МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2010
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2010
Proceedings of the Thirty Ninth Spring Conference of
the Union of Bulgarian Mathematicians
Albena, April 6–10, 2010

METHOD OF LINEAR PROGRAMMING AS AN IDEA FOR HIGH SCHOOL CURRICULUM*

Emil Kelevedjiev, Iordanka Gortcheva

Since the middle of the 20th century, mathematical modeling by means of optimization problems has been widespread in scientific applications. Together with the software developed, it has a real impact on practice, thus changing the ideas of society about the role of mathematics, computer science, and information technologies.
The present paper is aimed to show the accessibility of some basic notions and methods of linear programming and integer linear programming to the high school students. It also gives credit to R. Kaltinska and G. Hristov [1] who around three decades ago included for a short period of time these methods into the high school mathematics curriculum in Bulgaria. Besides classical applications, here we show almost unknown in educational practices methods for solving some mathematical problems. Among the examples are recreational problems helpful for building connections between mathematics and the humanities. Free software, appropriate for the considered problems is also introduced: it can be used separately or in combination with elementary program codes.

1. Introduction. Linear equations and systems of linear equations are studied in the frame of mathematics school curriculum, including the cases when they have no or more than one solution. Linear inequalities and systems of linear inequalities are also considered with due attention. As a result, the knowledgeable 8th and 9th graders can define, for example, the set of points in the plane, whose coordinates (x, y) satisfy the system of inequalities (1): it is the convex polygon G shown in Fig. 1.

$$(1)\quad \left|\begin{array}{l} 4x + 4y \leq 160 \\ 35x + 20y \leq 1190 \\ 5x + 15y \leq 480 \\ x \geq 0; y \geq 0 \end{array}\right.$$

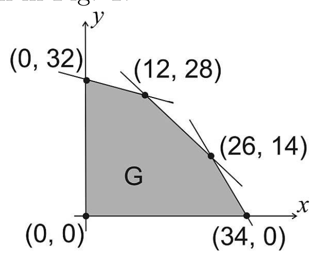

Fig. 1. Domain G, described by system of linear inequalities (1)

*2010 Mathematics Subject Classification: 97D40, 97M10, 97M40, 97N60, 97N80, 97R80
Key words: Linear programming, High school mathematics curriculum, Humanities

However, the same students often do not feel confident when they are to solve problems beyond the standard textbook wording. They come across difficulties even when a simple mathematical model of only one linear equation is to be written. Problems with more than one solution also puzzle the students and they tend to consider such situations their own mistake.

A way to help the high school students internalize the content of linear equations, linear inequalities, and systems of these is to make mathematics curriculum more application oriented. A proper device to achieve that is linear programming*. It adds modern 20th century sounding to the concepts of school mathematics originating in Euclid's time and unites them with the achievements of information and computing technologies.

2. The idea of maximum and minimum value. The study of linear functions provides a variety of opportunities to illustrate the notion of minimum/maximum of a function in the plane. Unfortunately they are not yet fully exploited in the high school curriculum. Set G, for example, whose elements are the solutions of system (1) allows the students to explore whether some of its elements can be marked by an additional property. On an oriented line L given by vector with coordinates (13, 23), they can look for a point of G, whose orthogonal projection on L is maximum (Fig. 2). Questions like: does such a point exist, is it unique, etc. are also worth discussing.

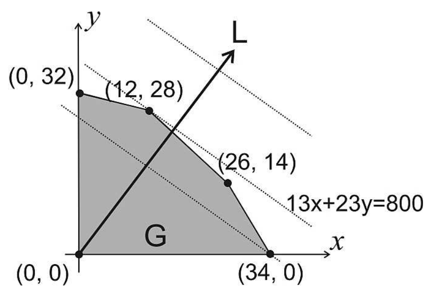

Fig. 2. From all points of G, the orthogonal projection of (12, 28) on L is maximum

Through accessible reasoning the students can infer that the maximum of the projection on the oriented line is equal to 800. The more observant ones also notice that this maximum is attained when the point with coordinates (12, 28) is projected on L. The special feature of point (12, 28) is that it is vertex of domain G.

Examples of that kind enable the students to write out the canonical form of the linear programming (LP) problem. The wide range of specialized software for solving LP problems presumes that the users are proficient to use it. This gives meaning to the exercises of writing an input to LP software which, in its essence, is the canonical form of the LP problems. For example, in order to solve the LP problem shown in Fig. 2 by the free software package *Lp_Solve* [2], the students must prepare an input text file as the one shown in Fig. 3.

