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STABILITY IN TERMS OF TWO MEASURES FOR DIFFERENTIAL EQUATIONS WITH "MAXIMA"*

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Stability in terms of two measures for nonlinear differential equations with "maxima" is studied. Two different measures for the initial conditions and for the solution are employed. Method of Razumikhin as well as comparison method for scalar ordinary differential equations have been employed. The usefulness of the introduced definition and the obtained sufficient conditions is illustrated through an example.

1. Introduction. Differential equations with "maxima" first appeared as an object of investigation about thirty years ago in connection with the solution of some applied problems. For example, in the theory of automatic control of various technical systems it often occurs that the law of regulation depends on the maximum values of some regulated state parameters over certain time intervals [6].

The problems of stability of solutions of differential equations *via* Lyapunov functions have been successfully investigated in the past. One type of stability, very useful in real world problems, deals with two different measures. Stability in terms of two measures for differential equations has been studied by means of various types of Lyapunov functions ([1], [2], [4], [5], [7] and references therein).

In the paper the stability as well as uniform stability in terms of two different measures is defined for differential equations with "maxima". The Razumikhin method and comparison results for scalar ordinary differential equations are employed to study the introduced stability.

2. Preliminary notes and definitions. Let $\mathbb{R}^n$ be the n-dimentional Eucledian space with norm $|\cdot|$, $\mathbb{R}_+ = [0,\infty)$, and $r > 0$ be a given number.

Consider the following nonlinear differential equations with "maximum"

$$x' = F(t, x(t), \max_{s\in[t-r,t]} x(s)) \quad \text{for} \quad t \ge t_0 \tag{1}$$

with initial condition

$$x(t) = \phi(t - t_0), \quad t \in [t_0 - r, t_0], \tag{2}$$

where $x \in \mathbb{R}^n$, $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\phi : [-r, 0] \to \mathbb{R}^n$, $t_0 \in \mathbb{R}_+$.
We assume that the solution $x(t; t_0, \phi)$ of the initial value problem (1), (2) is defined on $[t_0 - r, \infty)$ for any initial function $\phi \in C([-r, 0], \mathbb{R}^n)$.

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Definition 1 ([5]). *The function $V(t,x)$ belongs to the class Λ if $V(t,x) \in C(\Delta \times \Omega, \mathbb{R}_+)$, $\Delta \subset [-r,\infty), \Omega \subset \mathbb{R}^n$, $0 \in \Omega$, $V(t,0) \equiv 0$ for $t \in \Delta$, and it is Lipschitz with respect to its second vector argument.*

Consider the following sets:

$$K = \{a \in C[\mathbb{R}_+, \mathbb{R}_+] \ : \ a(s) \text{ is strictly increasing and } a(0) = 0\};$$

$$\Gamma = \{h \in C[[-r,\infty) \times \mathbb{R}^n, \mathbb{R}_+] \ : \ \min_{x\in\mathbb{R}^n} h(t,x) = 0 \text{ for each } t \in [-r,\infty)\};$$

$$S(h,\rho) = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n : \quad h(t,x) < \rho\}; \quad \rho > 0, \quad h \in \Gamma.$$

We introduce a definition for stability in terms of two measures for a system of differential equations with "maxima" which follows the ideas of [5].

Definition 2. *Let $h, h_0 \in \Gamma$. A system of differential equations with "maxima"* (1), (2) *is said to be:*

($S1$) stable *in terms of two measures (h_0, h) if for every $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that for any $\phi \in C([-r,0], \mathbb{R}^n)$ the inequality $\max_{s\in[-r,0]} h_0(t_0+s, \phi(t_0+s)) < \delta$ implies $h(t, x(t; t_0, \phi)) < \epsilon$ for $t \geq t_0$;*

($S2$) uniformly stable *in terms of two measures (h_0, h) if δ in ($S1$) is independent on t_0.*

In our further investigations we use the following comparison scalar *ordinary* differential equation:

$$u' = g(t,u), \tag{3}$$

where $u \in \mathbb{R}$, $g(t,0) \equiv 0$. We denote the solution of (3) with initial condition $u(t_0) = u_0$ by $u(t; t_0, u_0)$.

