МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2011
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2011
Proceedings of the Fortieth Jubilee Spring Conference
of the Union of Bulgarian Mathematicians
Borovetz, April 5–9, 2011

ON AFFINE CONNECTIONS IN A RIEMANNIAN MANIFOLD WITH A CIRCULANT METRIC AND TWO CIRCULANT AFFINOR STRUCTURES*

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In the present paper it is considered a class V of 3-dimensional Riemannian manifolds M with a metric g and two affinor tensors q and S. It is defined another metric $\bar{g}$ in M. The local coordinates of all these tensors are circulant matrices. It is found: 1) a relation between curvature tensors R and $\bar{R}$ of g and $\bar{g}$, respectively; 2) an identity of the curvature tensor R of g in the case when the curvature tensor $\bar{R}$ vanishes; 3) a relation between the sectional curvature of a 2-section of the type $\{x, qx\}$ and the scalar curvature of M.

1. Introduction. In this paper we investigate the class V of manifolds admitting an additional structure q, such that it's cube degree is the identity. In the basic manifold M the metric g is positively defined and q is a parallel structure with respect to the affine connection ∇ of g. By g and q we construct another metric f which is non-degenerate. By f we obtain an affine connection $\overline{\nabla}$. Our main problem is to find a subclass of V, such that $\overline{\nabla}$ is a locally flat connection.

We consider a 3-dimensional Riemannian manifold M with a metric tensor g and two affinor tensors q and S such that: their local coordinates form circulant matrices. So these matrices are as follows:

$$g_{ij} = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}, \quad A > B > 0, \tag{1}$$

where A and B are smooth functions of a point $p(x^1, x^2, x^3)$ on some $F \subset R^3$,

$$q_i^{.j} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad S_i^j = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}. \tag{2}$$

Let ∇ be the connection of g. The following results have been obtained in [1]:

$$q^3 = E; \quad g(qx, qy) = g(x, y), \quad x,\ y \in \chi M. \tag{3}$$

$$\nabla q = 0 \quad \Leftrightarrow \quad \operatorname{grad} A = \operatorname{grad} B.S. \tag{4}$$

*2000 Mathematics Subject Classification: 53C15, 53B20.
Key words: Riemannian manifold, affinor structure, curvatures.
This work is partially supported by project RS09 - FMI - 003 of the Scientific Research Fund, Paisii Hilendarski University of Plovdiv, Bulgaria.

$$(5) \qquad 0 < B < A \quad \Rightarrow \ g \text{ is possitively defined.}$$

If M has a metric g from (1), affinor structures q and S from (2) and $\nabla q = 0$, then we note for brevity that M is in the class V.

Now, we give an example of a manifold of this class. Let

$$(6) \qquad A = (x^1)^2 + (x^2)^2 + (x^3)^2, \quad B = x^1x^2 + x^2x^3 + x^1x^3,$$

be two functions of a point $p(x^1, x^2, x^3) \neq (x, x, x)$. Then, $A > B > 0$ and

$$(7) \qquad g_{ij} = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}$$

is positively defined. Also, we obtain $\operatorname{grad} A = \operatorname{grad} B.S$ which implies $\nabla q = 0$. So, the manifold M with a metric g, defined by (6) and (7), and affinor structures q and S, defined by (2), is in the class V.

We denote $\tilde{q}_j^s = q_a^s q_j^a$, $\Phi_j^s = q_j^s + \tilde{q}_j^s$, and from (2) we have:

$$(8) \qquad \tilde{q}_j^{.s} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \Phi_j^s = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

2. Affine connections. Let M be in V. We denote $f_{ij} = g_{ik}q_j^k + g_{jk}q_i^k$, i.e.

$$(9) \qquad f_{ij} = \begin{pmatrix} 2B & A+B & A+B \\ A+B & 2B & A+B \\ A+B & A+B & 2B \end{pmatrix}.$$

We calculate $\det f_{ij} = 2(A-B)^2(A+2B) \neq 0$, so f is a non-degenerated symmetric tensor field. Evidently, we have that $\nabla q = 0$, which thank's to (2), (8) and (9), implies:

$$(10) \qquad \nabla \tilde{q} = 0, \qquad \nabla f = 0, \qquad \nabla S = 0, \qquad \nabla \Phi = 0.$$