3. Solving LP problems using basic knowledge of algebraic operations. According to mathematics curriculum in Bulgaria, the 8th graders have already learned

*The term "linear programming" is used as a synonym of "linear optimization"

```
max:  13x + 23y;
4x + 4y < 160;
35x + 20y < 1190;
5x + 15y < 480;
```

Fig. 3. An input file for *Lp_Solve*: it is assumed that x and y are nonnegative and the sign "<" means "≤"

the basic algebraic operations. The acquired knowledge and skills enable them to do LP problems. We show that through the following example:

$$(2)\qquad \left|\begin{array}{rcl} x+2y & \le & 7 \\ 2x-5y & \le & 6 \\ x \ge 0 & & \\ y \ge 0 & & \end{array}\right.$$

Objective function: max: $f(x,y) = 2x - 3y$

We introduce auxiliary variables $u \ge 0$ and $v \ge 0$ to represent the system of linear inequalities (2) as a system of linear equations:

$$(3)\qquad \left|\begin{array}{rcrcrcrcr} x & + & 2y & + & u & & & = & 7 \\ 2x & - & 5y & & & + & v & = & 6 \end{array}\right.$$

Then we find a point A, for which the value of the objective function $f(x,y)$ is equal to zero. For example, the coordinates of A can be $x=0$, $y=0$, $u=7$, and $v=6$.

At the next stage we try to improve the value of f. Since the system (3) is actually solved for u and v, we further solve it for another pair of variables, e.g. for x and v:

$$(4)\qquad \left|\begin{array}{rcrcrcr} x & = & 7 & - & u & - & 2y \\ v & = & -8 & + & 2u & + & 9y \end{array}\right.$$

After that we express the objective function f through the variables u and y and obtain $f(x,y) = 2x - 3y = 14 - 2u - 7y$. Since we already have a point A with a coordinate $u = 7$, we can improve the value of the objective function through decreasing u. It is clear that for fixed $y = 0$, the value of u cannot be less than 4. So we choose another point B from the domain, for which $u = 4$, $y = 0$, $x = 3$, and $v = 0$. The value of the objective function becomes $f(3,0) = 6$.

At the next step we express x and y by means of u and v:

$$(5)\qquad \left|\begin{array}{l} x = \dfrac{1}{9}(47 - 5u - 2v) \\[2ex] y = \dfrac{1}{9}(8 - 2u + v) \end{array}\right.$$

We write the objective function in the form $f(x,y) = \dfrac{1}{9}(70 - 4u - 7v)$. The value of f can be increased by taking a point C with coordinates $u = 0$, $v = 0$, $x = \dfrac{47}{9}$, $y = \dfrac{8}{9}$. At point C, $f(x,y) = \dfrac{70}{9} \approx 7.777\ldots$, $x \approx 5.222\ldots$, $y \approx 0.888\ldots$

Since we have already touched boundary points of u and v, further improving of the values of the objective function is not possible.

4. Comparison between the solutions in continuous and in integer case. Studying mathematics at school starts with positive integer numbers, called also natural. Later fractions, i.e. rational numbers are introduced. The set of rational numbers is closed under division (if the divisor is not zero), but the convenience to perform that operation might be misleading. It is not too farfetched to say that examples of the kind do exist: when the answer obtained is "$1\frac{2}{3}$ people needed to do a job", the students tend to round up the result and conclude undisturbed that "2 people are needed".

In LP problems one can observe even a sharper discrepancy between the integer solution obtained directly by *Lp_Solve* and the solution obtained by *Lp_Solve* in the continuous case and rounded after that to the nearest integer. Fig. 4 illustrates that contrary to intuition, rounding of non-integer results of continuous LP problems to integer does not make them transferable to integer LP problems.

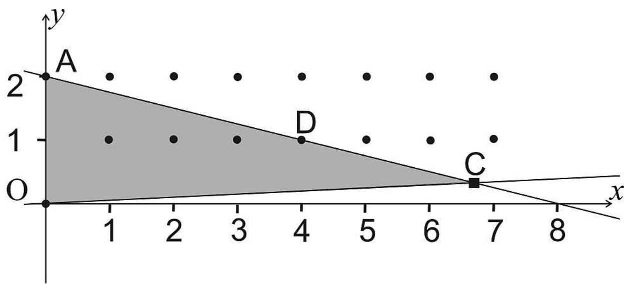

Fig. 4. The LP problem: max: x, subject to $x + 4y \leq 8$, $x - 20y \leq 0$, as depicted with its domain, the shaded triangle AOC, has a continuous solution at point C ($x = 6.6666\ldots$, $y = 0.3333\ldots$), while the integer solution of the same LP problem is at point D ($x = 4$, $y = 1$). The values of the pair (4, 1) can be found by the values of the pair ($6.6666\ldots$, $0.3333\ldots$) neither by truncating, nor by rounding

5. The idea of using information technologies: solving LP problems by ready made software. Modern information and computing technologies allow the accomplishment of the ambitious pedagogical task to help the students learn the ideas of linear programming by immediate doing. However, the abundance of ready made software for solving LP problems sometimes confuses the prospective users. The following classification may serve as a short guide in choosing a software package which fits both the educational purposes of the teachers and the hardware equipment available:

5.1. Console applications are command-line oriented applications that allow users to read characters from the console and write characters to the console. They are executed like commands in the DOS prompt window. Now they are mostly used in Linux prompt window. Such a console application usable in an integrated environment is the already mentioned open source linear programming software *Lp_Solve* [2].

