МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2011
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2011
Proceedings of the Fortieth Jubilee Spring Conference
of the Union of Bulgarian Mathematicians
Borovetz, April 5–9, 2011

CONVEXITY, $\mathbb{C}$-CONVEXITY AND PSEUDOCONVEXITY*

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We discuss different characterizations of various notions of convexity as well as we compare these notions.

1. Introduction. Geometric convexity of a domain in $\mathbb{C}^n$ is characterized by its intersection with real lines, and it is invariant under real affine maps. Pseudoconvexity is a generalization of that notion that designed, among other things, to be invariant under all biholomorphic maps. It can be characterized by the behavior of analytic disks (Kontinuitätsatz). Linear convexity and $\mathbb{C}$-convexity are intermediate notions that bring into play (respectively) complex hyperplanes and complex lines, and are invariant under complex affine maps.

In this survey, we exploit the parallels between all those notions to highlight their similarities and differences, and the crucial role played by smoothness of the domains being considered.

The exposition is based mainly on the paper [9].

2. Balanced indicatrixes. Let D be an open set in $\mathbb{C}^n$.

We say that D is *balanced, centered at* a if $z \in D$, $\zeta \in \mathbb{C}$ with $|\zeta| \le 1$, then $a+\zeta(z-a) \in D$.

Denote by $d_D(z,X)$ the distance from $z \in D$ to the boundary ∂D in the complex direction $X \in \mathbb{C}^n$ (possibly $d_D(z,X)=\infty$), i.e.:

$$d_D(z,X) = \sup\{r>0 : z+\lambda X \in D \text{ if } |\lambda|<r\}.$$

Recall now that D is called pseudoconvex if it has a plurisubharmonic exhaustion function. For many equivalent definitions of this notion and its role in the so-called $\bar\partial$-problem see e.g. [5, 2]. We only point out that D is pseudoconvex if and only if D is a domain of holomorphy which, roughly speaking, means that there exists a holomorphic function on D that cannot be extended outside D.

When ∂D is C^2-smooth, the pseudoconvexity is equivalent to the Levi pseudoconvexity (which follows, for example, by Proposition 3 below).

*2000 Mathematics Subject Classification: 32F17.
Key words: pseudoconvex domain, (weakly) linearly convex domain, convex domain.

If r is a C^2-smooth defining function, then the restriction of the Levi form of r on the complex tangent plane at any boundary point is semipositive definite, i.e.

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(a) X_j \overline{X_k} \geq 0 \tag{1}$$

for any $a \in \partial D$ and any vector $X \in \mathbb{C}^n$ with $\langle \partial r(a), X \rangle = 0$.

Note also that if $-\log d_D(\cdot, X)$ is a plurisubharmonic function for any $X \in \mathbb{C}^n$, then D is pseudoconvex, and *vice versa*. In particular, any pseudoconvex set D admits a continuous exhaustion function; for example, $-\log \operatorname{dist}(\cdot, \partial D) = \sup_X -\log d_D(\cdot, X)$ (the function is identically $-\infty$ if $D = \mathbb{C}^n$).

Closely related to our considerations is the largest balanced domain centered at z and contained in D, i.e. $B_{D,z} = z + I_{D,z}$, where $I_{D,z}$ is the balanced indicatrix of D at z :

$$I_{D,z} = \{X \in \mathbb{C}^n : z + \lambda X \in D \text{ if } |\lambda| \leq 1\}.$$

Finally, consider the global version of this, the Hartogs-like domain

$$H_D = \{(z, w) \in D \times \mathbb{C}^n : w \in I_{D,z}\}.$$

If D is pseudoconvex, then $-\log d_D$ is a plurisubharmonic function on $D \times \mathbb{C}^n$ (cf. [5, Proposition 2.2.21]), thus H_D is pseudoconvex.

Now we consider pseudoconvexity of an open set D in $\mathbb{C}^n$ in terms of pseudoconvexity of $B_{D,z}$, $z \in D$, i.e. in terms of pseudoconvexity in the "vertical" directions of H_D.