Definition 3 ([5]). *Let $h, h_0 \in \Gamma$. The function $h_0(t,x)$ is* uniformly finer *than $h(t,x)$, if there exists a constant $\delta > 0$ and a function $a \in K$ such that $h_0(t,x) < \delta$ implies $h(t,x) \leq a(h_0(t,x))$.*

Let $V \in \Lambda$, $t \in \Delta$ and $\phi \in C([-r,0], \mathbb{R}^n)$. We define derivative of the function V along the trajectory of solution of (1) as follows:

$$D_{(1)}V(t,\phi(0)) = \limsup_{\epsilon\to 0} \frac{1}{\epsilon}\{V(t+\epsilon, \phi(0) + \epsilon F(t, \phi(0), \max_{s\in[-r,0]} \phi(s))) - V(t,\phi(0))\}.$$

We introduce the conditions:

H1. Function $F \in C[\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$, $F(t,0,0) \equiv 0$.

H2. Function $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$, $g(t,0) \equiv 0$.

H3. Functions $h_0, h \in \Gamma$, $\quad h_0$ is uniformy finer than h.

Lemma 1. *Let the conditions* (H1), (H2) *be fulfilled and the function $V \in \Lambda$, $\Delta = [t_0, T)$, $T \leq \infty$, be such that for any $t \in [t_0, T)$ and $\psi \in C([-r,0], \mathbb{R}^n)$: $\quad V(t, \psi(0)) > V(t+s, \psi(s))$ for $s \in [-r, 0)$ the inequality $D_{(1)}V(t,\psi(0)) \leq g(t, V(t,\psi(0)))$ holds. Then, the inequality $\max_{s\in[-r,0]} V(t_0+s, \phi(0+s)) \leq u_0$ implies $V(t, x(t; t_0, \phi)) \leq u^*(t)$ for $t \in [t_0, T)$, where $u^*(t) = u^*(t; t_0, u_0)$ is the maximal solution of* (3) *with initial condition $u^*(t_0) = u_0$ which is defined for $t \in [t_0, T)$.*

The proof of Lemma 1 is a partial case of that of Lemma 1 [3] and we omit it.

3. Main results.

Theorem 1. *Let conditions (H1)-(H3) be fulfilled and there exist a function* $V(t,x) \in \Lambda$ *and a constant* $\rho > 0$ *such that:*

(*i*) *for any number* $t \geq 0$ *and any function* $\psi \in C([-r,0],\mathbb{R}^n)$, *such that* $\psi(0) \in S(h,\rho)$ *and* $V(t,\psi(0)) > V(t+s,\psi(s))$ *for* $s \in [-r,0)$, *the inequality* $D_{(1)}V(t,\psi(0)) \leq g(t,V(t,\psi(0)))$ *holds;*

(*ii*) $b(h(t,x)) \leq V(t,x) \leq a(h_0(t,x))$ *for* $(t,x) \in S(h,\rho)$, *where* $a,b \in K$.

Then, if the zero solution of the scalar differential equation (3) *is stable (uniformly stable), the system of differential equations with "maxima"* (1) *is stable (uniformly stable) in terms of two measures* (h_0,h).