5.2. Add-ins for MS-Excel help integrate LP problem solving activities with spreadsheets [3, 4].

5.3. Modeling environments as the FICO free version of Xpress-IVE also support building the students' experience in LP problems [5].

5.4. Online solvers, including professional ones, can be helpful too in education and self-education of the high school students in LP problem solving [6, 7].

6. Mathematical modeling activities for the high school students.

6.1. The diet problem stems from situations when, for example, one needs to find the most economical way to feed a group of children or a number of animals in a zoo. The diet has to bring all nutritional components to the members of the group in accordance with the prescribed requirements and availability of products. To formulate the mathematical model of that problem, we assume that n different kinds of food are in stock and the diet must provide at least the minimum of m different nutrients. We use the following notations:

d_j – the available amount of the j-th food,

b_j – the required minimum amount of the j-th food,

v_i – the required quantity minimum of the i-th nutrient,

a_{ij} – the amount of the i-th nutrient contained in one unit of the j-th food,

c_j – the price per unit of the j-th food,

x_j – the amount of the j-th food participating in the diet.

The LP problem is to find the values of x_j $(j = 1, 2, \dots, n)$ so that the cost of the diet (the objective function) is minimum and the constraints for the nutrition and food supply are satisfied:

$$\begin{aligned} \min: \quad & \sum_j c_j x_j \\ & \sum_{ij} a_{ij} x_j \geq v_i \text{ for } i = 1, 2, \dots, m; \\ & b_j \leq x_j \leq d_j \text{ for } j = 1, 2, \dots, n; \\ & x_j \geq 0 \text{ for } j = 1, 2, \dots, n. \end{aligned}$$

6.2. The backpack problem can be illustrated by the following example:

There are 4 items available which are worth 8, 11, 6, and 4 levs ("lev" is the name of Bulgarian currency). We know the weights of the items: 5, 7, 4, and 3 kilograms respectively. There is also a backpack where to put some of the items in a way that their total weight does not exceed 14 kilograms. Which items are to be put into the backpack in order to ensure the maximum value of its content?

To make an adequate mathematical model, prior to solving the backpack problem it is important to specify what kinds of items are available. If they are packages of sugar, flour, salt, or bottles of water, ice tea, etc., their content is separable and the LP problem allows fractional solutions. However, if the items are inseparable, like tins or clothes, for example, only integer solutions make sense. Here is the formulation of the backpack LP problem in case integer solutions (int) are sought:

$$\begin{aligned} \max: \quad & 8x_1 + 11x_2 + 6x_3 + 4x_4 \\ & 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ & 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 1,\ 0 \leq x_3 \leq 1,\ 0 \leq x_4 \leq 1, \\ & \text{int } x_1, x_2, x_3, x_4 \end{aligned}$$

The mathematical model of the backpack problem finds its applications in finances. Such an example is an investment portfolio design: to plan a maximum profit, taking into account both the investor's own financial resources and the financial instruments available, i.e. shares of stocks, bonds, certificates of deposit, etc.

6.3. The traveling salesman problem (TSP) has probably interested people for centuries. According to Homer, Odysseus the Cunning left his home island of Ithaca to go to the Trojan War and returned back twenty years later. Grötschel and Padberg [8] modeled Odysseus' crisscrossing the seas by choosing 16 exotic places on the map which, as in Homer's poem, have been visited by the ancient hero exactly once. Dependent on the whims of gods and nature, Odysseus can't have been trying to find the shortest route among the 653 837 184 000 different roundtrips connecting the 16 places chosen. That enormous number of roundtrips illustrates the importance of the TSP and the class of combinatorial optimization problems in general ([9], pp. 849-853).

If Odysseus had sailed in our modern times, it would not have been unusual if he had expanded his trip to new destinations and looked for an optimal TSP solution. Such an approach to the problem would attract the interests of the students keen on humanities and draw their attention to topics from literature, history, geography, mathematics, and information technologies. Also it enables the students to reach the general formulation of the TSP on their own: to find the shortest closed route which connects n given locations, provided that each location is to be visited exactly once, and the costs to each location from the remaining ones are given.

Figure 5 shows Grötschel and Padberg's design of Odysseus' roundtrip journey. After being optimized (Fig. 6), the route becomes around 30% shorter.

