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THE DARBOUX PROBLEM FOR A CLASS OF 3-D WEAKLY HYPERBOLIC EQUATIONS*

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Some three-dimensional analogues of the plane Darboux problems for weakly hyperbolic equations are studied. In 1952 M. Protter formulated new 3-D boundary value problems for a class of weakly hyperbolic equations, as well as for some hyperbolic-elliptic equations. In the contrast of the well-posedness of the Darboux problem in 2-D case, the new problems are strongly ill-posed. For weakly hyperbolic equation, involving lower order terms, we find sufficient conditions for existence and uniqueness of generalized solutions with isolated power-type singularities as well as for uniqueness of quasi-regular solutions to the Protter problem.

1. Introduction. Let Ω be the simply connected domain in $\mathbb{R}^3$, expressed in Cartesian coordinates (x_1, x_2, t) :

$$\Omega := \left\{ (x_1, x_2, t) : \ 0 < t < d, \ \int_0^t \sqrt{K(\tau)}\, d\tau < \sqrt{x_1^2 + x_2^2} < 1 - \int_0^t \sqrt{K(\tau)}\, d\tau \right\},$$

where $K : [0, d] \to \mathbb{R}$, $K \in C^3((0, d)) \cap C([0, d])$, $K(0) = 0$, $K(t) > 0$, for $t > 0$ and d is the unique solution of the equation $2 \displaystyle\int_0^t \sqrt{K(\tau)}\, d\tau = 1$.

The boundary of Ω is $\partial\Omega = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$, where Σ_0 is the disc $\Sigma_0 := \{(x_1, x_2, t) : t = 0, \ \sqrt{x_1^2 + x_2^2} < 1\}$ and

$$\Sigma_1 := \left\{ (x_1, x_2, t) : \ 0 < t < d, \ \sqrt{x_1^2 + x_2^2} = 1 - \int_0^t \sqrt{K(\tau)}\, d\tau \right\},$$

$$\Sigma_2 := \left\{ (x_1, x_2, t) : \ 0 < t < d, \ \sqrt{x_1^2 + x_2^2} = \int_0^t \sqrt{K(\tau)}\, d\tau \right\}.$$

In Ω we consider the weakly hyperbolic equation

$$L[u] := K(t)[u_{x_1x_1} + u_{x_2x_2}] - u_{tt} + b_1 u_{x_1} + b_2 u_{x_2} + b u_t + cu = f. \tag{1.1}$$

We have to mention that Σ_1 and Σ_2 are characteristic surfaces of the equation (1.1).

We investigate the following boundary value problem.

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Problem P_α. *Find a solution to* (1.1) *in* Ω *that satisfies the boundary conditions*

$$u|_{\Sigma_1} = 0, \quad [u_t + \alpha u]|_{\Sigma_0 \backslash O} = 0,$$

where $\alpha \in C^1(\bar{\Sigma}_0 \backslash O)$.

This problem is formulated by Protter, as multidimensional analogue of the Darboux problem in the plane. The following problems are known as Protter problems [15, 16].

Proter's problems. *Find solution to the equation*

$$K(t)[u_{x_1x_1} + u_{x_2x_2}] - u_{tt} = f, \tag{1.2}$$

that satisfies one of the following boundary conditions

$$P1: \quad u|_{\Sigma_0 \cup \Sigma_1} = 0, \qquad P1^*: \quad u|_{\Sigma_0 \cup \Sigma_2} = 0\,;$$
$$P2: \quad u_t|_{\Sigma_0} = 0,\ u|_{\Sigma_1} = 0, \quad P2^*: \quad u_t|_{\Sigma_0} = 0,\ u|_{\Sigma_2} = 0\,.$$

He worked with the wave equation corresponding to $K(t) = 1$ or $K(t) = t^m$, $m \in \mathbb{R}$, $m > 0$ and also investigated equation (1.2) in a domain which contained Ω in its hyperbolic part and contained a set in which (1.2) is elliptic. For equation (1.2), which is of mixed hyperbolic-elliptic type, he formulated certain other problems, which are three-dimensional analogues of the Guderley-Morawetz plane problems, appeared from the transonic fluid dynamic models [13]. Are that 3-D Protter problems well posed, or ill posed is still an open question.

