c. Let s be the choice function for ϕ^*. Setting $j : K \ni k \mapsto \delta_k \in \mathcal{P}(K)$, it is easy to see that $v = sj$ is the required map.

1c $\Rightarrow$ 1a. Set $u : C(L) \to C(K)$ by $u(f)(k) = v(k)(f)$. It is easy to see that $u(f)$ is indeed a continuous function on K and that u is a linear bounded operator such that $u(f) \geq 0$ whenever $f \geq 0$ and $u(1_L) = 1_K$. Moreover for $f \in C(K)$, $u\phi^o(f)(k) = v(k)(\phi^o(f)) = \phi^* v(k)(f) = \delta_k(f) = f(k)$ for all $k \in K$, so that $u\phi^o(f) = f$ and hence $u\phi^o = \mathrm{id}_{C(K)}$.

2. 2a $\Rightarrow$ 2b. Let u be the regular extension operator for ϕ. Then $\phi^o u = \mathrm{id}_{C(L)}$ and consequently $u^*(\phi^o)^* = \mathrm{id}_{\mathcal{M}(L)}$. As before, if $p \in \mathcal{P}(K)$, then $u^*(p) \in \mathcal{P}(L)$ and hence $s = u^*|\mathcal{P}(K)$ is the required map.

2b $\Rightarrow$ 2c. Let $s : \mathcal{P}(K) \to \mathcal{P}(L)$ such that $s\phi^* = \mathrm{id}_{\mathcal{P}(L)}$. For $j : K \ni k \mapsto \delta_k \in \mathcal{P}(K)$ simply set $v = sj$.

2c $\Rightarrow$ 2a. Set $u : C(L) \to C(K)$ by $u(f)(k) = v(k)(f)$. Then u is a regular operator and for $f \in C(L)$ and $\ell \in L$, $\phi^o u(f)(\ell) = u(f)(\phi(\ell)) = v\phi(\ell)(f) = \delta_\ell(f) = f(\ell)$, hence $\phi^o u = \mathrm{id}_{C(L)}$. $\square$

Proposition 3.5. *Assume that L, K are Hausdorff compact spaces, and $\phi : L \to K$ a continuous onto map which admits a regular averaging operator. Moreover assume that M is a closed subset of K. Then the map $\phi|\phi^{-1}(M) : \phi^{-1}(M) \to M$ admits a regular averaging operator.*

P r o o f. Since ϕ admits a regular averaging operator, there is a continuous $s : K \to \mathcal{P}(L)$ such that $s(k)$ is supported by $\phi^{-1}(k)$. Thus if $k \in M$, $s(k)$ is always supported by $\phi^{-1}(M)$ and therefore $s(M) \subset \mathcal{P}(\phi^{-1}(M))$. Hence $s|M$ is the required by Proposition 3.4, case 1d map. $\square$

The next Lemma gives a concrete continuous map $\phi : \{-1,0,1\}^{\mathbb{N}} \to [-1,1]$ that admits a regular averaging operator.

What we are planning to do here, is to fix a sequence $\{r_n\}$ of positive real numbers that sum up to one, so that for any $x \in \{-1,0,1\}^{\mathbb{N}}$, $|\phi(x)| =$

$\sum\limits_{n\in\mathbb{N}} |x(n)| \cdot r_n$. This will be very useful in the sequel, since we are planning to use this map in weak-* compact subsets of $\ell_1(\Gamma)$ for some set Γ. By that point of view, the above relation shows that modulo the sequence $\{r_n\}$, the ℓ_1-norm remains the same between x and $\phi(x)$.

For appropriately chosen sequence $\{r_n\}$, ϕ admits a regular averaging operator. The only problem here is to find for a particular x whether it will be mapped on $\sum\limits_{n\in\mathbb{N}} |x(n)|r_n$ or on $-\sum\limits_{n\in\mathbb{N}} |x(n)|r_n$. This can be fixed by observing the sign of $x(n)$ for the least n such that $x(n) \neq 0$.

Lemma 3.6. *The map $\phi : \{-1,0,1\}^{\mathbb{N}} \to [-1,1]$ defined by*

$$\phi(x) = \begin{cases} \sum\limits_{n\in\mathbb{N}} |x(n)|r_n & \textit{if} \quad x(\min\{n\in\mathbb{N} : x(n)\neq 0\}) = 1 \\ -\sum\limits_{n\in\mathbb{N}} |x(n)|r_n & \textit{if} \quad x(\{\min\{n\in\mathbb{N} : x(n)\neq 0\}) = -1 \\ 0 & \textit{else} \end{cases}$$

where $r_n = \dfrac{1}{3}\left(\dfrac{2}{3}\right)^{n-1}$, is continuous onto and admits a regular averaging operator.

P r o o f. The fact that ϕ is continuous onto can be easily checked. For the rest, let $t \in \{-1,0,1\}^k$. Set

$$k_0 = \text{least } n \leq k \text{ such that } t(n) \neq 0,$$

if there exists such a k_0, and

$$V_t = \{x \in \{-1,0,1\}^{\mathbb{N}} : x(1) = t(1), \ldots, x(k) = t(k)\}$$

the usual clopen subset of $\{-1,0,1\}^{\mathbb{N}}$. Since $\sum\limits_{n=k+1}^{\infty} r_n = \left(\dfrac{2}{3}\right)^k$, an easy calculation shows that

$$\phi(V_t) = \begin{cases} \left[\sum\limits_{n=1}^{k} |t(n)|r_n,\ \sum\limits_{n=1}^{k} |t(n)|r_n + \left(\dfrac{2}{3}\right)^k\right], & \text{if} \quad t(k_0) = 1 \\ \left[-\sum\limits_{n=1}^{k} |t(n)|r_n - \left(\dfrac{2}{3}\right)^k,\ -\sum\limits_{n=1}^{k} |t(n)|r_n\right], & \text{if} \quad t(k_0) = -1 \\ \left[-\left(\dfrac{2}{3}\right)^k,\ \left(\dfrac{2}{3}\right)^k\right], & \text{else, if } k_0 \text{ does not exist.} \end{cases}$$

and moreover

$$(\phi(V_t))^o = (\phi(V_{t\frown(-1)})^o \cup (\phi(V_{t\frown 0}))^o \cup (\phi(V_{t\frown 1}))^o.$$

Here, we denote by X^o the interior of a set X, and by $t \frown i$ the element of $\{-1,0,1\}^{k+1}$ extending t by i.

