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MODELING OF CYLINDRICAL COUETTE FLOW OF RAREFIED GAS BETWEEN ROTATING CYLINDERS*

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The cylindrical Couette flow of a rarefied gas is studied in the case of two cylinders rotating with different velocities. Velocity, density and temperature profiles are investigated by a Direct Monte Carlo Simulation method and a numerical solution of the Navier-Stokes equations for compressible flow is found. The results obtained by both methods are in an excellent agreement at a small Knudsen number Kn = 0.02. The calculations show that the gas lags or outstrips in comparison with the walls or it has the elastic rigid body behavior at the variety velocity boundary conditions. These results are important for applications in non-planar microfluidic problems.

1. Introduction. Fluid transport in micro and macro channels yields the necessity to study flow in a cylindrical coordinate system. Note that Couette cylindrical flow is a fundamental problem in the rarefied gas dynamics [1, 6, 9–11]. As such, its modeling and numerical solving is of a great importance for the microfluidics, which is the theoretical background for analysis of new emerging Micro Electro Mechanical Systems MEMS [2, 3, 12].

The design of adequate mathematical models of gaseous flows in micro devices is one of the most important tasks of the studies. We consider both molecular and continuum models treating the gaseous flow by using different level of mathematical description. Both models take into account the specific microfluidic effects of gas rarefaction and slip-velocity regime at the solid boundaries.

In the present paper we compare results obtained by using the molecular Direct Simulation Monte Carlo (DSMC) method with those calculated by a numerical solution of the continuum Navier-Stokes equations for compressible flow (NS) [7, 8]. Both methods are used to model the cylindrical Couette flow for Knudsen number 0.02 and for different rotating velocities of the both cylinders.

The aim of the present paper is to study the gas flow in the gap depending on the variety velocity boundary conditions. The results show that the gas lags or outstrips in comparison with the velocity walls or it behaves as an elastic rigid body.

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Key words: Fluid Dynamics, Kinetic theory, Rarefied gas, DSMC.
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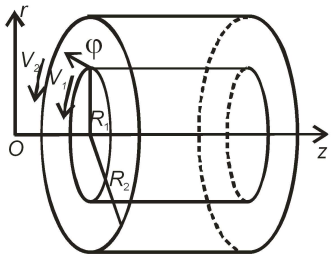

Fig. 1. Flow geometry

2. Formulation of the problem and methods of solving. We study a rarefied gas flow between two coaxial cylinders (one dimensional, axially-symmetrical problem) with equal temperatures $T_1 = T_2$. The inner cylinder has radius R_1 and the outer – R_2. The outer cylinder rotates with a constant velocity V_2 and the inner one – with constant velocity V_1.

Continuous Model and Numerical Simulation. The continuous model is based on the Navier-Stokes equations for compressible fluid, completed with the equations of continuity and energy transport. The governing equations are written in dimensionless form as follows:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho u) = 0, \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} - \frac{v^2}{r}\right) = -\frac{1}{2}\frac{\partial \rho T}{\partial r} + \frac{4}{3}C_\mu \mathrm{Kn}\sqrt{T}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru)\right] + \frac{2}{3}C_\mu \mathrm{Kn}\left(2\frac{\partial u}{\partial r} - \frac{u}{r}\right)\frac{\partial \sqrt{T}}{\partial r}, \tag{2}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{uv}{r}\right) = C_\mu \mathrm{Kn}\sqrt{T}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv)\right] + C_\mu \mathrm{Kn}\, r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\frac{\partial \sqrt{T}}{\partial r}, \tag{3}$$

$$\rho\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r}\right) = C_k \mathrm{Kn}\gamma\frac{1}{r}\frac{\partial}{\partial r}\left(r\sqrt{T}\frac{\partial T}{\partial r}\right) - (\gamma - 1)\frac{\rho T}{r}\frac{\partial}{\partial r}(ru) + \frac{4}{3}2C_\mu \mathrm{Kn}\,(\gamma - 1)\sqrt{T}\left[\left(\frac{\partial u}{\partial r}\right)^2 - \frac{u}{r}\frac{\partial u}{\partial r} + \left(\frac{u}{r}\right)^2\right] + 2C_\mu \mathrm{Kn}\,(\gamma\text{ - }1)\sqrt{T}\left[r\frac{\partial}{\partial r}\left(\frac{u}{r}\right)\right]^2, \tag{4}$$

$$P = \rho R T, \tag{5}$$

where u is the velocity along axis $r$and v is the velocity along axis φ. A rather standard notation is used in Eqs (1)–(5): ρ is density and T is the temperature. $\rho, P, T, u, v = f(r, t)$.