When equation (1.2) is of changing type, the problems given by Protter or some different statements of Darboux type problems in $\mathbb{R}^3$ were studied by Aziz and Schneider [2], Bitsadze [4], Edmunds and Popivanov [5] and others. In [11] one finds results for mixed type equations including some special nonlinearity with supercritical exponent term in various situations. For more publications in this area see, for example: [3], [10] – [13].

For equation (1.2) with $K(t) = t^m$, $m \in \mathbb{R}, m > 0$

$$t^m[u_{x_1x_1} + u_{x_2x_2}] - u_{tt} = f(x_1, x_2, t) \tag{1.3}$$

there are several interesting results. When $m = 0$, in 1960 P. Garabedian [6] proved uniqueness of the classical solution of Protter's problem $P1$ in $\mathbb{R}^4$. In contrast to Darboux problems on the plane, the corresponding problems $P1$ and $P2$ in $\mathbb{R}^3$ are not well-posed set, because the problems $P1$ and $P2$ for (1.3) have infinite-dimensional cokernels. Here the result is represented in polar coordinates (ϱ, φ, t), where $x_1 = \varrho\cos\varphi$, $x_2 = \varrho\sin\varphi$; $m \geq 0$.

Theorem 1.1 [14]**.** *For all* $n \in \mathbb{N}$, $n \geq 4$*;* a_n, b_n *arbitrary constants, the functions*

$$v_{n,m}(\varrho, \varphi, t) = t\varrho^{-n}\Big[\varrho^2 - \Big(\frac{2}{m+2}\Big)^2 t^{m+2}\Big]^{n-1-\frac{1}{m+2}}(a_n\cos n\varphi + b_n\sin n\varphi)$$

are classical solutions of the homogeneous problem $P1^*$ *for* (1.3) *and the functions*

$$w_{n,m}(\varrho, \varphi, t) = \varrho^{-n}\Big[\varrho^2 - \Big(\frac{2}{m+2}\Big)^2 t^{m+2}\Big]^{n-1+\frac{1}{m+2}}(a_n\cos n\varphi + b_n\sin n\varphi)$$

are classical solutions of the homogeneous problem $P2^*$ *for* (1.3).

Khe Kan Cher has found [9] some non-trivial solutions for the homogeneous Problems $P1^*$ and $P2^*$, but for the Euler-Poison-Darboux equation.

N. Popivanov and M. Schneider [14] studied problem $P1$ for the degenerating hyperbolic equation (1.3). They introduced a new class of generalized solutions to the problem $P1$ and proved that such solution exists and it is unique, but it has a very strong power-type singularity on the characteristic surface Σ_2. For results concerning equation (1.3) with lower order terms see [8] and the references cited there.

S. A. Aldashev [1] studied the Proter problems for the equation (1.1), and he climed that the homogeneous problems $P1^*$ and $P2^*$ have infinitely many classical solutions and homogeneous problems $P1$ and $P2$ have only trivial solutions.

For the weakly hyperbolic equation involving lower order terms, it is well known that the plane problems of Cauchy, Darboux and Goursat are well posed if the coefficients satisfy the so called Protter's condition (see [7], [16]). More precisely, for the 2-D equation

$$K(t)u_{xx} - u_{tt} + au_x + bu_t + cu = f$$

the Protter's condition is

$$\frac{ta(x,t)}{\sqrt{K(t)}} \to 0 \text{ as } t \to 0. \tag{1.4}$$

It is clear that the Problem $\boldsymbol{P_\alpha}$ is not classically solvable for each smooth right-hand side function. Our aim is to give notions for quasi–regular solution of this problem and for generalized solution in case $K(t) = t^m$, $m \in \mathbb{R}$, $m > 0$. Then under some conditions, analogues to (1.4), for lower order terms in (1.1) we prove existence and uniqueness of such generalized solution and find right-hand side functions for which this solution has strong power-type singularity isolated at the vertex O of Σ_2. Further, we find sufficient condition for uniqueness of quasi–regular solution to Problem $\boldsymbol{P_\alpha}$.