Thus, assigning to each t, the open subset $U_t = (\phi(V_t))^o$ of K, the conditions of Theorem 3.1 are fulfilled so that there exists a continuous map $s : [-1,1] \to \mathcal{P}(\{-1,0,1\}^{\mathbb{N}})$ such that for any k, $s(k)$ is supported by

$$\{x \in \{-1,0,1\}^{\mathbb{N}} : k \in \cap_{n\in\mathbb{N}}(\phi(V_{x|n}))^o\}.$$

Since now ϕ is continuous and $\{V_t\}_t$ form a basis for $\{-1,0,1\}^{\mathbb{N}}$, it follows that for any such x, $\phi(x) = k$, so that $s(k)$ is also supported by $\phi^{-1}(k)$. Thus by Proposition 3.4, ϕ admits a regular averaging operator. □

In what follows, for any set Γ, we will denote by $\phi_\Gamma : \{-1,0,1\}^{\Gamma\times\mathbb{N}} \to [-1,1]^\Gamma$, the combined map $(\phi)_{\gamma\in\Gamma}$ where ϕ is the map of the previous lemma. By Theorem 3.2, ϕ_Γ admits also a regular averaging operator. Thus for any $\gamma \in \Gamma$,

$$\phi_\Gamma(x)(\gamma) = \begin{cases} \sum\limits_{n\in\mathbb{N}} |x(\gamma,n)|r_n & \text{if} \quad x(\gamma, \min\{n\in\mathbb{N} : x(\gamma,n)\neq 0\}) = 1 \\ -\sum\limits_{n\in\mathbb{N}} |x(\gamma,n)|r_n & \text{if} \quad x(\gamma, \min\{n\in\mathbb{N} : x(\gamma,n)\neq 0\}) = -1 \\ 0 & \text{else.} \end{cases}$$

In any case, denoting by $\|\cdot\|$ the ℓ_1-norm we get that

$$\|\phi_\Gamma(x)\| = \sum_{\gamma\in\Gamma}\sum_{n\in\mathbb{N}} |x(\gamma,n)|r_n.$$

For any closed subset K of $[-1,1]^\Gamma$, the restriction of ϕ_Γ to $\phi_\Gamma^{-1}(K)$, also admits a regular averaging operator, by Proposition 3.5. This restricted map will be also denoted by ϕ_Γ, for the shake of simplicity in notation.

The next lemma is crucial:

Lemma 3.7. *Let K be a compact space, Γ a set and $F^\gamma_{-1}, F^\gamma_0, F^\gamma_1$, $\gamma\in\Gamma$ a family of closed subsets of K, such that for each $\gamma\in\Gamma$*

$$(F^\gamma_{-1})^o \cup (F^\gamma_0)^o \cup (F^\gamma_1)^o = K.$$

Then there exists a continuous map $s : K \to \mathcal{P}(\{-1,0,1\}^\Gamma)$, such that for each $k \in K$, $s(k)$ is supported by the set

$$x \in \{-1,0,1\}^\Gamma : k \in \cap_{\gamma\in\Gamma} K^\gamma_{x(\gamma)}\}.$$

Proof. For each $\gamma \in \Gamma$, consider the compact space L_γ which is the disjoint union of F^γ_{-1}, F^γ_0 and F^γ_1. We can visualize the space L_γ as the set

$$\{(i,k) : i = -1, 0 \text{ or } 1 \text{ and } k \in K^\gamma_i\},$$

equipped with the product topology.

Next we argue that the natural projection map $\phi_\gamma : L_\gamma \ni (i,k) \mapsto k \in K$ admits a regular averaging operator. (This is a result due to S. Ditor in [3].) Indeed, fixing a partition of unity $\{f^\gamma_{-1}, f^\gamma_0, f^\gamma_1\}$ subordinated to $\{(F^\gamma_{-1})^o, (F^\gamma_0)^o, (F^\gamma_1)^o\}$, the map

$$s_\gamma : K \ni k \mapsto \sum_{i \in \{-1,0,1\}} f^\gamma_i(k)\delta_{(i,k)} \in \mathcal{P}(L_\gamma),$$

where we denote by δ_x the Dirac measure supported by x, is continuous and since $\phi_\gamma^{-1}(k) = \{(-1,k), (0,k), (1,k)\} \cap L_\gamma$, it has the property that for every $k \in K$, $s_\gamma(k)$ is supported by $\phi_\gamma^{-1}(k)$. Then by Proposition 3.4, ϕ_γ admits a regular averaging operator.

By Pełczyński's Theorem 3.2, the combined map

$$\phi : \prod_{\gamma \in \Gamma} L_\gamma \ni (x_\gamma)_{\gamma \in \Gamma} \mapsto (\phi_\gamma(x_\gamma))_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} K$$

admits also a regular averaging operator. Identifying every $k \in K$ with the element $(k)_{\gamma \in \Gamma}$ of $\prod_{\gamma \in \Gamma} K$ we can view K as a subset of $\prod_{\gamma \in \Gamma} K$ and by Proposition 3.5, the restriction of ϕ to $\phi^{-1}(K)$ admits also a regular averaging operator. Now, again by Proposition 3.4, this means that there exists a continuous map $R : K \to \mathcal{P}(\prod_{\gamma \in \Gamma} L_\gamma)$ such that for each $k \in K$, $R(k)$ is supported by $\phi^{-1}(k)$ which in this case is the set

$$\{((x(\gamma), k))_{\gamma \in \Gamma} : k \in F^\gamma_{x(\gamma)} \text{ for all } \gamma\}.$$

Letting moreover

$$t : \prod_{\gamma \in \Gamma} L_\gamma \ni ((x(\gamma), k_\gamma))_{\gamma \in \Gamma} \mapsto (x(\gamma))_{\gamma \in \Gamma} \in \{-1, 0, 1\}^\Gamma$$

we get that t is a continuous map and for every $k \in K$, $t^* \circ R(k)$ will be supported by

$$\{(x(\gamma))_{\gamma \in \Gamma} : k \in F^\gamma_{x(\gamma)}, \text{ for all } \gamma\}.$$

Therefore setting $s = t^* \circ R$ we get what we want. $\square$

4. $C(B_H)$ admits a Pełczyński decomposition. The following Theorem shows that if K is the closed unit ball of a Hilbert space $(B_{\ell_2(\Gamma)}, w)$, we can use Pełczyński' s decomposition method since $C(K) = (\bigoplus_{n\in\mathbb{N}} C(K))_0$.

