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ON THE SEMIGROUP OF ALL INCREASING FULL TRANSFORMATIONS*

Ilinka Dimitrova

Let T_n be the full transformation semigroup on an n-element set and let Inc_n be the subsemigroup of T_n consisting of all increasing transformations. This paper considers some algebraic and rank properties of the semigroup Inc_n. It is shown that the semigroup Inc_n is an $\mathcal{R}$-trivial idempotent-generated semigroup. Finally, the maximal subsemigroups of Inc_n are characterized.

For $n \in \mathbb{N}$ let $X_n = \{1 < 2 < \cdots < n\}$ be a finite n-element chain and let T_n denote the full transformation semigroup, i.e. the semigroup of all mappings $\alpha : X_n \to X_n$ under composition. We say that a transformation $\alpha \in T_n$ is *increasing* (*decreasing*) if $x \le x\alpha$ ($x \ge x\alpha$) for all $x \in X_n$. This paper investigates algebraic and rank properties of the semigroup Inc_n of all increasing mappings of X_n. Algebraic and rank properties of T_n have been studied over a long period and many interesting results have emerged (see, for example [2], [4], [7]). In [6], Umar considered algebraic and rank properties of the semigroup D_n of all decreasing transformations. Moreover, Umar ([5]) showed that the semigroups Inc_n and D_n are isomorphic. Finally, we characterize all maximal subsemigroups of Inc_n.

We begin by recalling some notations and definitions that are used in the paper. For the standard terms and concepts in semigroup theory we refer to [3]. For every transformation $\alpha \in T_n$, we denote by $\ker\alpha$ and $\operatorname{im}\alpha$ the kernel and the image of α, respectively. The number $\operatorname{rank}\alpha := |X_n/\ker\alpha| = |\operatorname{im}\alpha|$ is called the rank of α. Let U be a subset of T_n, we denote by $E(U)$ the set of all idempotents in the set U.

Let $A \subseteq X_n$ and let π be an equivalence relation on X_n. We say that A is a *transversal* of π (denoted by $A \# \pi$) if $|A \cap \bar{x}| = 1$ for every equivalence class $\bar{x} \in X_n/\pi$.

Now, for $0 \le r \le n$, we consider the two-sided ideals of Inc_n and D_n

$$K^+(n,r) := \{\alpha \in Inc_n : |\operatorname{im}\alpha| \le r\},$$

$$K^-(n,r) := \{\alpha \in D_n : |\operatorname{im}\alpha| \le r\}$$

respectively. Also, for $2 \le r \le n$, we consider the Rees quotient semigroups of Inc_n and D_n

$$P_r^+ := K^+(n,r)/K^+(n,r-1),$$

$$P_r^- := K^-(n,r)/K^-(n,r-1),$$

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respectively. The elements of P_r^+ (P_r^-) may be thought as the elements of Inc_n (D_n) of rank r precisely. The product of two elements of P_r^+ (P_r^-) is 0 whenever their product in Inc_n (D_n) is of rank strictly less than r. The set $P_n^+ = P_n^-$ contains exactly one element, namely the identity, which we denote by ϵ.

Now, we recall some results for the semigroup D_n that are useful in what follows.

Proposition 1 ([6]). *Let $0 \le r \le n$. Then, every $\alpha \in P_r^-$ is expressible as a product of idempotents in P_r^-.*

Proposition 2 ([6]). *Let $0 \le k \le n-1$ and let $\varepsilon \in E(K^-(n,r))$. Then, ε is expressible as a product of idempotents in P_r^-.*

Therefore, from the isomorphism between Inc_n and D_n, it follows:

Proposition 3. *Let $0 \le r \le n$. Then every $\alpha \in P_r^+$ is expressible as a product of idempotents in P_r^+.*

Proposition 4. *Let $0 \le k \le n-1$ and let $\varepsilon \in E(K^+(n,r))$. Then, ε is expressible as a product of idempotents in P_r^+.*

From Proposition 3 and Proposition 4, we have:

Corollary 1. *Let $0 \le r \le n-1$. Then, $K^+(n,r) = \langle E(P_r^+)\rangle$ as well as $Inc_n = \langle E(P_{n-1}^+), \epsilon\rangle$.*

For the definition of Green's relations, see for example [3].