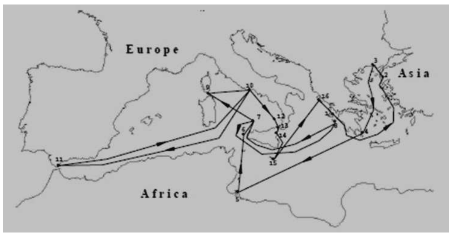

Fig. 5. About 9 900 kilometer roundtrip of Odysseus as designed in [8]

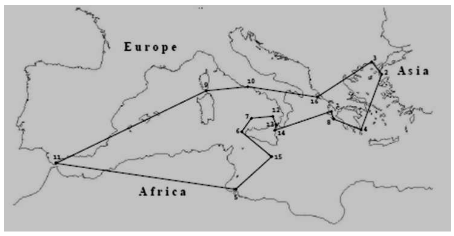

Fig. 6. About 3 000 kilometers shorter TSP solution to Odysseus' roundtrip [8]

In many cases the TSP is not symmetric: the cost of the trip from location A to location B differs from the cost of the return trip from B to A. Such costs are called *oriented* or *asymmetric distances*. They appear, for example, when we estimate the time in hiking routes. We illustrate an asymmetric TSP via the following example:

Four cities numbered by 1, 2, 3, and 4 are given. The oriented distances between each pair of cities are shown in Table 1. Find a closed route of minimum length which passes through each of these cities exactly once.

To write the mathematical model of the problem, we use nonnegative integer variables x_{ij} whose values are $x_{ij} = 1$, if the optimal route connects City i and City j in the direction from City i to City j, and $x_{ij} = 0$ otherwise. By default, *Lp_Solve* seeks nonnegative values of the variables. Therefore, it is not necessary to list the inequalities $x_{ij} \geq 0$ $(i = 1, 2, 3, 4;\ j = 1, 2, 3, 4)$ in the input file. Also, since *Lp_Solve* interprets strict inequalities $(x_{ij} < 0)$ as non-strict ones $(x_{ij} \leq 0)$, we write the objective function and the constraints in the form shown in Figure 7.

	1	2	3	4
1	0	12	11	12
2	14	0	13	10
3	5	17	0	23
4	17	14	20	0

Table 1. An example of an asymmetric TSP: Distances between each pair of towns

```
min:       12x12 + 11x13 + 12x14
 + 14x21         + 13x23 + 10x24
 +  5x31 + 17x32 +         23x34
 + 17x41 + 14x42 + 20x43;

           x12<1;  x13<1;  x14<1;
x21<1;             x23<1;  x24<1;
x31<1;     x32<1;          x34<1;
x41<1;     x42<1;  x43<1;

        x12 + x13 + x14 = 1;
x21 +         x23 + x24 = 1;
x31 +   x32 +       x34 = 1;
x41 +   x42 + x43       = 1;

/* no subtours: */

x13 + x31 <= 1;
x24 + x42 <= 1;
x12 + x23 + x31 <=2;

int
     x12, x13, x14,
x21,      x23, x24,
x31, x32,      x34,
x41, x42, x43;
```

Fig. 7. An input file to run *LP_Solve* for TSP shown in Table 1.

6.4. The approximation problem we consider here consists of finding the best approximation of a given decimal number by a fraction with a bounded denominator. As an example we take the decimal number 3.141592, which is a truncation of π, and seek the closest fraction to it with a denominator, e.g. less than 1 000. It means finding two integers p and q, where $1 < q < p$ and $q < 1000$ such that their ratio p/q is the best approximation to 3.141592. In other words, the distance from 3.141592 to p/q, which is the absolute value $d(p, q) = |3.141592 - p/q|$ has to be minimum.

This problem offers several challenges to the students. First, they have to recognize that it is not linear, in contrast to the previous examples. Second, they have to reveal where nonlinearity appears. Third, the students are to figure out how to reduce the problem to an LP type.

In the example being considered, nonlinearity appears in the denominator q and in the absolute value function. Fortunately, it can be successfully overcome. For that purpose we consider the difference $3.141592q - p$, multiplied to our convenience by the constant 1 000 000. Thus we work with the function $f(p, q) = 1\,000\,000\, qd(p, q)$. An appropriate technique here is the one of representing functions through their positive and negative parts ([10], pp. 149-150): $f = f^+ - f^-$, where $f^+ \geq 0$, $f^- \geq 0$ and $|f| = f^+ + f^-$. So, we write the expression $f(p, q) = 3141592q - 1000000p$ in the form $f = u - v$, where $u \geq 0$ and $v \geq 0$. Thus looking for min: $u + v$, we can find the minimum value of $|f|$. Fig. 8 shows an input to the *Lp_Solve*. The output lists the numbers $p = 355$ and $q = 113$. It allows the students to infer that the fraction $\dfrac{355}{113}$ is the best approximation to the given decimal number 3.141592 provided that the denominator is less than 1 000. Therefore, the fraction we found is a good approximation to π.

```
min: u+v;
3141592q-1000000p = u-v;
q>=1;
int p,q;
```

Fig. 8. An input to run *Lp_Solve* for problem 6.4.