Theorem 4.1. *Let K be the closed unit ball of an infinite dimensional Hilbert space with its weak topology. Then $C(K)$ admits a 0-Pełczyński decomposition.*

Proof. Let $\mathcal{C} = \{-1,1\}^{\mathbb{N}}$ be the Cantor space and $I = [-1,1]$. We define the following Banach spaces:

$$C_{00}(I \times K) = \{f \in C(I \times K) : f(1,k) = f(-1,k) = 0, \text{ for all } k \in K\}.$$

$$C_{00}(\mathcal{C} \times K) = \{f \in C(\mathcal{C} \times K) : f(t_1,k) = f(t_{-1},k) = 0, \text{ for all } k \in K\},$$

where t_1 and t_{-1} are the constant 1 and the constant -1 sequences in $\mathcal{C}$.

We proceed by proving some facts about these spaces:

Fact 1. $C(\mathcal{C} \times K)$ is isomorphic to $(C(\mathcal{C} \times K) \oplus C(\mathcal{C} \times K) \oplus \cdots)_0$.

Set $V_n = \{t \in \mathcal{C} : t(1) = \cdots = t(n-1) = -1 \text{ and } t(n) = 1\}$ and $t_{-1} = (-1,-1,\ldots) \in \mathcal{C}$. Define now

$$T : C(\mathcal{C} \times \mathcal{C} \times K) \to C(\{t_{-1}\} \times \mathcal{C} \times K) \oplus (\bigoplus_{n=1}^{\infty} C(V_n \times \mathcal{C} \times K))_0$$

where $T(f) = (f_0, (f_n)_n)$ and $f_0 = f \restriction \{t_{-1}\} \times \mathcal{C} \times K$, $f_n(t,s,k) = f(t,s,k) - f(t_{-1},s,k)$. Clearly every f_n is a continuous function on $V_n \times \mathcal{C} \times K$ and it can be easily checked that $\|f_n\| \to 0$. Moreover T is linear and $\|T(f)\| \le 2\|f\|$. We define also

$$S : C(\{t_{-1}\} \times \mathcal{C} \times K) \oplus \left(\bigoplus_{n=1}^{\infty} C(V_n \times \mathcal{C} \times K)\right)_0 \to C(\mathcal{C} \times \mathcal{C} \times K)$$

with

$$S(f_0,(f_n)_n)(t,s,k) = \begin{cases} f_0(t_{-1},s,k), & \text{if} \quad t = t_{-1} \\ f_n(t,s,k) + f_0(t_{-1},s,k) & \text{if} \quad t \in V_n. \end{cases}$$

We can easily check that $S(f_0,(f_n)_n)$ is a continuous function on $\mathcal{C}\times\mathcal{C}\times K$ and moreover $S \circ T = T \circ S = \mathrm{id}$. Therefore T is an isomorphism.

Since every one of $\{t_{-1}\} \times \mathcal{C}, V_n \times \mathcal{C}, \mathcal{C} \times \mathcal{C}$ is homeomorphic to $\mathcal{C}$, we conclude that

$$C(\mathcal{C} \times K) \cong (C(\mathcal{C} \times K) \oplus C(\mathcal{C} \times K) \oplus \cdots)_0.$$

Fact 2. $C_{00}(\mathcal{C} \times K)$ is isomorphic to $C(\mathcal{C} \times K)$.

Using Fact 1 and Pełczyński's decomposition method, it suffices to prove that $C_{00}(\mathcal{C} \times K) \hookrightarrow_{\perp} C(\mathcal{C} \times K)$ and $C(\mathcal{C} \times K) \hookrightarrow_{\perp} C_{00}(\mathcal{C} \times K)$.

Since $C_{00}(\mathcal{C} \times K)$ is a closed subspace of $C(\mathcal{C} \times K)$, it suffices to show that it is also complemented. Set $V_{(1)} = \{T \in \mathcal{C} : t(1) = 1\}$ and $V_{(-1)} = \{t \in \mathcal{C} : t(1) = -1\}$. Define $T : C(\mathcal{C} \times K) \to C_{00}(\mathcal{C} \times K)$ with

$$T(f)(t,k) = \begin{cases} f(t,k) - f(t_1,k), & \text{if} \quad t \in V_{(1)}, \\ f(t,k) - f(t_{-1}), & \text{if} \quad t \in V_{(-1)}. \end{cases}$$

Here t_1 and t_{-1} are respectively the constant 1 and the constant -1 sequence. Clearly $T(f) \in C_{00}(\mathcal{C} \times K)$, T is linear, $\|T\| \leq 2$ and $T(f) = f$ for every $f \in C_{00}(\mathcal{C} \times K)$. Therefore $C_{00}(\mathcal{C} \times K)$ is indeed a complemented subspace of $C(\mathcal{C} \times K)$.

For the other direction, let V_{-1}, V_0, V_1 be three pairwise disjoint clopen subsets of $\mathcal{C}$ such that $V_{-1} \cup V_0 \cup V_1 = \mathcal{C}$ and let $t_{-1} \in V_{-1}$, $t_1 \in V_1$. $C(V_0 \times K)$ can be naturally embedded in $C_{00}(\mathcal{C} \times K)$ by extending every function to be zero outside V_0, i.e. let $T : C(V_0 \times K) \to C_{00}(\mathcal{C} \times K)$ with

$$T(f)(t,k) = \begin{cases} f(t,k) & \text{if } \ t \in V_0, \\ 0, & \text{otherwise.} \end{cases}$$

We can also project $C_{00}(\mathcal{C} \times K)$ on $C(V_0 \times K)$ by $P : C_{00}(\mathcal{C} \times K) \to C(V_0 \times K)$ where $P(g) = g \restriction V_0 \times K$. Since $P \circ T = \mathrm{id}$ and they are both linear and bounded, we get that $C(V_0 \times K) \hookrightarrow_{\perp} C_{00}(\mathcal{C} \times K)$. Since V_0 is homeomorhpic to $\mathcal{C}$, we obtain the desired result.

Fact 3. $C_{00}(I \times K)$ is isomorphic to $C_{00}(\mathcal{C} \times K)$ and $C(I \times K)$ is isomorphic to $C(\mathcal{C} \times K)$.