Theorem 1. *Let D be a proper open set of $\mathbb{C}^n$. Then the following properties of D are equivalent:*

(1) *D is pseudoconvex.*

(2) *H_D is pseudoconvex.*

(3) *$B_{D,z}$ is pseudoconvex, for any $z \in D$.*

We have already seen that (1) implies (2), and (2) implies (3) is trivial (slice by the sets $\{z\} \times \mathbb{C}^n$, for $z \in D$). The implication (3) $\Rightarrow$ (1) follows by

Proposition 2. *Let D be a proper open set of $\mathbb{C}^n$ and let U be a neighborhood of ∂D. If $I_{D,a}$ is a pseudoconvex domain for any $a \in D \cap U$, then D is itself pseudoconvex.*

One of the main points in the proof of Proposition 2 (and other propositions below), is [2, Theorem 4.1.25][1]

Proposition 3. *An open set D in $\mathbb{C}^n$ is not pseudoconvex if and only if there is a point $a \in \partial D$, say the origin, and a real-valued quadratic polynomial q such that $q(a) = 0$, $\partial q(a) \neq 0$,*

$$\sum_{j,k=1}^{n} \frac{\partial^2 q}{\partial z_j \partial \overline{z_k}}(a) X_j \overline{X_k} < 0$$

[1] The first inequality on p. 242 in the proof must contains an obvious extra term. Otherwise, it is not true in general; a counterexample is given by the domain $D = \{\operatorname{Re} z < (\operatorname{Re} w)^2\}$ in $\mathbb{C}^2$.

for some vector $X \in \mathbb{C}^n$ *with* $\langle \partial q(a), X\rangle = 0$, *and* D *contains the set* $\{q < 0\}$ *near* a.

Therefore, if D *is not pseudoconvex, then, after an affine change of coordinates, we may assume* $0 \in \partial D$ *and, near this point,* D *contains the set*

$$\{z \in \mathbb{C}^n : 0 > \operatorname{Re} z_1 + (\operatorname{Im} z_1)^2 + |z_2|^2 + \cdots + |z_{n-1}|^2 + c(\operatorname{Im} z_n)^2 - (\operatorname{Re} z_n)^2\},$$

where $c < 1$.

It is interesting to note that a similar statement holds for linear convexity. Recall that (cf. [1, 2]) an open set D in $\mathbb{C}^n$ is called weakly linearly convex (resp. linearly convex) if for any $a \in \partial D$ (resp. $a \in \mathbb{C}^n \setminus D$) there exists a complex hyperplane T_a through a which does not intersect D (such a set is necessarily pseudoconvex). We call T_a a supporting complex hyperplane.

A domain D in $\mathbb{C}^n$ is said to be $\mathbb{C}$-convex if any nonempty intersection of D with a complex line is connected and simply connected. It clear that convexity implies $\mathbb{C}$-convexity. On the other hand, $\mathbb{C}$-convexity implies linearly convexity and this notation is tightly related to the linear partial differential equations in the class of holomorphic functions.

All three notions mentioned above coincide for C^1-smooth open sets. There is a simple differential characterization of $\mathbb{C}$-convexity in the C^2 case which is similar, but stronger to that of pseudoconvexity given by (1):

$$\text{Hess } r_D(a)|_{T_a^{\mathbb{C}}} \geq 0, \quad a \in \partial D.$$

Of course, convexity is characterized by the strongest similar conditions, namely

$$\text{Hess } s_D(a)|_{T_a^{\mathbb{R}}} \geq 0, \quad a \in \partial D.$$

The next two propositions demonstrate the role of circularity in the (weakly) linearly convexity.

Proposition 4. (i) (*cf.* [7]) *An open balanced set is weakly linearly convex if and only if it is convex.*

(ii) (*cf.* [8]) *If* D *is a weakly linearly convex open set, then* $B_{D,z}$ *is a convex domain for any* $z \in D$ (*i.e. the Minkowski function* $1/d_D(z, \cdot)$ *of* $I_{D,z}$ *is convex*).

Theorem 5. *Consider the following three properties:*

(1) D *is weakly linearly convex* (*resp. linearly convex*).

(2) H_D *is weakly linearly convex* (*resp. linearly convex*).

(3) $B_{D,z}$ *is* (*weakly linearly*) *convex, for any* $z \in D$.

Then, (1) *and* (2) *are equivalent, and imply* (3). *If* D *is a* $C^{1,1}$*-smooth bounded domain, then* (3) *implies* (1).