Proof. Let $0 < \epsilon < \rho$. There exist $\delta_0 > 0$ and a function $\psi_1 \in K$ such that $h_0(t,x) < \delta_0$ implies $h(t,x) \leq \psi_1(h_0(t,x))$. There exists $\delta_1 \in (0,\rho)$, $\delta_1 = \delta_1(t_0,\epsilon)$ such that $|u_0| < \delta_1$ implies

$$|u(t;t_0,u_0)| < b(\epsilon), \quad t \geq t_0. \tag{4}$$

Since $a \in K$ and $\psi_1 \in K$, we can find a $\delta_2 = \delta_2(\epsilon) > 0$, $\delta_2 < \min(\delta_0,\rho)$ such that

$$\psi_1(\delta_2) < \epsilon \qquad a(\delta_2) < \delta_1. \tag{5}$$

Choose $t_0 \in \mathbb{R}_+$ and $\phi \in C([-r,0],\mathbb{R}^n)$ such that $\max\limits_{s\in[-r,0]} h_0(t_0+s,\phi(s)) < \delta_2 < \delta_0$. Then

$$\max_{s\in[-r,0]} h(t_0+s,\phi(s)) \leq \psi_1(\max_{s\in[-r,0]} h_0(t_0+s,\phi(s))) \leq \psi_1(\delta_2) < \epsilon. \tag{6}$$

We prove that $h(t,x(t;t_0,\phi)) < \epsilon$ holds for $t \in [t_0-r,\infty)$. Assume the contrary. Then there exist a point $t^* > t_0$ such that

$$h(t^*,x(t^*;t_0,\phi)) = \epsilon, \quad h(t,x(t;t_0,\phi)) < \epsilon, \quad t \in [t_0-r,t^*). \tag{7}$$

Since $\epsilon < \rho$, it follows that $(t,x(t)) \in S(h,\rho),\ t \in [t_0-r,t^*]$. From (7) it follows that

$$b(h(t^*,x(t^*;t_0,\phi))) = b(\epsilon). \tag{8}$$

Define $x(s) = x(s;t_0,\phi)$, $s \in [t_0-r,t^*]$. If we assume that $h_0(t^*,x(t^*)) \leq \delta_2$, then from the choice of δ_2 and inequality (5) it follows

$$h(t^*,x(t^*)) \leq \psi_1(h_0(t^*,x(t^*))) < \psi_1(\delta_2) < \epsilon,$$

which contradicts (7). Hence, $h_0(t^*,x(t^*)) > \delta_2$. Therefore, there exist a point $t_0^* \in (t_0,t^*)$ such that $h_0(t_0^*,x(t_0^*)) = \delta_2$, and $h_0(s,x(s)) < \delta_2$ for $s \in [t_0-r,t_0^*)$.

Let $u^*(t;t_0^*,u_0^*)$ be the maximal solution of (3), with initial condition $u_0^* = \max\limits_{t\in[t_0^*-r,t_0^*]} V(t,x(t))$. Then, from condition (i) of Theorem 1 and Lemma 1 it follows

$$V(t,x(t)) \leq u^*(t;t_0^*,u_0^*), \quad t \in [t_0^*,t^*]. \tag{9}$$

From condition (ii) of Theorem 1 it follows

$$V(t_0^*+s,x(t_0^*+s)) \leq a(h_0(t_0^*+s,x(t_0^*+s)) \leq a(\delta_2) < \delta_1, \quad s \in [-r,0].$$

Hence, $V(t_0^*+s,x(t_0^*+s)) < \delta_1,\quad s \in [-r,0]$. Therefore, $|u_0^*| < \delta_1$ and according to (4) the inequality

$$|u^*(t;t_0^*,u_0^*)| < b(\epsilon), \quad t \geq t_0^* \tag{10}$$

holds.

From (8), (9), (10) and condition (ii) of Theorem 1 it follows

$$b(\epsilon) = b(h(t^*, x(t^*))) \le V(t^*, x(t^*)) \le u^*(t^*; t_0^*, u_0^*) < b(\epsilon).$$

The obtained contradiction proves the validity of Theorem 1. $\square$

Remark 1. Partial cases:

1. For $r = 0$ the results of Theorem 1 reduce to stability in terms of two measures for differential equations.

2. For the measures $h(t, x) = h_0(t, x) \equiv |x|$ for $t \in [-r, \infty)$, the results of Theorem 1 reduce to stability for differential equations with "maximum".