We fix a $\theta : \mathcal{C} \to I$ which is continuous onto and admits a regular averaging operator. Moreover we assume that $\theta^{-1}(1) = t_1 = (1, 1, \ldots)$ and $\theta^{-1}(-1) = t_{-1} = (-1, -1, \ldots)$. Such a map is for example

$$\theta : \{-1, 1\}^{\mathbb{N}} \ni x \mapsto \frac{1}{4} \sum_{n=1}^{\infty} x(n) \Big(\frac{3}{4}\Big)^{n-1} \in [-1, 1].$$

Using Theorem 3.1 it can be easily checked as we have already done for ϕ in Lemma 3.6, that θ admits a regular averaging operator.

We fix also an embedding $\pi : \mathcal{C} \to I$ such that $\pi(t_1) = 1$ and $\pi(t_{-1}) = -1$. Let

$$(\theta, i) : \mathcal{C} \times K \ni (t, k) \mapsto (\theta(t), k) \in I \times K \quad \text{and}$$
$$(\pi, i) : \mathcal{C} \times K \ni (t, k) \mapsto (\pi(t), k) \in I \times K.$$

By Theorem 3.2, (θ, i) admits a regular averaging operator and (π, i) a regular extension one. Using Fact 1 and Pełczyński's decomposition method, we conclude that $C(I \times K) \cong C(\mathcal{C} \times K)$. Next, since $(\theta, i)(\{t_1\} \times K) = \{1\} \times K$ and $(\theta, i)(\{t_{-1}\} \times K) = \{-1\} \times K$ we have that $(\theta, i)^o(C_{00} \times K)) \subset C_{00}(\mathcal{C} \times K)$.

On the other hand, if u is a regular averaging operator for (θ, i), since

$$(\theta, i)^{-1}(\{1\} \times K) = \{t_1\} \times K \quad \text{and} \quad (\theta, i)^{-1}(\{-1\} \times K) = \{t_{-1}\} \times K$$

we get that $u(C_{00}(\mathcal{C} \times K)) \subset C_{00}(I \times K)$. Therefore $C_{00}(I \times K)$ is a complemented subspace of $C_{00}(\mathcal{C} \times K)$.

Using similar arguments we show that $C_{00}(\mathcal{C} \times K)$ is a complemented subspace of $C_{00}(I \times K)$.

By Facts 1 and 2 we gave that $C_{00}(\mathcal{C} \times K) \cong (\oplus_{n \in \mathbb{N}} C_{00}(\mathcal{C} \times K))_0$ and therefore by Pełczyński's decomposition method we conclude that $C_{00}(\mathcal{C} \times K) \cong C_{00}(I \times K)$.

Since Γ is infinite, for an element δ not in Γ, we have that

$$K = (B_{\ell_2(\Gamma)}, w) \sim (B_{\ell_1(\Gamma)}, w^*) \sim (B_{\ell_1(\{\delta\} \cup \Gamma)}, w^*). \tag{1}$$

Therefore we may assume that $K = (B_{\ell_1(\{\delta\} \cup \Gamma)}, w^*)$.

Let x_1, x_{-1} be the unique elements of K such that $x_1(\delta) = 1$ and $x_{-1}(\delta) = -1$ and set $C_{00}(K) = \{f \in C(K) : f(x_1) = f(x_{-1}) = 0\}$. Then

Fact 4. $C_{00}(I \times K)$ is isomorphic to $C_{00}(K)$.

According to (1), it is the same and more convenient in notation to prove that $C_{00}(I \times B_{\ell_1(\Gamma)}) \cong C_{00}(B_{\ell_1(\{\delta\} \cup \Gamma)})$. We define a function $\rho : I \times B_{\ell_1(\Gamma)} \to B_{\ell_1(\{\delta\} \cup \Gamma)}$ with $\rho(t, x) = (t, (1 - |t|)x)$ i.e.

$$\rho(t, x)(\gamma) = \begin{cases} t, & \text{if} \quad \gamma = \delta \\ (1 - |t|) \cdot x(\gamma) & \text{if} \quad \gamma \neq \delta \end{cases}.$$

ρ is continuous and we claim that $\rho^o : C(B_{\ell_1(\{\delta\} \cup \Gamma)}) \ni f \mapsto f \circ \rho \in C(I \times B_{\ell_1(\Gamma)})$ restricted to $C_{00}(B_{\ell_1(\{\delta\} \cup \Gamma)})$ is an isomorphism between $C_{00}(B_{\ell_1(\{\delta\} \cup \Gamma)})$ and $C_{00}(I \times B_{\ell_1(\Gamma)})$. First since ρ maps $(1, x)$ to $(1, 0) = x_1$ and $(-1, x)$ to

$(-1,0) = x_{-1}$ we get that ρ^o maps $C_{00}(B_{\ell_1(\{\delta\}\cup\Gamma)})$ to $C_{00}(I \times B_{\ell_1(\Gamma)})$. Next we can easily check that for any $(t,y) \in B_{\ell_1(\{\delta\}\cup\Gamma)})$, $\rho^{-1}(t,y)$ is non empty. So ρ^o is one-to-one. Moreover in the case where $|t| \neq 1$, $\rho^{-1}(t,y)$ is the single point set $\left\{(t, \dfrac{1}{1-|t|}y)\right\}$. Therefore if $g \in C_{00}(I \times B_{\ell_1(\Gamma)})$, we may define $f : B_{\ell_1(\{\delta\}\cup\Gamma)} \to \mathbb{R}$ by:

$$f(t,y) = \begin{cases} 0, & \text{if } |t| = 1 \\ g(\rho^{-1}(t,y)), & \text{if } \quad |t| < 1. \end{cases}$$

It is easy to check that since $g \in C_{00}(I \times B_{\ell_1(\Gamma)})$, f is continuous and in fact in $C_{00}(B_{\ell_1(\{\delta\}\cup\Gamma)})$. Moreover

$$\rho^o(f)(t,x) = f(\rho(t,x)) = \begin{cases} 0, & \text{if } \quad |t| = 1 \\ g(t,x), & \text{if } \quad |t| < 1 \end{cases} = g(t,x).$$

Therefore ρ^o is indeed an isomorphism between $C_{00}(B_{\ell_1(\{\delta\}\cup\Gamma)})$ and $C_{00}(I \times B_{\ell_1(\Gamma)})$.