The last statement follows from [8]. Note that in this case, the domain D is in fact $\mathbb{C}$-convex. The domain H_D, however, does not share the smoothness of D, and may fail to be $\mathbb{C}$-convex.

Example 6. If $D = \{z \in \mathbb{C} : |z - 1| < 2 \text{ or } |z + 1| < 2\}$, then H_D is not $\mathbb{C}$-convex.

If we turn to the third, and more usual notion of convexity, it is clear that a domain D in $\mathbb{R}^n$ is convex if and only if H_D is convex in $\mathbb{R}^n \times \mathbb{R}^n$.

3. Defining functions.

We point out that proof that the convexity of $B_{D,z}$ implies linear convexity for $\mathcal{C}^{1,1}$ domains [8, Proposition 1 & introduction] is based on the following result which can be easily deduced from [3]. Let s_D stand for the signed distance to ∂D.

Proposition 7. *If D is a $C^{1,1}$-smooth bounded domain in $\mathbb{C}^n$ and*

$$\liminf_{T_a^{\mathbb{C}} \ni z \to a} \frac{s_D(z)}{|z-a|^2} \geq 0$$

for $a \in \partial D$ almost everywhere, then D is linearly convex.

Proposition 7 has an obvious convex analog.

Proposition 8. *A proper domain D in $\mathbb{R}^n$ is convex if and only if for any $a \in \partial D$ there exists a (real) hyperplane S_a through a such that*

$$\liminf_{S_a \ni x \to a} \frac{s_D(x)}{|x-a|^2} \geq 0.$$

If D is convex, then obviously S_a is a (real) supporting hyperplane.

Note that these two propositions are known in the C^2-smooth case, since the limits are equal to the minimal eigenvalue of $2\mathrm{Hess}_{s_D}(a)|_{T_a^{\mathbb{C}}}$, respectively $2\mathrm{Hess}_{s_D}(a)|_{T_a^{\mathbb{R}}}$.

We also point out that the proof of Proposition 8 is based on the "convex" version of Proposition 3. This version implies the following simple, but useful characterization of (non)convexity:

A domain D is not convex if and only there exists a segment in $\overline{D}$ such that only its midpoint lies on ∂D.

Clearly, the relationship between a domain and its defining function is not symmetric, as convexity of one sublevel set (or indeed, of all of them) cannot imply convexity of the function: simply compose by a monotone increasing function from the real line to itself. Given a convex domain, the question arises of how to choose a convex defining function, and of how much choice one may have.

By [4, Proposition], a smooth bounded domain D is convex if and only if $-\log s_D$ is convex near ∂D. Thanks to [2, Theorem 2.1.27], this result can be easily generalized.

Proposition 9. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a nonconstant decreasing and convex function. Let U be a neighborhood of the boundary of a proper domain D in $\mathbb{R}^n$. Then, D is convex if and only if $f \circ s_D$ is a convex function on $D \cap U$.*

In particular, if one of the defining functions given above is convex on a neighborhood of ∂D, then all the others are convex too.

Note that the conditions the function f to be decreasing and convex are necessary as the following example shows.

Example 10. Let $D = \mathbb{R}^+ \times \mathbb{R}^+$ and let $f : \mathbb{R}^+ \to \mathbb{R}$ be a nonconstant function such that $f \circ s_D$ is a convex function on D. Then f is decreasing and convex.

The pseudoconvex analog of Proposition 9 is the following

Proposition 11. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonconstant increasing and convex function. Let U be a neighborhood of the boundary of a proper domain D in $\mathbb{C}^n$. Then, D is* *pseudoconvex if and only if* $f \circ q_D$ *is a plurisubharmonic function on* $D \cap U$, *where* $q_D = -\log s_D$.

Note that there is a smooth bounded pseudoconvex domain in $\mathbb{C}^2$ having no defining function which is plurisubharmonic on a two-sided neighborhood of the boundary. We do not know under which general conditions on f the plurisubharmonicity of $f \circ s_D$ is equivalent to the pseudoconvexity of D.