Proposition 5 ([6]). *The semigroup D_n is $\mathcal{R}$-trivial.*

Again, using the isomorphism between Inc_n and D_n, we obtain:

Proposition 6. *The semigroup Inc_n is $\mathcal{R}$-trivial.*

To identify the classes of semigroups to which Inc_n belongs, we consider the starred Green's relations studied in [1]. Recall that on a semigroup S the relation $\mathcal{L}^*$ ($\mathcal{R}^*$) is defined by the rule that $\alpha\mathcal{L}^*\beta$ ($\alpha\mathcal{R}^*\beta$) if and only if the elements α, β are related by the Green's relation $\mathcal{L}$ ($\mathcal{R}$) in some oversemigroup of S. The intersection of the equivalences $\mathcal{L}^*$ and $\mathcal{R}^*$ is denoted by $\mathcal{H}^*$. To define $\mathcal{J}^*$ we first denote the $\mathcal{L}^*$-class containing the element α of the semigroup S by L_α^*. (The corresponding notation is used for the classes of the other relations.) Then, a left (right) $*$-ideal of a semigroup S is defined to be a left (right) ideal I of S such that $L_\alpha^* \subseteq I$ ($R_\alpha^* \subseteq I$), for all $\alpha \in I$. A subset I of S is a $*$-ideal of S if it is both a left $*$-ideal and a right $*$-ideal. We also recall from [1], that the principal $*$-ideal $J^*(\alpha)$ generated by the element α of S is the intersection of all $*$-ideals of S to which α belongs. The relation $\mathcal{J}^*$ is defined by the rule that $\alpha\mathcal{J}^*\beta$ if and only if $J^*(\alpha) = J^*(\beta)$.

We begin our investigations by the observation that $K^+(n,r)$ is a $*$-ideal of Inc_n and for $\alpha, \beta \in Inc_n$ it holds

$$\begin{aligned} \alpha\mathcal{L}^*\beta &\iff \operatorname{im}\alpha = \operatorname{im}\beta \\ \alpha\mathcal{R}^*\beta &\iff \ker\alpha = \ker\beta \\ \alpha\mathcal{J}^*\beta &\iff \operatorname{rank}\alpha = \operatorname{rank}\beta \\ \mathcal{H}^* \;\; &= \;\; \mathcal{L}^* \cap \mathcal{R}^*. \end{aligned}$$

Thus the semigroup Inc_n, like T_n itself, is the union of $\mathcal{J}^*$-classes $J_0^*, J_1^*, \ldots, J_n^*$, where

$$J_r^* := \{\alpha \in Inc_n : \operatorname{rank}\alpha = r\} \text{ for } r = 0, \ldots, n.$$

The $\mathcal{J}^*$-class J_n^* contains exactly one element, namely ϵ.

For $0 \leq r \leq n$, we denote by E_r the set of all idempotents in J_r^*.

Further, we pay attention to the $\mathcal{J}^*$-class J_{n-1}^*. The $\mathcal{R}^*$-, $\mathcal{L}^*$- and $\mathcal{H}^*$-classes in J_{n-1}^* have the following form

$$R_{(i,j)}^* := \{\alpha \in J_{n-1}^* : \text{ the sole non-singleton class of } \ker\alpha \text{ is } \{i,j\}\},$$

$1 \leq i < j \leq n$.

$$L_k^* := \{\alpha \in J_{n-1}^* : \operatorname{im}\alpha = X_n \setminus \{k\}\}, \quad 1 \leq k \leq n-1.$$

$$H_{(i,j),k}^* := R_{(i,j)}^* \cap L_k^*, \quad 1 \leq k \leq i < j \leq n.$$

Clearly, $|H_{(i,j),k}^*| = 1$ for $k = i-1, i$. Moreover, J_{n-1}^* has exactly $\dfrac{n(n-1)}{2}$ different $\mathcal{R}^*$-classes and exactly $n-1$ different $\mathcal{L}^*$-classes. It is obvious that an element α in the $\mathcal{H}^*$-class $H_{(i,j),k}^*$ is idempotent if and only if $\operatorname{im}\alpha \ \# \ \ker\alpha$, i.e. if and only if $k = i$. We denote by $\varepsilon_{(i,j)}$ the idempotent of the class $H_{(i,j),i}^*$. Therefore, we have that