The equations are normalized by using the following scales: for density, $\rho_0 = mn_0$, for velocity, $V_0 = \sqrt{2RT_0}$, R is the gas constant, for length – the distance between the cylinders $L = R_2 - R_1$, for time $t_0 = L/V_0$, for temperature $T_0 = T_w$- the wall temperature of both cylinders. In (2) the pressure is eliminated by using the perfect gas law (5)

For a perfect monatomic gas, the viscosity and the heat transfer coefficient read

as [13]:

$$\mu = \mu(T) = C_\mu \rho_0 l_0 V_0 \sqrt{T}, \quad C_\mu = \frac{5}{16}\sqrt{2\pi}, \tag{6}$$

$$\lambda = \lambda(T) = C_\lambda \rho_0 l_0 V_0 \sqrt{T}, \quad C_\lambda = \frac{15}{32}\sqrt{2\pi}. \tag{7}$$

The Knudsen number in (2)–(5) is $Kn = l_0/L$, where the mean free path is l_o and $\gamma = c_P/c_V = 5/3$.

For the problem (1)-(5) and $t > 0$, first-order slip boundary conditions are imposed at both walls, which can be written as follows [13–14]:

$$u \mp 1.1466\mathrm{Kn}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) = \bar{u}_i, \tag{8}$$

$$v = 0, \tag{9}$$

$$T \pm 2.1904\,\mathrm{Kn}\,\frac{\partial T}{\partial r} = \bar{T}_i, \tag{10}$$

at $r = R_i$, $i = 1, 2$. In Eqs. (8)–(10) $\bar{u}_i = u_{w,i}/V_0$ and $\bar{T}_i = T_W/T_o = 1$ are the dimensionless wall velocity and temperature for both cylinders.

The equations of transfer (1)–(4), together with the boundary conditions (8)–(10) and zero initial distributions for u, v and T, formulate the initial nonstationary boundary-value problem. A second order of approximation, implicit difference scheme to solve numerically the formulated problem is used. Starting from the inner cylinder wall M grid nodes are introduced along the coordinate r. Thus, the difference value problem, for a given time t, is reduced to the solution of 4 linearized systems of M algebraic equations. Every algebraic system has a diagonal and weakly filled matrix. Due to problem non-linearity, an internal iteration process is used. After attaining a previously prescribed accuracy for the current time, the next time step is considered.

Direct Simulation Monte Carlo (DSMC) Method. The gas considered is simulated as a stochastic system of Nparticles [4, 5]. All quantities used are non-dimensional, so that the mean free path at equilibrium is equal to 1. The basic steps of simulation are as follows:

A. The time interval $\left[0; \hat{t}\right]$ over which the solution is found, is subdivided into subintervals with step Δt.

B. The space domain is subdivided into cells with sides $\Delta z, \Delta r$.

C. Gas molecules are simulated in gap G using a stochastic system of N points (particles) having position $z_i(t), r_i(t)$ and velocities $\overrightarrow{\xi_i(t)}$

D. N_m particles are located in the m-th cell at any given time. This number varies during the computer simulation by the following two stages:

   Stage 1. Binary collisions in each cell are calculated, whereas particles do not move. Collision modeling is realized using Bird's scheme "no time counter".

   Stage 2. Particles move with new initial velocities acquired after collisions, and no external forces act on particles. No collisions are accounted for at this stage.