Fact 5. $C_{00}(K)$ is isomorphic to $C(K)$.

Since K is Eberlein compact, we get that for some X complemented subspace of $C(K)$,

$$C(K) \cong X \oplus c_0(\mathbb{N}) \cong X \oplus c_0(\mathbb{N} \setminus \{1,2\}) \cong C(K)/\mathbb{R}^2 \cong C_{00}(K).$$

So, by fact 5, $C(K) \cong C_{00}(K)$, by Fact 4, $C_{00}(K) \cong C_{00}(I \times K)$, by Fact 3, $C_{00}(I \times K) \cong C_{00}(\mathcal{C} \times K)$, by Fact 2, $C_{00}(\mathcal{C} \times K) \cong C(\mathcal{C} \times K)$ and finally by Fact 1, $C(\mathcal{C} \times K) \cong (C(\mathcal{C} \times K) \oplus C(\mathcal{C} \times K) \oplus \cdots)_0$ and this concludes the proof. $\square$

Remark 4.2. Notice that the proof implies that $C(K)$ is isomorphic to $C(I \times K)$ although it is not clear if the spaces K and $I \times K$ are homeomorphic.

Let us observe that all steps in proving the previous theorem, except Fact 4 are valid for any Eberlein compact set. Fact 4 requires some additional properties for K, namely if K is a closed subspace of $(B_{\ell_1(\Gamma)}, w^*)$ we moreover need the following two properties:

- K is homeomorphic to the subspace of $I \times K$, $K' = \{(t,k) : t \in [-1,1], k \in K$ and $|t| + \sum_{\gamma\in\Gamma} |k(\gamma)| \leq 1\}$.
- K must be starshaped, in order to ensure that $\rho : I \times K \ni (t,k) \mapsto (t,(1-|t|k)$ takes values in K'.

So, the following proposition generalizes Theorem 4.1:

Proposition 4.3. *Let K be a starshaped weak* compact subset of $\ell_1(\Gamma)$ for some set Γ, which is homeomorphic to the space $K' = \{(t,k) \in I \times K : |t| + \sum_{\gamma\in\Gamma} |k(\gamma)| \leq 1\}$.*

Then $C(K)$ admits a 0–Pełczyński decomposition.

5. A starshaped weakly compact subset of B_H.

Theorem 5.1. *Let H be a Hilbert space, $\epsilon > 0$. Then there exists a symmetric, starshaped weakly compact set K and a totally disconnected compact space L such that $B_H(1-\epsilon) \subset K \subset B_H(1)$ and $C(K) \cong C(L)$.*

Here, we denote by $B_H(\delta)$ the closed δ–ball of H centered at 0.

Proof. Let $H = \ell_2(\Gamma)$ for a set Γ. It is well known that the map

$$B_H(\delta) \ni (x(\gamma))_{\gamma\in\Gamma} \mapsto ((x(\gamma))^2 \cdot \operatorname{sgn} x(\gamma))_{\gamma\in\Gamma} \in B_{\ell_1(\Gamma)}(\delta),$$

where $\operatorname{sgn} x(\gamma)$ is the sign of $x(\gamma)$, is a weak-weak* homeomorphism of $B_H(\delta)$ and $B_{\ell_1(\Gamma)}(\delta)$. Therefore, it suffices to prove the statement replacing H by $\ell_1(\Gamma)$, in which case the closed balls are endowed with the weak* topology.

The proof will follow by an inductive construction. Let us indicate the first step of it.

For any set E, let us denote by $\phi_E : \{-1,0,1\}^{E\times\mathbb{N}} \to [-1,1]^E$ the map defined by the rule

$$\phi_E(x)(e) = \begin{cases} \sum\limits_{n\in\mathbb{N}} |x(e,n)| r_n & \text{if} \quad x(e, \min\{n \in \mathbb{N} : x(e,n) \neq 0\}) = 1 \\ -\sum\limits_{n\in\mathbb{N}} |x(e,n)| r_n & \text{if} \quad x(e, \min\{n \in \mathbb{N} : x(e,n) \neq 0\}) = -1 \\ 0 & \text{else} \end{cases}$$

where $\{r_n\}$ is an appropriately chosen sequence of real numbers, so that ϕ_E is onto and admits a regular averaging operator, according to the comments that follow Lemma 3.6. Since obviously we can consider $B_{\ell_1(\Gamma)}(1)$ as a closed subset of $[-1,1]^\Gamma$, the map

$$\phi_\Gamma | \phi_\Gamma^{-1}(B_{\ell_1(\Gamma)}(1)) : \phi_\Gamma^{-1}(B_{\ell_1(\Gamma)}(1)) \to B_{\ell_1(\Gamma)}(1),$$

which we will also denote by ϕ_Γ, does admit a regular averaging operator by Proposition 3.5. Observe now that thanks to the appropriate choice of ϕ_Γ, if

$L_1 = \phi_\Gamma^{-1}(B_{\ell_1(\Gamma)}(1))$ then

$$\begin{aligned} x \in L_1 \quad &\Longleftrightarrow \quad \phi_\Gamma(x) \in B_{\ell_1(\Gamma)}(1) \Longleftrightarrow \sum_{\gamma\in\Gamma} |\phi_\Gamma(x)(\gamma)| \le 1 \\ &\Longleftrightarrow \quad \sum_{\gamma\in\Gamma}\sum_{n\in\mathbb{N}} |x(\gamma,n)| r_n \le 1. \end{aligned}$$

The last equivalence indicates that we can consider $\phi_\Gamma^{-1}(B_{\ell_1(\Gamma)}(1))$ as a closed subset of $B_{\ell_1(\Gamma\times\mathbb{N})}(1)$ by corresponding to each $x \in \phi_\Gamma^{-1}(B_{\ell_1(\Gamma)}(1))$ the element x' of $B_{\ell_1(\Gamma\times\mathbb{N})}(1)$ with $x'(\gamma,n) = x(\gamma,n)\cdot r_n$. The problem is that we can't show that this embedding admits a regular extension operator. (Observe that for infinite Γ, $B_{\ell_1(\Gamma)}(1)$ and $B_{\ell_1(\Gamma\times\mathbb{N})}(1)$ are the same as topological spaces.) We must restrict ourselves to a slightly smaller part of $B_{\ell_1(\Gamma\times\mathbb{N})}$, let us call it K_1, in order to assure that it does. But yet $\phi_{\Gamma\times\mathbb{N}}^{-1}(K_1)$ will not be the same as L_1. So, the purpose of our construction is to find in the n-th step a totally disconnected space L_{n+1} that maps onto K_n by a map that admits a regular averaging operator and to alter a bit K_n into K_{n+1} such that L_{n+1} can be embedded in K_{n+1} by an embedding that admits a regular extension operator. At the end, having done this infinitely many times, we end up to L and K which have the desired properties.