Lemma 1. *Every $\mathcal{R}^*$-class of J_{n-1}^* contains a unique idempotent.*

Since the product $\alpha\beta$, for all $\alpha, \beta \in J_{n-1}^*$, belongs to the class J_{n-1}^* (if and only if $\alpha\beta \in R_\alpha^* \cap L_\beta^*$) if and only if $\operatorname{im}\alpha \ \# \ \ker\beta$, it is easy to show:

Lemma 2. *Let $\alpha, \beta \in J_{n-1}^*$. Then,*

$$H_{(i,j),k}^* H_{(p,q),l}^* = \begin{cases} H_{(i,j),l}^*, & \text{if } k = p, k = q \\ 0, & \text{otherwise.} \end{cases}$$

We say that the element $\alpha \in Inc_n$ is *undecomposable* in Inc_n if there are no elements $\beta, \gamma \in Inc_n \setminus \{\alpha\}$ such that $\alpha = \beta\gamma$.

Proposition 7. *Idempotents of the class J_{n-1}^* are undecomposable in Inc_n.*

Proof. Let $\varepsilon_{(i,j)} \in E_{n-1}$ for some $1 \leq i < j \leq n$. Suppose that there exist transformations $\beta, \gamma \in Inc_n \setminus \{\varepsilon_{(i,j)}\}$ such that $\varepsilon_{(i,j)} = \beta\gamma$. Clearly, the elements β and γ have to be contained in J_{n-1}^* or in $J_n^* = \{\epsilon\}$. But ϵ can be excluded as factor. Then, $R_\beta^* = R_{(i,j)}^*$, $L_\gamma^* = L_i^*$, $\operatorname{im}\beta \ \# \ \ker\gamma$, and $\beta \in L_q^*$ for some $1 \leq q \leq n$. Thus $\beta \in H_{(i,j),q}^*$ with $q \leq i < j$. Since $\operatorname{im}\beta \ \# \ \ker\gamma$, it follows that $\gamma \in H_{(p,q),i}^*$ for some $1 \leq p < q \leq n$ or $\gamma \in H_{(q,s),i}^*$ for some $1 \leq q < s \leq n$. If $\gamma \in H_{(p,q),i}^*$ then $i \leq p < q$, which contradicts $q \leq i < j$. Thus $\gamma \in H_{(q,s),i}^*$ with $i \leq q < s$ and from $q \leq i < j$ we obtain $q = i$. Therefore, $\beta \in H_{(i,j),i}^*$ and since $|H_{(i,j),i}^*| = 1$, we have $\beta = \varepsilon_{(i,j)}$, a contradiction. $\square$

From Corollary 1 and Proposition 7 we have

Corollary 2. *If $M \subset E_{n-1}$ then $\langle M \rangle \subset K^+(n, n-1)$, i.e. no proper subset of E_{n-1} can generate $K^+(n, n-1)$.*

The rank of a finite semigroup S is usually defined by

$$\operatorname{rank} S = \min\{|A| : A \subseteq S, \langle A \rangle = S\}.$$

If S is idempotent generated, then the idempotent rank of S is defined by

$$\operatorname{idrank} S = \min\{|A| : A \subseteq E(S), \langle A \rangle = S\}.$$

Since there are exactly $\dfrac{n(n-1)}{2}$ different $\mathcal{R}^*$-classes in J_{n-1}^* and each $\mathcal{R}^*$-class contains a unique idempotent (see Lemma 1), we have $|E_{n-1} \cup \{\epsilon\}| = \dfrac{n(n-1)}{2} + 1$. Then, Corollary 1 and Corollary 2 provide

$$\operatorname{rank} Inc_n = \operatorname{idrank} Inc_n = \frac{n(n-1)}{2} + 1.$$

Now we are able to characterize the maximal subsemigroups of Inc_n.