The problem discussed encourages the students to do numerical experiments and interpret the results. If, for example, the constraint $q < 10$ is added to the input file, they will get a different answer: the fraction $\frac{22}{7}$. This approximation of number π is taught to the Bulgarian students in middle school and later many of them treat π and $\frac{22}{7}$ as equal numbers. We hope that working on such type of problems on their own, the high school students gain deeper conceptual understanding of irrational numbers.

6.5. There are **recreational problems** in newspapers, on TV shows, in popular mathematical books, etc. which can be formulated as LP problems [11]. As an example we consider the following problem by Kordemsky ([12], pp. 233–234; [11], p. 1):

6.5.1. There are several eggs in a basket. If I take them out in groups of 2, or 3, or 4, or 5, or 6, there is always one egg left in the basket. However, if I take them out in groups of seven, I empty the basket. What is the least number of eggs in the basket?

To represent the problem as an LP one, we denote by n the number of eggs and by x_i – the ratio of the integer division of n by i ($i = 2, \ldots, 7$). Then we write the objective function and the constraints to run *Lp_Solve* as it is shown in the left box of Fig. 9. The solution produced by the software is shown in the right box of Fig. 9.

```
min:  n;
n=2*x2+1;
n=3*x3+1;
n=4*x4+1;
n=5*x5+1;
n=6*x6+1;
n=7*x7;
int x2, x3, x4, x5, x6, x7;
```

```
Value of objective function:  301
n        301
x2       150
x3       100
x4        75
x5        60
x6        50
x7        43
```

Fig. 9. Solving problem 6.5.1 by *Lp_Solve*: an input (left) and output (right).

6.5.2. Sixteen coins are arranged to form a square with four (horizontal) rows and four (vertical) columns. How can six coins be removed so that an even number of coins remains in each row and each column? ([12], p. 18)

Figure 10 contains an input file to run *Lp_Solve*, Figure 11 – the output of *Lp_Solve* and the respective solution to problem 6.5.2. The notations remain for the reader to decode.

```
max: x11;

2r1+2r2+2r3+2r4=10;

x11+x12+x13+x14=2r1;            x11<1; x12<1; x13<1; x14<1;
x21+x22+x23+x24=2r2;            x21<1; x22<1; x23<1; x24<1;
x31+x32+x33+x34=2r3;            x31<1; x32<1; x33<1; x34<1;
x41+x42+x43+x44=2r4;            x41<1; x42<1; x43<1; x44<1;

x11+x21+x31+x41=2s1;            int
x12+x22+x32+x42=2s2;            x11,x12,x13,x14,x21,x22,x23,x24,
x13+x23+x33+x43=2s3;            x31,x32,x33,x34,x41,x42,x43,x44;
x14+x24+x34+x44=2s4;            int r1,r2,r3,r4,s1,s2,s3,s4;
```

Fig. 10. An input to *Lp_Solve* for problem 6.5.2.

```
Value of objective function: 1

x11  1      x31  1      r1  1
x12  0      x32  1      r2  1
x13  0      x33  1      r3  2
x14  1      x34  1      r4  1
x21  0      x41  0      s1  1
x22  1      x42  0      s2  1
x23  0      x43  1      s3  1
x24  1      x44  1      s4  2
```

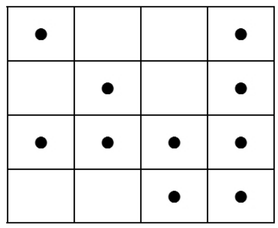

Fig. 11. The output of *Lp_Solve* (left) and the solution of problem 6.5.2 (right)

6.6. Sudoku is a very popular game played (in the classical case) within a square divided into 81 small square cells by 9 rows and 9 columns. All these 81 small square cells are congruent. They are grouped into 9 adjacent and non-overlapping square fields, each one consisting of 9 small squares. In some of the square cells there is a one-digit positive integer written. The goal of the game is to write one positive integer from 1 to 9 inclusive in each empty cell in such a way that each of the rows, columns, and square fields contains all of the integers from 1 to 9 exactly once.

Brader [13] calculated that there exist 6 670 903 752 021 072 936 960 different 9×9 Sudoku grids. Thus solving a particular Sudoku puzzle may require sophisticated combinatorial considerations and is in itself a kind of mental exhaustive search.