3. For the measures $h(t, x) = h_0(t, x) \equiv |x|$ for $t \in \mathbb{R}_+$ and $r = 0$ the results of Theorem 1 reduce to stability for differential equations.

Now we illustrate some applications of the obtained sufficient conditions.

Example. Consider the linear system of differential equations perturbed by the maximum function

$$x' = -x + 2y + C_1 \max_{s \in [t-r, t])} x(s) \tag{11}$$

$$y' = -x - y + C_2 \max_{s \in [t-r, t])} y(s), \tag{12}$$

where $x, y \in \mathbb{R}$, C_1, C_2 are constants such that $C_1 + C_2 < 1$.

Consider the function $V : \mathbb{R}^2 \to \mathbb{R}_+$, $V(x, y) = \dfrac{1}{2}x^2 + y^2$. Note that the condition (ii) of Theorem 1 is satisfied for $a(s) = s^2$, $b(s) = \dfrac{1}{2}s^2$, $h(x, y) = \sqrt{x^2 + y^2}$, $h_0(x, y) = |x| + |y|$.

Let $\psi \in C([-r, 0], \mathbb{R}^2)$, $\psi = (\psi_1, \psi_2)$ be such that $V(\psi_1(0), \psi_2(0)) > V(\psi_1(s), \psi_2(s))$, $s \in [-r, 0)$. Then,

$$\begin{aligned}
&\mathcal{D}_{(11),(12)} V(\psi_1(0), \psi_2(0)) \\
&\qquad = -\psi_1^2(0) - 2\psi_2^2(0) + C_1\psi_1(0) \max_{s \in [-r, 0]} \psi_1(s) + 2C_2\psi_2(0) \max_{s \in [-r, 0]} \psi_2(s).
\end{aligned}$$

Note that

$$\begin{aligned}
\psi_1(0) \max_{s \in [-r, 0]} \psi_1(s) &\le |\psi_1(0)| \, | \max_{s \in [-r, 0]} \psi_1(s)| = \sqrt{(\psi_1(0))^2} \sqrt{(\max_{s \in [-r, 0]} \psi_1(s))^2} \\
&\le \sqrt{(\psi_1(0))^2 + 2(\psi_2(0))^2} \sqrt{(\psi_1(\xi))^2 + 2(\psi_1(\xi))^2} \le (\psi_1(0))^2 + 2(\psi_2(0))^2
\end{aligned}$$

and $2\psi_2(0) \max_{s \in [-r, 0]} \psi_2(s) \le (\psi_1(0))^2 + 2(\psi_2(0))^2$, where $\xi \in [-r, 0]$.

Hence

$$\begin{aligned}
\mathcal{D}_{(11),(12)} V(\psi_1(0), \psi_2(0)) &\le -\psi_1^2(0) - 2\psi_2^2(0) + C_1(\psi_1^2(0) + 2\psi_2^2(0)) + C_2(\psi_1^2(0) + 2\psi_2^2(0)) \\
&= (\psi_1^2(0) + 2\psi_2^2(0))(-1 + C_1 + C_2) \le 0.
\end{aligned}$$

Consider the scalar differential equation $u' = 0$ which zero solution is stable. Therefore, according to Theorem 1 the solution of (11), (12) is uniformly stable in terms of the measures (h_0, h).

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УСТОЙЧИВОСТ НА ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ С "МАКСИМУМИ" ПО ОТНОШЕНИЕ НА ДВЕ МЕРКИ

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Изследвана е устойчивостта на нелинейни диференциални уравнения с "максимуми" по отношение на две мерки. Приложени са две различни мерки за началните условия и за решението. Използван е методът на Разумихин, а също така и методът на сравнението на обикновени скаларни диференциални уравнения. Приложението на получените резултати и достатъчни условия за устойчивост е илюстрирано с пример.