We also point out that the proofs of Propositions 8 and 11 imply a result similar to Propositions 7 and 8 in the pseudoconvex case:

Proposition 12. *If* D *is a proper open set in* $\mathbb{C}^n$ *and for any* $a \in \partial D$ *there exists a complex hyperplane* S_a *through* a *such that*

$$\liminf_{S_a \ni z \to a} \frac{s_D(z) + s_D(a + J(z-a))}{|z-a|^2} \geq 0,$$

where J *is the standard complex structure, then* D *is pseudoconvex.*

The converse is also true if D is a C^2-smooth open set. We do not know if the smoothness can be weakened.

4. Slicing. It is known that an open set D in $\mathbb{C}^n$ ($n \geq 3$) is pseudoconvex if and only if any two-dimensional slice of D is pseudoconvex [6]. We would like to restrict the family of slices that has to be used in order to detect pseudoconvexity, namely we would like to consider the family of complex planes passing through a point $a \in \mathbb{C}^n$. As the next results show, it will be enough generically. Given an open non-pseudoconvex set D in $\mathbb{C}^n$, call *a exceptional* with respect to D if for any 2-dimensional complex plane $P \ni a$, $P \cap D$ is pseudoconvex. The next proposition shows that the set of exceptional points has to be contained in a complex hyperplane.

Proposition 13. *Let* D *be an open non-pseudoconvex set in* $\mathbb{C}^n$ *(*$n \geq 3$*). Let* S *be the union of all* 2*-dimensional complex planes with non-empty and non-pseudoconvex intersections with* D, *so the set of exceptional points is* $\mathbb{C}^n \setminus S$. *Then, there exists a complex hyperplane* T *such that* $\mathbb{C}^n \setminus S \subset T$.

If D is C^2-smooth, then we have more for the set of exceptional points with respect to D.

Proposition 14. *Let* D *and* S *be as in Proposition* 13*. If* D *is* C^2*-smooth, then there is a complex plane* T *of codimension* 3 *such that* $\mathbb{C}^n \setminus S \subset T$.

The proof of this proposition is based on the following lemma of independent interest.

Lemma 15. *Let* M *be a* C^1*-smooth hypersurface in* $\mathbb{C}^2$*. Then, the complex tangent line at some point of* M *does not contain the origin.*

The following example shows that there can be an exceptional point even in the C^∞-smooth 3-dimensional case.

Example 16. There exists a bounded, $\mathcal{C}^\infty$-smooth domain Ω in $\mathbb{C}^3$ such that the set of non-pseudoconvex points is a nonempty relatively open subset of $\partial\Omega$, but such that $P \cap \Omega$ is pseudoconvex for any complex plane P containing the origin.

In the non-smooth 3-dimensional case we may have more than one exceptional point.

Example 17. Let $a \in \mathbb{C}^3$, G be a pseudoconvex set in $\mathbb{C}^3$, and let l_1, l_2 be distinct complex lines in $\mathbb{C}^3$ that intersect G. Then:

(i) the intersection of $G \setminus l_1$ with any 2-dimensional complex plane through a is pseudoconvex if and only if $a \in l_1 \setminus G$.

(ii) the intersection of $G \setminus (l_1 \cup l_2)$ with any 2-dimensional complex plane through a is pseudoconvex if and only if $G \not\ni a = l_1 \cap l_2$.

Using Proposition 7, similar arguments as in the proof of Proposition 13 imply that if a is a point in C^2-smooth domain D such that any non-empty intersection of D with a 2-dimensional complex plane through a is weakly linearly convex, then D is $\mathbb{C}$-convex.

The following example shows that we have no such a phenomenon in general.

Example 18. Let

$$D = \{z \in \mathbb{C}^3 : |z| < \sqrt{2} \max\{|z_1|, |z_2|, |z_3|\}\}.$$

Then, D is a union of three disjoint linearly convex domains and D has a non-empty linearly convex intersection with any complex plane through 0 (in particular, D is pseudoconvex and not weakly linearly convex).

In spite of Example 18, one may also conjecture the following:

If D is an open set in $\mathbb{C}^n$ such that any non-empty intersection with 2-dimensional complex plane is (weakly) linearly convex, then D is (weakly) linearly convex.

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ИЗПЪКНАЛОСТ, $\mathbb{C}$-ИЗПЪКНАЛОСТ И ПСЕВДОИЗПЪКНАЛОСТ

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Разгледани са характеризации на различни понятия за изпъкналост, като тези понятия са сравнени.