We also consider the game of Sudoku as an opportunity for the high school students to experience themselves the effectiveness of the software for linear programming. As a first step, a correspondent mathematical model is to be built. In our model we use $9^3 = 729$ variables which take values 0 or 1. We denote them by x_{ijk}, where each subscript i, j, and k takes values from 1 to 9. The correspondence between these variables and the one-digit integers written in the Sudoku-grid is the following: $x_{ijk} = 1$, if in the cell with coordinates (i, j) the digit k is written; otherwise $x_{ijk} = 0$. This leads to the following constraints:

$$\sum_{k=1}^{9} x_{ijk} = 1, \quad \text{for } i = 1, \ldots, 9;\ j = 1, \ldots, 9.$$
$$\sum_{j=1}^{9} x_{ijk} = 1, \quad \text{for } i = 1, \ldots, 9;\ k = 1, \ldots, 9.$$
$$\sum_{i=1}^{9} x_{ijk} = 1, \quad \text{for } j = 1, \ldots, 9;\ k = 1, \ldots, 9.$$
$$\sum_{i,j=1}^{3} x_{3p+i\ 3q+j} = 1, \quad \text{for } p = 0, 1, 2;\ q = 0, 1, 2;\ k = 1, \ldots, 9.$$
$$x_{ijk} = 1, \text{ if the digit } k \text{ is initially written in the cell } (i, j).$$

The long description of that mathematical model can easily persuade the students that it is much more appropriate if they write a program code which generates automatically the input file to *LP_Solve* software. Such an activity is a good exercise in studying programming languages.

6.7. Other examples. Fiction and linear programming. Many other classical examples of optimization problems, as the Transportation problem, the Maximum flow problem, Diophantine approximations, Matrix games, etc. can be found by the students and solved on their own by means of LP software. A rich source of such problems is [14].

Unexpected connections between mathematics and fiction are revealed in a small book by Elin Pelin and Dimityr Podvarzachov [15]. Elin Pelin is Bulgaria's "most popular interwar author", who has written "superb children's stories and poems" ([16], p. 395). What remains less-known about him is his passion for recreational mathematics. One of the story-problems is unique with its mathematical topic and fairy tale content ([15], pp. 29-30). Elin Pelin uses the plot of the old German tale "One Eye, Two Eyes, Three Eyes" by the German story tellers Brothers Grimm [17] with no trace of witty mathematics inside. In this canvas he intertwines an amusing moral and ... marketing situation to show the young readers how ingenuity and proficiency in mathematics help overcome injustice in life. Below we offer our edited version of the story-problem, keeping close to the flavor of the original:

Once upon a time there lived an old and cruel witch. She had three daughters. The first had only one eye placed in the middle of her forehead; the second had three eyes and the third two eyes, just like other people do. The old witch loved the One-Eyed and the Three-Eyed daughters, but hated the Two-Eyed one: maybe because she was the most beautiful and her two eyes were just in their right places.

One day the witch called her three girls and gave them apples: 50 wonderful apples to the One-Eyed, 30 wonderful apples to the Three-Eyed and only 10 unripe and wrinkled apples to the Two-Eyed.

"Now, girls, go immediately to the market and sell all your apples," she ordered them. "However, make sure that each of you earns the same amount of money!"

The Two-eyed daughter was scared: "O Mom! I cannot earn as much money for my wrinkled 10 apples as my sisters for their pretty 50 and 30 apples."

"Shut up!" chided the old hag. "If you bring even a penny less than your sisters, you will be in trouble!"

The Two-Eyed girl began to cry: "Then let me separate from my sisters and sell my apples at a higher price than them."

The old woman scolded again: "Listen! If you dare sell at a price even a penny higher than your sisters' prices, I will punish you. Go!"

The three girls headed for the market: The One-Eyed and the Three-Eyed sisters, dressed up and dolled up, ran ahead, laughing. Their Two-Eyed sister, wearing her only faded cotton dress, dragged behind, crying miserably.

Thus the wretch was left a whole kilometer behind her sisters. Meanwhile, her beloved golden-horned goat caught up with her and turned the ten wrinkled apples into ten irresistibly fresh and juicy ones.

What happened later? Soon the Two-Eyed sister reached the market and stood there next to her sisters. Whatever price they had asked for their apples, the same price she wanted for hers. At the end of the day, although The One-Eyed sister sold 50 apples, The Three-Eyed sold 30 apples, and The Two-Eyed one sold only 10 apples, the three girls earned an equal amount of money and brought it back to their mother.

At what price, do you think, the three sisters sold their apples to fulfill the demands of their witch mother?

Elin Pelin has provided only one answer to the problem ([15], pp. 81-82), probably a result of guesses and observations of an unknown mathematical talent. Approaching that problem by means of mathematical modeling allows finding many solutions, each of them with respective interpretation. Models with more than one solution are highly beneficial for the students, as they are not often discussed in mathematics classes.