Lemma 3. *Every maximal subsemigroup of Inc_n contains the ideal $K^+(n, n-2)$.*

Proof. Let S be a maximal subsemigroup of Inc_n. Assume that $J_{n-1}^* \subset S$, then $K^+(n, n-2) \subset K^+(n, n-1) = \langle J_{n-1}^* \rangle \subset S$. If $J_{n-1}^* \not\subseteq S$, then $S \cup K^+(n, n-2)$ is a proper subsemigroup of Inc_n since $K^+(n, n-2)$ is an ideal, and, hence, $S \cup K^+(n, n-2) = S$ by maximality of S. This implies $K^+(n, n-2) \subset S$. $\square$

Theorem 1. *A subsemigroup S of Inc_n is maximal if and only if it belongs to one of the following types:*
1) $S_\epsilon = Inc_n \setminus \{\epsilon\}$;
2) $S_{(i,j)} = Inc_n \setminus \{\varepsilon_{(i,j)}\}$, *for* $1 \le i < j \le n$.

Proof. It is clear that both S_ϵ and $S_{(i,j)}$ are maximal subsemigroups of Inc_n, since $S_\epsilon \cup \{\epsilon\} = S_{(i,j)} \cup \{\varepsilon_{(i,j)}\} = Inc_n$.

For the converse part, let S be a maximal subsemigroup of Inc_n. Then, $S = K^+(n, n-2) \cup T$, where $T \subset (J_{n-1}^* \cup J_n^*)$ (see Lemma 3). If $J_n^* \not\subseteq T$ then $S \subseteq S_\epsilon$ since $J_n^* = \{\epsilon\}$ and thus $S = S_\epsilon$ by the maximality of S. Let $J_n^* \subseteq T$. Then, $J_{n-1}^* \not\subseteq T$. Since $J_{n-1}^* \subseteq \langle E_{n-1} \rangle$, by Proposition 3, the set T does not contain at least one idempotent $\delta \in E_{n-1}$. Assume that there exists an element $\alpha \in J_{n-1}^* \setminus T$ and $\alpha \neq \delta$. Then, from $\epsilon\alpha = \alpha\epsilon = \alpha$ and Proposition 7, it follows that $\delta \notin \langle S, \alpha \rangle$. This contradicts the maximality of S and thus we have $T = J_n^* \cup (J_{n-1}^* \setminus \{\delta\})$. Therefore, $S = S_{(i,j)}$ for a suitable $1 \le i < j \le n$. $\square$

Since the idempotents in J_{n-1}^* are exactly $\dfrac{n(n-1)}{2}$, we have

Corollary 3. *The semigroup Inc_n contains exactly $\dfrac{n(n-1)}{2} + 1$ maximal subsemigroups.*

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Ilinka Dimitrova
Faculty of Mathematics and Natural Science
South-West University "Neofit Rilski"
2700 Blagoevgrad, Bulgaria
e-mail: ilinka_dimitrova@yahoo.com

ВЪРХУ ПОЛУГРУПАТА ОТ ВСИЧКИ НАРАСТВАЩИ ПЪЛНИ ПРЕОБРАЗУВАНИЯ

Илинка А. Димитрова

Полугрупата T_n от всички пълни преобразувания върху едно n-елементно множество е изучавана в различни аспекти ог редица автори. Обект на разглеждане в настоящата работа е полугрупата Inc_n състояща се от всички нарастващи пълни преобразувания. Очевидно Inc_n е подполугрупа на T_n. Доказано е, че всеки елемент на полугрупата Inc_n от ранг r може да се представи като произведение на идемпотенти от същия ранг и всеки идемпотент от ранг по-малък или равен на r може да се представи като произведение на идемпотенти от ранг r. С помощта на тези твърдения е показано, че полугрупата Inc_n се поражда от множеството на всички идемпотенти от ранг $n - 1$ и тъждественото преобразувание. Освен това е доказано, че идемпотентите от ранг $n - 1$ са неразложими в полугрупата Inc_n. В резултат на това е получено, че рангът и идемпотичниат ранг на разглежданата полугрупа са равни. Като са използвани тези твърдения е направена пълна класификация на маскималните подполугрупи на полугрупата Inc_n.