2. Generalized and quasi–regular solutions. Now, in order to obtain our results, we give the following definitions of a generalized solution to Problem $\boldsymbol{P_\alpha}$ with a possible singularity at point O and of a quasi-regular solution.

Definition 2.1. *A function $u = u(x_1, x_2, t)$ is called generalized solution of problem $\boldsymbol{P_\alpha}$ with $K(t) = t^m$, $m \in \mathbb{R}$, $m > 0$ in Ω, if*

(1) $u \in C(\bar{\Omega}\backslash O) \cap C^1(\bar{\Omega}\backslash\{\bar{\Sigma}_1 \cup O\})$, $u\big|_{\Sigma_1} = 0$, $[u_t + \alpha(x)u]\big|_{\Sigma_0\backslash O} = 0$;

(2) *For each $0 < \varepsilon < 1$ there exists a constant $C(\varepsilon)$, such that*

$$|u(x_1,x_2,t)| \le C(\varepsilon)\Big(1 - |x| - \frac{2}{m+2}t^{\frac{m+2}{2}}\Big)^{1-2\beta},$$
$$|u_{x_i}(x_1,x_2,t)| \le C(\varepsilon)\Big(1 - |x| - \frac{2}{m+2}t^{\frac{m+2}{2}}\Big)^{-2\beta}, \quad i = 1, 2,$$
$$|u_t(x_1,x_2,t)| \le C(\varepsilon)\Big(1 - |x| - \frac{2}{m+2}t^{\frac{m+2}{2}}\Big)^{-2\beta}$$

in $\Omega_\varepsilon := \Omega \cap \Big\{|x| > \varepsilon + \frac{2}{m+2}t^{\frac{m+2}{2}}\Big\}$, $\beta = \frac{m}{2(m+2)}$;

(3) *The identity*

$$\int_\Omega \Big\{u_t v_t - t^m(u_{x_1}v_{x_1} + u_{x_2}v_{x_2}) + (b_1 u_{x_1} + b_2 u_{x_2} + bu_t + cu - f)v\Big\} dx_1 dx_2 dt$$
$$= \int_{\Sigma_0} \alpha(x)(uv)(x,0)dx_1 dx_2$$

holds for all v from

$$V := \{v \in C^1(\bar{\Omega}) : [v_t + (\alpha + b)v]\big|_{\Sigma_0} = 0,\ v = 0 \ \textit{in a neighbourhood of}\ \Sigma_2\}.$$

Definition 2.2. *We call a function $u(x_1, x_2, t) \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \{\Sigma_1 \cup \Sigma_2\}) \cap C(\bar{\Omega})$ a quasi-regular solution of Problem $\boldsymbol{P_\alpha}$ with $\alpha \in C(\bar{\Sigma}_0)$ if*

(1) *$u(x_1, x_2, t)$ satisfies $L[u] = f$ in Ω, $u|_{\Sigma_1} = 0$, $[u_t + \alpha u]|_{\Sigma_0} = 0$.*

(2) *If $\Omega(\varepsilon)$ are regions with boundaries $\partial\Omega(\varepsilon)$ lying entirely in Ω, then the integrals along $\partial\Omega(\varepsilon)$ which result from the application of Green's theorem to*

$$\iint_{\Omega(\varepsilon)} uL[u]\, d\tau,\ \iint_{\Omega(\varepsilon)} u_t L[u]\, d\tau,\ \iint_{\Omega(\varepsilon)} u_{x_i} L[u]\, d\tau,\ i = 1, 2$$

have a limit when $\partial\Omega(\varepsilon)$ approach the boundary of Ω for $\varepsilon \to 0$.

3. Existence and uniqueness theorems. Let consider equation (1.1) in polar coordinates (ϱ, φ, t) :

$$L[u] = K(t)\left[\frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2}u_{\varphi\varphi}\right] - u_{tt} + a_1 u_\varrho + a_2 u_\varphi + b u_t + c u = f, \tag{3.5}$$

where $x_1 = \varrho\cos\varphi$, $x_2 = \varrho\sin\varphi$, $a_1 = b_1 cos\varphi + b_2 sin\varphi$, $a_2 = \varrho^{-1}(b_2 cos\varphi - b_1 sin\varphi)$. Here we assume that all coefficients of (3.5) depend only on ϱ and t, and $\alpha(x_1, x_2) = \alpha(\varrho)$.