To formulate our mathematical model, we assume that the trade has two stages and the apples are sold at two different prices during them. We use the notation x_{ij} for the number of apples sold by the i-eyed sister during the j-th period of trade ($i = 1, 2, 3$; $j = 1, 2$), p_j for the constant value of the price used in the j-th period of trade, and a_i for the number of apples given to the i-th sister. Our aim is to find integer values of x_{ij} such that the following conditions are fulfilled:

$$\left| \begin{array}{l} v_i = p_1 x_{i1} + p_2 x_{i2}, \quad i = 1, 2, 3; \\ v_1 = v_2 = v_3; \\ x_{i1} + x_{i2} = a_i, \quad i = 1, 2, 3; \ x_{ij} \geq 1, \ p_j > 0. \end{array} \right.$$

Because there exist infinitely many solutions to the problem, it makes sense to search for the minimum values of some of the variables. For example, we may minimize the common value $v_1 = v_2 = v_3$ and set such an optimization problem. However, nonlinearity appearing in the constraints, causes some difficulties in search of a solution. Additional difficulties arise because of the integer type of the solution.

Nevertheless, an integer linear programming software still can be applied: to avoid nonlinearity, we just call the LP solver many times, each time with some given constant values of p_1 and p_2.

A typical input file for *Lp_Solve* is shown in Fig. 12. There the values of p_1 and p_2 are inserted as constant coefficients 1 and 6, respectively. Since they are subject to repeated change, it is appropriate to generate automatically the input file for the LP solver. The students may write their own programs, e.g. in C/C++, which is again a successful training activity in studying programming languages. As an example we consider the output of *LP_Solve*, obtained for $p_1 = 1$ and $p_2 = 6$. It shows that the first stage of the trade passed at the price of 1 lev per apple, and The One-Eyed sister sold 49 apples, the Two-Eyed sold 1, and the Three-Eyed sold 25. At the second stage the sisters sold the rest of their apples at the price of 6 levs per apple. The One-Eyed sister sold 1 apple, the Two-Eyed sold 9, the Three-Eyed sold 5. Thus the revenue each sister brought home amounted to 55 levs.

```
min: v1;

v1=v2;
v2=v3;                          x11>=1;
v1=1*x11+6*x12;                 x12>=1;
v2=1*x21+6*x22;                 x21>=1;
v3=1*x31+6*x32;                 x22>=1;
                                x31>=1;
x11+x12=50;                     x32>=1;
x21+x22=10;
x31+x32=30;                     int x11,x12,x21,x22,x31,x32;
```

Fig. 12. An input file to run *LP_Solve* for $p_1 = 1$, $p_2 = 6$

7. Exhaustive search and integer linear programming: either or both? Many students who are familiar with a programming language of the type C/C++ would prefer to approach the problems discussed so far by writing an exhaustive search program. The method of integer linear programming gives them another perspective which emphasizes the constructing of a mathematical model rather than the writing of a program code.

It often happens that the number of variables and constraints becomes too large when mathematical models describe real life problems. In such cases, instead of writing the input files to LP software "by hand", it would be better for the students to create a program code to generate them. Although this requires more advanced knowledge in programming languages, the principle of writing such codes remains the same.

Nonlinearity of mathematical models is also an important issue since it reflects the nature of the modeled processes or phenomena. It may appear, for example, as a product of two or more variables. In such cases it is not correct to use directly any LP software. Instead, an exhaustive search through a part of the variables can be combined with repeated solving of a family of LP problems, as each of these LP problems has a fixed separate value of one or more variables of the model. Here the students again have to create a program code, this time to generate a family of input text files for the LP software, to call it repeatedly, and to process its output.

8. Concluding remarks. Most of the above ideas have already been shared with the participants of the summer schools, organized by the High School Students Institute of Mathematics and Informatics, whose activities are described in [18]. With their capacity to model real world processes, the LP problems attracted these students' attention. It is therefore crucial to gain also the audience of the students who are keen on humanities. The best way to do that is to offer them problems close to their interests. In this case the high school curriculum can only benefit from implementing the free professional software for solving LP problems. Leaving the tedious computations to the information and computing technologies help all the students to be active learners who produce, adjust, carry out, and verify their own ideas in a creative educational process.

REFERENCES

[1] G. Ivanov, R. Kaltinska. Numerical methods 10th grade. Narodna prosveta state publishers, Sofia, Bulgaria, 1976 (in Bulgarian).

[2] M. Berkelaar, K. Eikland, P. Notebaert. LP_Solve: Open source (mixed-integer) linear programming system. Version 5.1.0.0, May 1, 2004.
http://lpsolve.sourceforge.net/5.5/ (January 25, 2010).