In order to give the details of the construction, let us introduce some notation.

We set $\Gamma_0 = \Gamma$ and for $n \in \mathbb{N}$,

$$\begin{aligned} \Gamma_n &= \Gamma \cup (\Gamma\times\mathbb{N}) \cup \ldots \cup (\Gamma\times\mathbb{N}^n), \\ \Delta_n &= (\Gamma\times\mathbb{N}) \cup \ldots \cup (\Gamma\times\mathbb{N}^n). \end{aligned}$$

Since we can view Δ_{n+1} as $\Gamma_n \times \mathbb{N}$, we can consider that $\phi_{\Gamma_n} : \{-1,0,1\}^{\Delta_{n+1}} \to [-1,1]^{\Gamma_n}$. We will use the sequence $\{r_n\}$ defined earlier and also a decreasing sequence $\{h_n\}$ of real numbers greater than 1 such that $\prod_{n\in\mathbb{N}} h_n = \dfrac{1}{1-\epsilon}$ where ϵ is a given positive number.

Given $x \in [-1,1]^{\Gamma_n}$ we define the corresponding element $\overline{x}$ in $\{-1,0,1\}^{\Delta_n}$ as follows: For any $(\gamma,k_1,k_2,\ldots,k_m) \in \Delta_n$,

$$\overline{x}(\gamma,k_1,k_2,\ldots,k_m) = \begin{cases} 1, & \text{if} \quad x(\gamma,k_1,k_2,\ldots,k_m) > \dfrac{r_{k_m}}{h_m} \\ 0, & \text{if} \quad -\dfrac{r_{k_m}}{h_m} \le x(\gamma,k_1,k_2,\ldots,k_m) \le \dfrac{r_{k_m}}{h_m} \\ -1, & \text{if} \quad x(\gamma,k_1,k_2,\ldots,k_m) < -\dfrac{r_{k_m}}{h_m} \end{cases}.$$

It will be clear later on that $\overline{x}$ is a measure of whether or not x can be considered as an element of K_n. The sets K_n and L_n will be defined as subsets of $[-1,1]^{\Gamma_n}$ and $\{-1,0,1\}^{\Delta_n}$ respectively:

Let $K_0 = B_{\ell_1(\Gamma)}(1) \subset [-1,1]^{\Gamma_0}$.

Having defined $K_n \subset [-1,1]^{\Gamma_n}$ and $L_n \subset \{-1,0,1\}^{\Delta_n}$, we define $L_{n+1} = \phi_{\Gamma_n}^{-1}(K_n) \subset \{-1,0,1\}^{\Delta_{n+1}}$ so that $\phi_{\Gamma_n} : L_{n+1} \to K_n$ is onto and admits a regular averaging operator. Now, we define K_{n+1},

$$K_{n+1} = \{x \in B_{\ell_1(\Gamma_{n+1})}(1) : \overline{x} \in L_{n+1}\}.$$

As mentioned earlier L_{n+1} is always embeddable in $B_{\ell_1(\Gamma_{n+1})}(1)$. The last condition will assure us that this embedding does admit a regular extension operator.

Next we prove some useful facts about K_n and L_n.

Fact 1. For every n, K_n is starshaped and symmetric and L_n is adequate and symmetric.

Clearly K_0 is starshaped and symmetric being the unit ball of $\ell_1(\Gamma)$. Admitting that this holds for K_n, the adequacy and symmetricity of L_{n+1} follows immediately from its definition, the starshapeness and symmetricity of K_n. Then it can be easily proved that K_{n+1} is also starshaped and symmetric.

Fact 2. For every n, K_n and L_n are compact.

K_0 is definitely compact and the compactness of K_n yields the compactness of L_{n+1}. To prove that K_{n+1} is compact, let $x \in [-1,1]^{\Gamma_{n+1}} \setminus K_{n+1}$. Since $B_{\ell_1(\Gamma_{n+1})}(1)$ is closed in $[-1,1]^{\Gamma_{n+1}}$ we can assume that $\overline{x} \notin L_{n+1}$. Since L_{n+1} is a compact subset of $\{-1,0,1\}^{\Delta_{n+1}}$ there must exist a finite subset A of Δ_{n+1} such that no element of L_{n+1} agrees with $\overline{x}$ on A. Since moreover by Fact 1, L_{n+1} is adequate we can subtract from A all its coordinates a such that $\overline{x}(a) = 0$, so that we can assume that for every $a \in A$, $\overline{x}(a) \neq 0$. Then the set

$$U = \Bigg\{ y \in [-1,1]^{\Gamma_{n+1}} : \text{For all } a = (\gamma, k_1, \ldots, k_m) \text{ in } A$$

$$|y(\gamma, k_1, \ldots, k_m)| > \frac{r_{k_m}}{h_m} \iff |x(\gamma, k_1, \ldots, k_m)| > \frac{r_{k_m}}{h_m} \Bigg\}$$

is an open subset of $[-1,1]^{\Gamma_{n+1}}$ containing x and having the property that for every $y \in U$, $\overline{y} \notin L_{n+1}$, therefore $U \cap K_{n+1} = \emptyset$.

Fact 3. The map $\phi_{\Gamma_n} : L_{n+1} \to K_n$ is onto and admits a regular averaging operator.

This follows immediately from the definition of L_{n+1} and Proposition 3.5.

Fact 4. For every n, $K_{n+1}|\Gamma_n = K_n$ and $L_{n+1}|\Delta_n = L_n$.