Theorem 3.1. *Let $b \in C^2(\bar{\Omega}\backslash O)$, $c \in C^1(\bar{\Omega}\backslash O)$, $\alpha \in C^1((0, 1])$ and b_1 and b_2 have the form*

$$b_i = t^{\frac{m}{2}}\tilde{b}_i,\ \tilde{b}_i \in C^2(\bar{\Omega}\backslash O),\ i = 1, 2. \tag{3.6}$$

Then, there exists at most one generalized solution of Problem $\boldsymbol{P_\alpha}$ with $K(t) = t^m$, $m \in \mathbb{R}$, $m > 0$ in Ω.

Theorem 3.2. *Let conditions of Theorem 3.1 be fulfilled and the function $f \in C^1(\bar{\Omega}\backslash O)$ for some $k \in \mathbb{N} \cup \{0\}$ has the form*

$$f(\varrho, \varphi, t) = \sum_{n=0}^{N}\left\{f_n^{(1)}(\varrho, t)\cos n\varphi + f_n^{(2)}(\varrho, t)\sin n\varphi\right\}.$$

Then, there exists one and only one generalized solution of Problem $\boldsymbol{P_\alpha}$ with $K(t) = t^m$, $m \in \mathbb{R}$, $m > 0$ in Ω.

Remark 3.3. Note that the condition (3.6) is a little stronger than Protter's condition. We suppose that for $m < 2$ it is possible to have existence and uniqueness of generalized solution to Problem $\boldsymbol{P_\alpha}$ under some weaker condition. For example, this might be true for equation (1.1) with Tricomi operator in its main part ($K(t) = t$).

Theorem 3.4. *The Problem $\boldsymbol{P_\alpha}$ for equation (3.5) with $a_1, b \in C^2(\Omega)$, $c \in C^1(\Omega)$*

$$L[u] = f(\varrho, \varphi, t) = \sum_{n=0}^{N}\left\{f_n^{(1)}(\varrho, t)\cos n\varphi + f_n^{(2)}(\varrho, t)\sin n\varphi\right\}$$

has at most one quasi-regular solution

$$u(\varrho, \varphi, t) = \sum_{n=0}^{N}\left\{u_n^{(1)}(\varrho, t)\cos n\varphi + u_n^{(2)}(\varrho, t)\sin n\varphi\right\}$$

in Ω, *if for* $(\varrho, \varphi, t) \in \Omega$

$$a_2 = 0,\ K'(t) - \sqrt{K(t)}\left(a_1 + \frac{K(t)}{\varrho}\right) + K(t)b \geq 0,$$
$$\lim_{t\to 0+0}[\gamma_t^0 + \gamma^0(b + 2\alpha)] \geq 0,$$
$$a_{00} > 0 \text{ for all } 0 \leq n \leq N,$$

where

$$a_{00} = -2\gamma^0 R + (R\gamma^1)_\varrho + (R\gamma^2)_t - K(t)\gamma^0_{\varrho\varrho} + \gamma^0_{tt} + \left(\gamma^0\left(a_1 + \frac{K(t)}{\varrho}\right)\right)_\varrho + (\gamma^0 b)_t,$$

$$\gamma^1 = \mu K(t), \quad \gamma^2 = -\mu\sqrt{K(t)}, \quad 2\gamma^0 = \mu\left\{-\frac{K'(t)}{2\sqrt{K(t)}} + a_1 + \frac{K(t)}{\varrho} + b\sqrt{K(t)}\right\},$$

$$\mu = \varrho - \int_0^t \sqrt{K(\tau)}\, d\tau, \quad R = c(\varrho, t) - \frac{n^2}{\varrho^2}K(t).$$

4. On the singularity of solutions of Problem $\boldsymbol{P_\alpha}$. In this section we find sufficient conditions for the coefficients and appropriate functions on the right-hand side, for which the corresponding unique generalized solution to Problem $\boldsymbol{P_\alpha}$ has strong power type singularity isolated at the vertex O of the characteristic surface Σ_2. It is interesting that this singularity does not propagate along the bi-characteristics of Σ_2 and does not depend on the power of degeneration m.