E. Stage 1 and Stage 2 are repeated until $t = \hat{t}$.

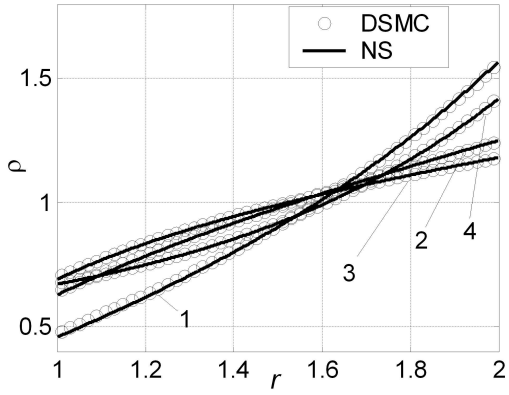

Fig. 2. Density profile in the cases 1–4

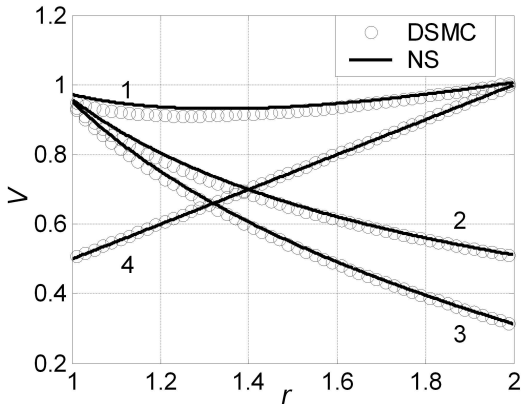

Fig. 3. φ velocity profile in the cases 1–4

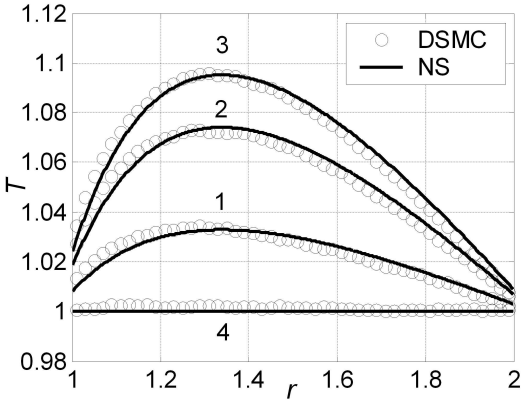

Fig. 4. Temperature profile in the cases 1–4

F. Flow macro-characteristics (density, velocity, temperature) are calculated as time-averaged when steady regime is attained.

G. Boundary conditions are diffusive over the cylinders and periodical along axis Oy.

All magnitudes used are non-dimensional so that the mean free path in equilibrium state is equal to 1. The modeling particles number for DSMC method is 3 200 000.

3. Numerical results. Let us note $\Delta = V_1 \dfrac{R_2}{R_1} - V_2$.

We study the four typical cases of rarefied gas between rotating cylinders:
Case 1: V_1=1, V_2=1, $T_1 = T_2$=1, Δ=1.5;
Case 2: V_1=1, V_2=0.5, $T_1 = T_2$=1, Δ=1.0;
Case 3: V_1=1, V_2=0.3, $T_1 = T_2$=1, Δ=1.7;
Case 4: V_1=0.5, V_2=1, $T_1 = T_2$=1, Δ=0.0.

The numerical results for the φ velocity show that the gas velocity on the walls lags from the inner cylinder velocity and it outstrips from the outer cylinder velocity in the case 1 and 2 (Figure 3). It is observed the reverse effect at lower velocity of the outer cylinder – the gas velocity lags from the outer cylinder velocity and it outstrips from the inner cylinder velocity – case 3 (Figure 3).

It is interest the φ velocity profile in the case 4. The gas moves as an elastic rigid body together with the inner and outer wall. The velocity profile is linear (Figure 3) and the temperature is constant (Figure 4). The density profile (Figure 2) is non-linear. It is inversely proportional to the gas pressure because in the presence of centripetal acceleration which is higher from the convection one, the gas pressure is proportional to r^2 and, therefore, the density is proportional to $1/r^2$. Here is only elastic deformation of the gas medium at Δ=0. It is necessary to denote that the numerical results show the same conduct of the gas medium between rotating cylinders at Kn=0.2, 0.3, 0.5 and the keeping of the velocity and temperature boundary conditions as in case 4.

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МОДЕЛИРАНЕ НА ЦИЛИНДРИЧНО ТЕЧЕНИЕ НА КУЕТ ЗА РАЗРЕДЕН ГАЗ МЕЖДУ ВЪРТЯЩИ СЕ ЦИЛИНДРИ

Петър Господинов, Добри Данков, Владимир Русинов, Стефан Стефанов

Изследвано е цилиндрично течение на Кует за разреден газ между два въртящи се цилиндъра. Получени са профилите на налягането, скоростта и температурата по метода на прякото статистическо моделиране (DSMC) и чрез числено решаване на уравненията на Навие-Стокс за свиваем флуид. Резултатите сочат много добро съвпадение за малки числа на Кнудсен Kn = 0.02. Показано е, че при различни кинематични гранични условия, газът изостава или избързва спрямо скоростта на стената, или има поведение на твърдо еластично тяло. Получените резултати са важни при решаването на неравнинни, задачи от микрофлуидиката с отчитане на ефектите на кривината.