For later use, from (1) – (9), we find next identities:

$$(11) \qquad \Phi_j^s g_{is} = f_{ji}, \qquad \Phi_j^s f_{is} = 2g_{ji} + f_{ji}, \qquad f_{ji}g^{is} = \Phi_j^s \qquad g_{ji}f^{is} = \frac{1}{2}S_j^s.$$

Further, we suppose that α and β are two smooth functions in F, such that $\alpha \neq \beta$, $\alpha + 2\beta \neq 0$. Now, we construct the metric $\bar{g}$ as follows

$$(12) \qquad \bar{g} = \alpha.g + \beta.f.$$

The local coordinates of $\bar{g}$ are

$$\bar{g}_{ij} = \begin{pmatrix} \alpha A + 2\beta B & \beta A + (\alpha+\beta)B & \beta A + (\alpha+\beta)B \\ \beta A + (\alpha+\beta)B & \alpha A + 2\beta B & \beta A + (\alpha+\beta)B \\ \beta A + (\alpha+\beta)B & \beta A + (\alpha+\beta)B & \alpha A + 2\beta B \end{pmatrix}.$$

Since $\det \bar{g}_{ij} = (\alpha - \beta)^2(A-B)^2(A+2B)(\alpha+2\beta) \neq 0$, $\bar{g}$ is a non-degenerated tensor field.

Let $\alpha > \beta > 0$, then $\alpha A + 2\beta B > \beta A + (\alpha+\beta)B > 0$. Analogously to (5) we state that $\overline{g}$ is positively defined.

Let $\alpha = 0, \beta \neq 0$.

(a) If $\beta > 0$, then the main minors of the matrix $\bar{g}_{ij}$ are: $2\beta B > 0$, $\beta^2(B-A)(A+3B) < 0$, $(-\beta)^2(A-B)^2(A+2B)2\beta > 0$. We state that $\overline{g}$ is an indefinite metric.

(b) If $\beta < 0$, then analogously to (a) we state that $\overline{g}$ is an indefinite metric.

In [2] it is proved the next assertion:

Theorem 2.1. *Let M be a manifold in V, g and $\bar{g}$ be two metrics of M, related by (12). Let ∇ and $\overline{\nabla}$ be the corresponding connections of g and $\bar{g}$. Then, $\overline{\nabla}q = 0$ if and only if, when*

$$\operatorname{grad}\alpha = \operatorname{grad}\beta.S. \tag{13}$$

Let $\beta = 0$ in (12). Then, we have

$$\bar{g} = \alpha.g. \tag{14}$$

The condition (14) defines the well-known conformal transformation in the Riemannian manifold M.

The general case, when $\alpha \neq 0$, $\beta \neq 0$ in (12), leads to very complex calculations and it will be an object of next investigations.

Now, we consider the case $\alpha = 0$ in (12). We obtain

$$\bar{g}_{ij} = \beta.f_{ij}. \tag{15}$$

Then, from (13) we can get that $\overline{\nabla}q = 0$ if and only if, when β is a constant. Further, we suppose $\overline{\nabla}q \neq 0$, i.e. β is not a constant. Thanks to (15) we get

$$\nabla_k \bar{g}_{ij} = \beta_k f_{ij}, \qquad \beta_k = \nabla_k \beta. \tag{16}$$

We have the well-known identities:

$$\overline{\nabla}_k \bar{g}_{ij} = \partial_k \bar{g}_{ij} - \overline{\Gamma}^a_{ki}\bar{g}_{aj} - \overline{\Gamma}^a_{kj}\bar{g}_{ai}, \tag{17}$$

$$\nabla_k \bar{g}_{ij} = \partial_k \bar{g}_{ij} - \Gamma^a_{ki}\bar{g}_{aj} - \Gamma^a_{kj}\bar{g}_{ai}, \tag{18}$$

$$\overline{\nabla}_k \bar{g}_{ij} = 0. \tag{19}$$

Using (11), (16) – (19), for the tensor $T^s_{ik} = \overline{\Gamma}^s_{ki} - \Gamma^s_{ki}$ of the affine deformation of ∇ and $\overline{\nabla}$ we find

$$T^s_{ik} = \beta_k \delta^s_i + \beta_i \delta^s_k - \frac{1}{2}\beta^a S^s_a f_{ik}, \qquad \beta_k \sim \frac{\beta_i}{2\beta}. \tag{20}$$

We have that $\overline{\nabla}_i q^k_j = \nabla_i q^k_j - \beta_j q^k_i + \tilde{\beta}_j \delta^k_i - \frac{1}{2}\beta^a S^k_a q^t_j f_{ti} + \frac{1}{2}\beta^a S^t_a q^k_t f_{ij}$.