Here, by $K_{n+1}|\Gamma_n$ we mean the set

$$\{x \in [-1,1]^{\Gamma_n} : \text{There exists } y \text{ in } K_{n+1},\ y|\Gamma_n = x\}.$$

Clearly $K_1|\Gamma_0 = K_0 = B_{\ell_1(\Gamma)}$ since L_1 is a subset of $\{-1,0,1\}^{\Gamma\times\mathbb{N}}$ and therefore it imposes no restriction on the coordinates of K_1 coming from Γ. Assuming now that $K_{n+1}|\Gamma_n = K_n$, we observe that for any γ in Γ_n and $x \in \{-1,0,1\}^{\Delta_{n+2}}$, the value of $\phi_{\Gamma_{n+1}}(x)$ in γ depends only on the coordinates of Δ_{n+2} that actually come from Δ_{n+1}, so that $\phi_{\Gamma_{n+1}}(x)|\Gamma_n = \phi_{\Gamma_n}(x|\Delta_{n+1})$ and therefore

$$L_{n+2}|\Delta_{n+1} = \phi^{-1}_{\Gamma_{n+1}}(K_{n+1})|\Delta_{n+1} = \phi^{-1}_{\Gamma_n}(K_{n+1}|\Gamma_n) = \phi^{-1}_{\Gamma_n}(K_n) = L_{n+1}.$$

Now, using the same argument as before, since $L_{n+2}|\Delta_{n+1} = L_{n+1}$, L_{n+2} imposes the same restrictions on the coordinates of K_{n+2} coming from Γ_{n+1}, as L_{n+1}imposes on the coordinates of K_{n+1}, therefore $K_{n+2}|\Gamma_{n+1} = K_{n+1}$.

Fact 5. For every n, if x is an element of $\ell_1(\Gamma_n) \setminus K_n$, then $\|h_1 h_2 \cdots h_n \cdot x\| > 1$.

If x is as above, define the following element x' of $\ell_1(\Gamma_n)$:

$$x'(\gamma, k_1, \dots, k_m) = h_1 \cdots h_m \cdot x(\gamma, k_1, \dots, k_m).$$

Then obviously for every $\delta \in \Gamma_n$, $|x'(\delta)| \le h_1 \cdots h_n |x(\delta)|$ and the fact will follow easily if we prove that $\|x'\| > 1$. The key observation here is that for any $(\gamma, k_1, \dots, k_m) \in \Gamma_n$,

$$r_{k_m} \cdot |\overline{x}(\gamma, k_1, \dots, k_m)| \le h_m \cdot |x(\gamma, k_1, \dots, k_m)| \tag{2}$$

and it follows easily from the definition of $\overline{x}$.

We prove inductively on n that $\|x'\| > 1$.

For $n = 1$ either $x \notin B_{\ell_1(\Gamma_1)}$ and therefore also $\|x'\| > 1$, or it must be the case that $\overline{x} \notin L_1$. Thus $\phi_{\Gamma_0}(\overline{x}) \notin B_{\ell_1(\Gamma)}(1)$ and therefore, by the definition of the map ϕ_{Γ_0},

$$\sum_{\gamma\in\Gamma} |\phi_{\Gamma_0}(\overline{x})(\gamma)| > 1 \Rightarrow \sum_{\gamma\in\Gamma}\sum_{k\in\mathbb{N}} |\overline{x}(\gamma,k)| r_k > 1.$$

Using (2) we get that $\sum\limits_{\gamma\in\Gamma}\sum\limits_{k\in\mathbb{N}} h_1 \cdot |x(\gamma,k)| > 1$ and thus

$$\sum_{\gamma\in\Gamma}\sum_{k\in\mathbb{N}} h_1 \cdot |x(\gamma,k)| + \sum_{\gamma\in\Gamma} |x(\gamma)| > 1$$

which shows that $\|x'\| > 1$.

Let now $x \in \ell_1(\Gamma_{n+1}) \setminus K_{n+1}$. As before this means that either $\|x\| > 1$ in which case also $\|x'\| > 1$, or $\phi_{\Gamma_n}(\overline{x}) \notin K_n$ and by the inductive hypothesis we infer that $\|(\phi_{\Gamma_n}(\overline{x}))'\| > 1$. Thus, setting for simplicity $\delta = (\gamma, k_1, \ldots, k_m) \in \Gamma \times \mathbb{N}^m$,

$$\begin{aligned} 1 < \|(\phi_{\Gamma_n}(\overline{x}))'\| &= \sum_{m=0}^{n} \sum_{\delta \in \Gamma \times \mathbb{N}^m} h_1 \cdots h_m |\phi_{\Gamma_n}(\overline{x})(\delta)| = \\ &= \sum_{m=0}^{n} \sum_{\delta \in \Gamma \times \mathbb{N}^m} h_1 \cdots h_m \cdot \sum_{l \in \mathbb{N}} r_l \cdot |x(\delta, l)| \end{aligned}$$

Using again (2) we get

$$1 < \sum_{m=0}^{n} \sum_{(\delta,l) \in \Gamma \times \mathbb{N}^{m+1}} h_1 \cdots h_m h_{m+1} |x(\delta, l)| \le \|x'\|.$$

Fact 6. For every n, there is an embedding $i_n : L_n \to K_n$ that admits a regular extension operator.

We define $i_n : L_n \to K_n$ by the following rule:

$$i_n(x)(\delta) = \begin{cases} 0 & \text{if} \quad \delta \in \Gamma \\ r_{k_m} \cdot x(\delta) & \text{if} \quad \delta = (\gamma, k_1, \ldots, k_m) \in \Gamma \times \mathbb{N}^m \end{cases}.$$

Clearly i_n is continuous and 1–1. Also it is easy to see that $\|i_n(x)\| = \|\phi_{\Gamma_{n-1}}(x)\|$ and therefore $i_n(x) \in B_{\ell_1(\Gamma_n)}(1)$. Moreover $\overline{i_n(x)} = x \in L_n$ and thus $i_n(x) \in K_n$.

We will make use of Lemma 3.7 and Proposition 3.4 in order to show that i_n admits a regular extension operator.