[3] Z. MacDonald. Teaching Linear Programming using Microsoft Excel Solver. Computers in higher education economics review, **9** (1995), No 3, ISSN 1358-5363.
http://www.economicsnetwork.ac.uk/cheer/ch9_3/ch9_3p07.htm (January 23, 2010).

[4] P. O. Jensen. Operations research models and methods.
http://www.me.utexas.edu/ jensen/ORMM/ 2004–2009 (January 23, 2010).

[5] FICO express optimization suite. Student edition. http://optimization.fico.com /student-version-of-fico-xpress.html (January 25, 2010).

[6] S. Waner. Finite mathematics utility: Simplex method tool. In: Finite mathematics and applied calculus. 2007–2009. http://www.zweigmedia.com/RealWorld/simplex.html (January 23, 2010).

[7] Network enabled optimization system 1997–2009. Version 5.0. http://neos.mcs.anl.gov/ (January 23, 2010).

[8] M. Grötschel, M. Padberg. Ulysses 2000: In search of optimal solutions to hard combinatorial problems. 1993. ZIB-Report SC 93-34. Konrad-Zuse Zentrum für Informationstechnik and Technische Universität, Berlin. New York University, NY.

[9] K. L. Hoffman, M. Padberg. Traveling salesman problem. In: S. I. Gass, C. M. Harris (Eds.). Encyclopedia of operation research and management science, 2nd ed. ISBN 0-7923-7827-X. Norwell, MA: Kluwer Academic Publishers, 2001.

[10] J. N. McDonald, N. A. Weiss. A course in real analysis. ISBN 0-12-742830-5. San Diego, CA: Academic Press, 1999.

[11] M. J. Chlond, C. Nolan. Puzzles: Integer programming in recreational mathematics.
http://www.chlond.demon.co.uk/academic/puzzles.html (January 20, 2010).

[12] B. A. Kordemsky. Mathematical ingenuity, 5th ed. Moscow, Russia: State publishers of physical and mathematical literature, 1958 (in Russian).

[13] M. Brader. Combinatorial question on 9×9.
http://groups.google.co.uk/group/rec.puzzles/ (January 20, 2010).

[14] H. Arsham. Integer optimization and the network model. 1994-2009. 8th ed.
http://home.ubalt.edu/ntsbarsh/opre640A/partIII.htm (January 20, 2010).

[15] E. Pelin, D. Podvarzachov. The Old Hand Stanyo: Story-problems, 2nd ed. ISBN 954-8161-07-9. Sofia, Bulgaria: Language and culture publishers, 1992 (in Bulgarian).

[16] J. Rothschild. East Central Europe between the two World Wars: Vol. 9. In: P. F. Sugar, D. W. Treadgold (Eds.) A history of East Central Europe. ISBN 0-295-95357-8. Seattle, WA: University of Washington Press, 1974.

[17] J. and W. Grimm. Household Tales. In: C. W. Eliot (Ed.). The Harvard Classics, **17**. Folklore and fable: Aesop, Grimm, Anderson. NY: P. F. Collier, 1909-1914.
www.bartleby.com/17/2/ (January 20, 2010).

[18] O. Mushkarov, A. Rangachev, E. Sendova. Entering the world of mathematics research at school age. In: *Mathematics and Education in mathematics*, Proceedings of the 38th Spring conference of the Union of Bulgarian mathematicians, pp. 90-102. ISBN 1313-3330, Sofia, Bulgaria, 2009.

Emil Kelevedjiev
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: keleved@math.bas.bg

Iordanka Gortcheva
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: gorcheva@math.bas.bg

МЕТОДЪТ НА ЛИНЕЙНОТО ОПТИМИРАНЕ КАТО ИДЕЯ В ГОРНИЯ УЧИЛИЩЕН КУРС

Емил Келеведжиев, Йорданка Горчева

От средата на двадесети век моделирането с апарата на оптимизационните задачи навлиза широко в научните приложения. Такива математически модели, заедно с разработения софтуер, имат конкретен принос в практиката и променят възгледите на обществото за ролята на математиката, информатиката и информационните технологии.

Целта на настоящия доклад е да покаже достъпността на някои основни понятия и методи от линейното и целочисленото линейно оптимиране за учениците от горния училищен курс и да представи в нова светлина започнатото преди 2–3 десетилетия пропагандиране на този апарат от Р. Калтинска и Г. Христов [1]. Освен класически приложения са показани и почти непознати за средното образование методи за решаване на задачи. Сред разгледаните примери са и занимателни задачи, които допринасят за изграждане на междупредметни връзки с хуманитарните науки. Представени са софтуерни средства, които самостоятелно или в комбинация с елементарно програмиране са мощен апарат за атакуване на разнообразни математически задачи.