Let R be the curvature tensor field of ∇. Let $\bar{R}$ be the curvature tensor field of $\overline{\nabla}$. It is well-known the relation (see [3])

$$\overline{R}^h_{ijk} = R^h_{ijk} + \nabla_j T^h_{ik} - \nabla_k T^h_{ij} + T^s_{ik}T^h_{sj} - T^s_{ij}T^h_{sk}. \tag{21}$$

From (20) and (21) after some calculations we obtain

$$\begin{aligned} \overline{R}^h_{ijk} &= R^h_{ijk} + \delta^h_k(\nabla_j\beta_i - \beta_i\beta_j + \varphi f_{ij}) - \delta^h_j(\nabla_k\beta_i - \beta_i\beta_k + \varphi f_{ik}) \\ &+ \frac{1}{2}f_{ij}S^h_t(\nabla_k\beta^t - \beta_k\beta^t) - \frac{1}{2}f_{ik}S^h_t(\nabla_j\beta^t - \beta_j\beta^t), \quad \varphi = \frac{1}{2}\beta^t\beta_s S^s_t. \end{aligned} \tag{22}$$

Theorem 2.2. *Let M be in V, ∇ and $\overline{\nabla}$ be the Riemannian connections of g and $\bar{g}$, related by (15). If $\overline{\nabla}$ is a locally flat connection, then the curvature tensor field R of ∇*

is

$$R(x,y,z,u)=\frac{\tau}{6}[(2g(x,u)g(y,z)-2g(x,z)g(y,u)$$
(23)
$$+(g(qx,u)+g(x,qu))(g(qy,z)+g(y,qz))$$
$$-(g(qx,z)+g(x,qz))(g(qy,u)+g(y,qu))],$$

where $x,\ y,\ z,\ u\in\chi M$.

Proof. We have $\overline{R}=0$. From (22) we find

(24) $$R^h_{ijk}=\delta^h_j P_{ki}-\delta^h_k P_{ij}-f_{ij}Q^h_k+f_{ik}Q^h_j,$$

where $P_{ki}=\nabla_k\beta_i-\beta_i\beta_k+\varphi f_{ik}$, $Q^h_k=\frac{1}{2}S^h_t(\nabla_k\beta^t-\beta_k\beta^t)$.

Now, we put $k=h$ in (24) and with the help of (11) we get

(25) $$R_{ij}=-P_{ij}-\psi f_{ij},\qquad \psi=\frac{1}{2}S^h_t\nabla_h\beta^t.$$

We note that $R_{ij}=R^k_{ijk}$ are the local coordinates of the Ricci tensor of ∇, also $\tau=R_{ij}g^{ij}$ and $\tau^*=R_{ij}f^{ij}$ are the first and the second scalar curvatures of M, respectively. The identity (25) implies

(26) $$\tau^*=-2\varphi-4\psi.$$

Using (11), we have that $Q^h_k=P_{ka}f^{ah}-\varphi\delta^h_k$, and from (25) we get

(27) $$Q^h_k=-R_{ka}f^{ah}-(\psi+\varphi)\delta^h_k.$$

We substitute (25) – (27) in (24), and find

(28) $$R^h_{ijk}=\delta^h_k\left(R_{ij}-\frac{\tau^*}{2}f_{ij}\right)-\delta^h_j\left(R_{ki}-\frac{\tau^*}{2}f_{ki}\right)+f_{ij}R_{ka}f^{ah}-f_{ik}R_{ja}f^{ah}.$$