For any $\delta = (\gamma, k_1, \ldots, k_m) \in \Gamma_n \setminus \Gamma$ we find two real numbers $\omega_0(\delta)$ and $\omega_1(\delta)$ such that

$$\frac{r_{k_m}}{h_m} < \omega_1(\delta) < \omega_0(\delta) < r_{k_m}. \tag{3}$$

Denoting by p_δ the projection of K_n onto the δ-coordinate, we define

$$F_{-1}^\delta = p_\delta^{-1}([-1, -\omega_1(\delta)]), \quad F_0^\delta = p_\delta^{-1}([-\omega_0(\delta), \omega_0(\delta)]), \quad F_1^\delta = p_\delta^{-1}([\omega_1(\delta), 1]).$$

Then clearly by (3),

$$(F_{-1}^\delta)^o \cup (F_0^\delta)^o \cup (F_1^\delta)^o = K_n$$

so that the hypothesis of Lemma 3.7 are fulfilled. Thus there exists a continuous $s_n : K_n \to \mathcal{P}(\{-1,0,1\}^{\Delta_n})$ such that for every $k \in K_n$, $s_n(k)$ is supported by

$$\{x \in \{-1,0,1\}^{\Delta_n} : k \in \cap_{\delta \in \Delta_n} F^{\delta}_{x(\delta)}\}.$$

Let x be in the support of $s_n(k)$ for some $k \in K_n$. Then for any $\delta = (\gamma, k_1, \ldots, k_m)$ in Δ_n if $|x(\delta)| = 1$, then $|k(\delta)| \geq \omega_1(\delta) > \dfrac{r_{k_m}}{h_m}$ and hence $|\overline{k}(\delta)| = 1$. Thus $|x(\delta)| \leq |\overline{k}(\delta)|$ for all $\delta \in \Delta_n$ and since $\overline{k} \in L_n$, it follows by Fact 1 that $x \in L_n$. Therefore actually $s_n : K_n \to \mathcal{P}(L_n)$. Now by Proposition 3.4 it suffices to show that $s_n \circ i_n(x) = \delta_x$, where δ_x is as usual the Dirac measure supported by x, for every $x \in L_n$. So let x' be in the support of $s_n \circ i_n(x)$. Then $i_n(x) \in \cap_{\delta \in \Delta_n} F_{x'(\delta)}$ and since for any $\delta \in \Delta_n$, $i_n(x)(\delta)$ can only take the values $-r_{k_m}, 0$ and r_{k_m} if $x(\delta) = -1, 0$ or 1 respectively it follows that for any such δ, $x(\delta) = x'(\delta)$ and hence $x = x'$.

We set $\Delta_\infty = \bigcup\limits_{n \in \mathbb{N}} \Delta_n$ and $\Gamma_\infty = \bigcup\limits_{n \in \mathbb{N}} \Gamma_n$ and we define $K \subset [-1,1]^{\Gamma_\infty}, L \subset \{-1,0,1\}^{\Delta_\infty}$ by the rules

$$x \in K \iff \text{for all } n \in \mathbb{N}, x|\Gamma_n \in K_n$$

$$x \in L \iff \text{for all } n \in \mathbb{N}, x|\Delta_n \in L_n.$$

It follows by Fact 4 that K and L are well defined compact subspaces of $[-1,1]^{\Gamma_\infty}$ and $\{-1,0,1\}^{\Delta_\infty}$ respectively.

Since $\Gamma_\infty \times \mathbb{N} = \Delta_\infty$, we can consider the map $\phi_{\Gamma_\infty} : \{-1,0,1\}^{\Delta_\infty} \to [-1,1]^{\Gamma_\infty}$. For any $x \in \{-1,0,1\}^{\Delta_\infty}$ and for any $\delta \in \Gamma_n$, $\phi_{\Gamma_\infty}(x)(\delta) = \phi_{\Gamma_n}(x|\Delta_{n+1})(\delta)$ and thus $\phi_{\Gamma_\infty}(x)|\Gamma_n = \phi_{\Gamma_n}(x|\Delta_{n+1})$. It follows that

$$\begin{aligned} x \in L &\iff \text{For all } n,\ x|\Delta_{n+1} \in L_{n+1} \iff \text{For all } n,\ \phi_{\Gamma_n}(x|\Delta_{n+1}) \in K_n \\ &\iff \text{For all } n,\ \phi_{\Gamma_\infty}(x)|\Gamma_n \in K_n \iff \phi_{\Gamma_\infty}(x) \in K. \end{aligned}$$

Therefore $\phi^{-1}_{\Gamma_\infty}(K) = L$ and thus $\phi_{\Gamma_\infty} : L \to K$ is onto and admits a regular averaging operator.

Consider now the map $i : L \to K$ defined by:

$$i(x)(\delta) = \begin{cases} 0 & \text{if} \quad \delta \in \Gamma \\ r_{k_m} \cdot x(\delta) & \text{if} \quad \delta = (\gamma, k_1, \ldots, k_m) \in \Gamma \times \mathbb{N}^m \end{cases}$$

Then exactly as in Fact 6, it turns out that i is one to one and admits a regular extension operator.

Thus $C(K)$ is a complemented subspace of $C(L)$ and $C(L)$ is a complemented subspace of $C(K)$.

It remains to prove that $B_{\ell_1(\Gamma_\infty)}(1-\epsilon) \subset K \subset B_{\ell_1(\Gamma_\infty)}(1)$. Since for every $x \in K$ and $n \in \mathbb{N}$, $x|\Gamma_n \in K_n$, it follows that $\sum_{\delta\in\Gamma_n} |x(\delta)| \leq 1$. Thus also $\sum_{\delta\in\Gamma_\infty} |x(\delta)| \leq 1$ and this settles that $K \subset B_{\ell_1(\Gamma_\infty)}(1)$.

For the other inclusion, if x is an element of $\ell_1(\Gamma_\infty)$ and not of K, then for some n, $x|\Gamma_n \in \ell_1(\Gamma_n)\setminus K_n$. By Fact 5 then $\|h_1h_2\cdots h_n\cdot x\| > 1$ and thus also

$$\|x\| > \frac{1}{\prod_{n\in\mathbb{N}} h_n} = 1-\epsilon,$$

by the choice of the sequence $\{h_n\}$.

It is easy to verify that the space K satisfies the two properies of Proposition 4.3 and therefore $C(K) \cong (C(K)\oplus C(K)\oplus\cdots)_0$. By applying now Pełczyński's decomposition method we conclude that $C(K) \cong C(L)$. $\square$

Remark 5.2. We would like to mention here that the construction can be carried out in the case where instead of a Hilbert space H, we have a reflexive Banach space X with an one symmetric and one unconditional basis.

We conclude this section with two questions which, for us, are open:

Problem 1. *If H is as in the above theorem is $C(B_H)$ isomorphic to $C(L)$ for some L totally disconnected?*

Problem 2. *More generally, does there exist K weakly compact, convex, non-metrizable subset of a Banach space X with $C(K) \cong C(L)$ for some L totally disconnected?*

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Received May 21, 2004