Theorem 4.1. *Let the conditions of Theorem* 3.1 *hold and for* $(x_1, x_2, t) \in \Omega$,

$$b_i(x_1, x_2, t) = t^{\frac{m}{2}} x_i |x|^{-1} a(|x|, t),\ a \in C^2(\bar\Omega \setminus O),$$

$$a(|x|, t) \geq 0,\ \partial_\varrho(a + b)(|x|, t) \geq 0,\ 2\alpha(|x|) + (a + b)(|x|, 0) \geq 0, \tag{4.7}$$

$$\left(m - 2|x|^{-1}t^{\frac{m+2}{2}}\right)a(|x|, t) \leq t\{a^2 - b^2 - 4c + 2\partial_\tau(a + b)\}(|x|, t),$$

where $\partial_\tau := t^{\frac{m}{2}}\partial_\varrho + \partial_t$, $\varrho = |x|$. *Then, for each function*

$$f_n(x, t) = |x|^{-2n}\left(x_1^2 + x_2^2 - \left(\frac{2}{m+2}\right)^2 t^{m+2}\right)^{n - \frac{m+1}{m+2}} Re(x_1 + ix_2)^n$$
$$\in C^{n-2}(\overline{\Omega}) \cap C^\infty(\overline{\Omega}\backslash\Sigma_2),$$

where $n \in \mathbb{N}$, $n \geq 4$, *there exists a unique generalized solution of Problem* $\boldsymbol{P_\alpha}$ *with* $K(t) = t^m$, $m \in \mathbb{R}$, $m > 0$ *in* Ω *and it satisfies on* $\Sigma_2 = \left\{t = t(|x|) = \left(2^{-1}(m + 2)|x|\right)^{\frac{2}{m+2}}\right\}$ *the estimate*

$$\left|u_n\big(x_1, x_2, t(|x|)\big)\right| \geq |x|^{-n}\left|\cos\left(n \arctan\frac{x_2}{x_1}\right)\right|. \tag{4.8}$$

Finally we give an example for existence of singular solutions of Problem $\boldsymbol{P_\alpha}$, where the conditions (4.7) look very simple.

Example 4.2. Consider the equation (1.1) in case $K(t) = t^m$, $m \in \mathbb{R}$, $m > 0$, $b_1 = a_0 t^\sigma |x|^{-1}x_1$, $b_2 = a_0 t^\sigma |x|^{-1}x_2$, where $\sigma \geq \max\{m, 2 + m/2\}$, $a_0 \geq 0$ and b are constants, $c = c(|x|, t) \in C^1(\bar\Omega)$, $\alpha(|x|) \in C^1([0, 1])$, $b^2 - 4c \geq 0$, $2\alpha + b \geq 0$.

In that case conditions (4.7) are fulfilled and Theorem 4.1 states that the corresponding generalized solution u_n of Problem $\boldsymbol{P_\alpha}$ in Ω with right-hand side $f_n(x,t)$, satisfies on Σ_2 the estimate (4.8).

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ЗАДАЧА НА ДАРБУ ЗА КЛАС ТРИМЕРНИ СЛАБО ХИПЕРБОЛИЧНИ УРАВНЕНИЯ

Недю Попиванов, Цветан Христов

Изследвани са някои тримерни аналози на задачата на Дарбу в равнината. През 1952 М. Протер формулира нови тримерни гранични задачи както за клас слабо хиперболични уравнения, така и за някои хиперболично-елиптични уравнения. За разлика от коректността на двумерната задача на Дарбу, новите задачи са некоректни. За слабо хиперболични уравнения, съдържащи младши членове, ние намираме достатъчни условия както за съществуване и единственост на обобщени решения с изолирана степенна особеност, така и за единственост на квази-регулярни решения на задачата на Протер.