From (28) and $R^h_k=R^h_{ijk}g^{ij}$ we have

(29) $$2R^h_k=\tau\delta^h_k+\frac{\tau^*}{2}\Phi^h_k-\Phi^t_kR_{ta}f^{ah}.$$

Now, we contract (29) with f_{ih}, and from identity $f_{ih}R^h_k=\Phi^a_iR_{ka}$ we obtain:

$$2\Phi^a_iR_{ka}=\left(\frac{\tau^*}{2}+\tau\right)f_{ki}+\tau^*g_{ki}-\Phi^t_kR_{ti}$$

and

$$2\Phi^a_kR_{ia}=\left(\frac{\tau^*}{2}+\tau\right)f_{ki}+\tau^*g_{ki}-\Phi^t_iR_{tk}.$$

The last system of two equations implies

(30) $$\Phi^a_kR_{ia}=\frac{1}{3}\left(\left(\frac{\tau^*}{2}+\tau\right)f_{ki}+\tau^*g_{ki}\right).$$

From (11) and (30) we find

(31) $$\Phi^a_kR_{ia}f^{ij}=\frac{1}{3}\left(\left(\frac{\tau^*}{2}+\tau\right)\delta^j_k+\tau^*S^j_k\right).$$

After substituting (31) in (29), we get

$$R^h_k=\frac{\tau}{3}\delta^h_k+\frac{\tau^*}{6}\Phi^h_k,$$

and also

(32) $$R_{ki}=\frac{\tau}{3}g_{ki}+\frac{\tau^*}{6}f_{ki},\quad R_{ki}f^{ih}=\frac{\tau}{6}S^h_k+\frac{\tau^*}{6}\delta^h_k.$$

From the last equations we find that

$$\tau^*=-\tau.$$

That's why (32) becomes

(33) $$R_{ki}=\frac{\tau}{6}(2g_{ki}-f_{ki}),\quad R_{ki}f^{ih}=\frac{\tau}{6}(S^h_k-\delta^h_k).$$

Finely we obtain:

$$R^h_{ijk}=\frac{\tau}{6}(2\delta^h_k g_{ij}-2\delta^h_j g_{ki}+(\delta^h_k+S^h_k)f_{ij}-(\delta^h_j+S^h_j)f_{ki})$$

and

$$R_{hijk}=\frac{\tau}{6}(2g_{kh}g_{ij}-2g_{hj}g_{ki}+f_{kh}f_{ij}-f_{hj}f_{ki}).$$

The last identity is equivalent to (23).

We note that $R^h_{ijk}\neq 0$, so ∇ isn't a locally flat connection. □

Let p be a point in M and x, y be two linearly independent vectors in T_pM. It is known that

$$\mu(x,y)=\frac{R(x,y,x,y)}{g(x,x)g(y,y)-g^2(x,y)}$$

is the sectional curvature of the 2-section $\{x,y\}$.

Corollary 2.3. *Let M satisfy the conditions of Theorem 2.2. Let x be an arbitrary vector in T_pM, and φ be the angle between x and qx. Then, the sectional curvature of the 2-section $\{x,qx\}$ is*

$$\mu(x,qx)=-\frac{\tau}{6}\tan^2\frac{\varphi}{2},\qquad \varphi\in(0,\frac{2\pi}{3}).$$

Corollary 2.4. *Let M satisfy the conditions of Theorem 2.2. Then, the Ricci tensor of g is degenerated.*

The proof follows from (33).

Note. Let $\{x,qx\}$ be a 2-section and $g(x,qx)=0$. Then, $\mu(x,qx)=-\frac{\tau}{6}$.

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ВЪРХУ АФИННИ СВЪРЗАНОСТИ В РИМАНОВО МНОГООБРАЗИЕ С ЦИРКУЛАНТНА МЕТРИКА И ДВЕ ЦИРКУЛАНТНИ АФИНОРНИ СТРУКТУРИ

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В настоящата статия е разгледан клас V оттримерни риманови многообразия M с метрика g и два афинорни тензора q и S. Дефинирана е и друга метрика $\bar{g}$ в M. Локалните координати на всички тези тензори са циркулантни матрици. Намерени са: 1) зависимост между тензора на кривина R породен от g и тензора на кривина $\bar{R}$ породен от $\bar{g}$; 2) тъждество за тензора на кривина R в случая, когато тензорът на кривина $\bar{R}$ се анулира; 3) зависимост между секционната кривина на прозволна двумерна q-площадка $\{x, qx\}$ и скаларната кривина на M.