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ON PARETO SETS IN MULTI-CRITERIA OPTIMIZATION*

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In this work we consider the Pareto solutions in continuous multi-criteria optimization problem. We discuss the role of some assumptions that affect the characteristics of Pareto sets. We have tried to remove the assumptions for concavity of the objective functions and convexity of the feasible domain which are usually used in multi-criteria optimization problems. The results are based on the construction of a retraction from the feasible domain onto the Pareto-optimal set.

1. Introduction. The topological properties of the Pareto sets in multi-criteria optimization problems have attracted much attention of the researchers. The topological properties are studied in [1–5], [8–9], [11], [13–18], [20], [22]. Information about these properties is very important for computational algorithms generating Pareto solutions [19].

The aim of this work is to present some new topological properties of Pareto-optimal and Pareto-front sets, shortly Pareto sets, in a multi-criteria optimization problem. The author has tried to remove the assumptions for concavity of the objective functions and convexity of the feasible domain usually using in this optimization problem.

The standard form of the multi-criteria optimization problem is to find a variable $x(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$, $m \geq 1$, so that to maximize $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ subject to $x \in X$, where the feasible domain X is nonempty, $J_n = \{1, 2, \ldots, n\}$ is the index set, $n \geq 2$, $f_i : X \to R$ is a given continuous function for all $i \in J_n$.

Since the objective functions $\{f_i\}_{i=1}^n$ may conflict with each other, it is usually difficult to obtain the global maximum for each objective function at the same time. Therefore, the target of the maximization problem is to achieve a set of solutions that are Pareto-optimal. Historically, the first reference to address such situations of conflicting objectives is usually attributed to Vilfredo Pareto (1848–1923).

Definition 1. (*a*) *A point $x \in X$ is called Pareto-optimal solution if and only if there does not exist a point $y \in X$ such that $f_i(y) \geq f_i(x)$ for all $i \in J_n$ and $f_k(y) > f_k(x)$ for some $k \in J_n$. The set of the Pareto-optimal solutions of X is denoted by $PO(X, f)$ and is called Pareto-optimal set. Its image $f(PO(X, f)) = PF(X, f)$ is called Pareto-front set.*

(*b*) *A point $x \in X$ is called strictly Pareto-optimal solution if and only if there does not exist a point $y \in X$ such that $f_i(y) \geq f_i(x)$ for all $i \in J_n$ and $x \neq y$. The set of*

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the strictly Pareto-optimal solutions of X is denoted by $SPO\,(X, f)$ and is called strictly Pareto-optimal set.

The strictly Pareto-optimal solutions are the multi-objective analogue of unique optimal solutions in the scalar optimization.

In this work let the feasible domain X be compact. It is well-known that $PO\,(X, f)$ and $PF\,(X, f)$ are nonempty, $SPO\,(X, f) \subset PO\,(X, f)$ and $PF\,(X, f) \subset bdf(X)$ [10].

Definition 2. (a) *The set $Y \subset X$ is a retract of X if and only if there exists a continuous function $r : X \to Y$ such that $r(x) = x$ for all $x \in Y$. The function r is called a retraction.*

(b) *The set Y is a deformation retract of X if and only if there exist a retraction $r : X \to Y$ and a homotopy $H : X \times [0; 1] \to X$ such that $H(x, 0) = x$ and $H(x, 1) = r(x)$ for all $x \in X$.*

Remark 1. Let X and Y be two topological spaces. A homotopy between two continuous functions $f, g : X \to Y$ is defined to be a continuous function $H : X \times [0; 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. Note that we can consider the homotopy H as a continuously deformation of f to g [7].

Remark 2. From a more formal viewpoint, a retraction is a function $r : X \to Y$ such that $r \circ r(x) = r(x)$ for all $x \in X$, since this equation says exactly that r is the identity on its image. Retractions are the topological analogs of projection operators in other parts of mathematics. Clearly, every deformation retract is a retract, but the converse does not hold in generally [7].

Remark 3. A property is called a topological property if and only if an arbitrary topological space X has this property, then Y has this property too, where Y is homeomorphic to X.

Of course, the topological properties of the Pareto-optimal set related to the topological properties of the Pareto-front set, respectively.

We introduce the following notations: for every two vectors $x, y \in \mathbb{R}^n$, $x(x_1, x_2, \ldots, x_n) = y(y_1, y_2, \ldots, y_n)$ means $x_i = y_i$ for all $i \in J_n$, $x(x_1, x_2, \ldots, x_n) \geqq y(y_1, y_2, \ldots, y_n)$ means $x_i \geq y_i$ for all $i \in J_n$ (weakly componentwise order), $x(x_1, x_2, \ldots, x_n) > y(y_1, y_2, \ldots, y_n)$ means $x_i > y_i$ for all $i \in J_n$ (strictly componentwise order), and $x(x_1, x_2, \ldots, x_n) \geq y(y_1, y_2, \ldots, y_n)$ means $x_i \geq y_i$ for all $i \in J_n$ and $x_k > y_k$ for some $k \in J_n$ (componentwise order).

2. Main result. As usually, the key idea is to transfer our multi-objective optimization problem to mono-objective optimization problem by defining a unique objective function.

First of all, begin with the following definition: define a multifunction $\psi : X \rightrightarrows X$ by $\psi\,(x) = \left\{y \in X | f(y) \geqq f(x)\right\}$ for all $x \in X$; define a function $s : X \to R$ by $s\,(x) = \sum\limits_{j=1}^{n} f_j\,(x)$ for all $x \in X$.

Let choose $x \in X$ and consider an optimization problem with single objective function as follows: maximize $s(y)$ subject to $y \in \psi(x)$. By choosing different $x \in X$ we can identify different Pareto-optimal solutions. This optimization technique allow us to find the whole Pareto-optimal solutions [6].

In this work, we discus the role of the following assumptions:

Assumption 1. *f is bijective.*

Assumption 2. $|\mathrm{Argmax}(s, \psi(x))| = 1$ *for all* $x \in X$.

Assumption 3. *If d is a metric in $\mathbb{R}^m$, $\{x_i\}_{i=0}^{\infty} \subset X$ and $\lim\limits_{k\to\infty} d(x_k, x_0) = 0$, then $\lim\limits_{k\to\infty} d(y_0, \psi(x_k)) = 0$ for all $y_0 \in \psi(x_0)$.*

Remark 4. Let $Cl(X)$ be a set of all nonempty compact subset of X. A sequence of sets $\{A_k\}_{k=1}^{\infty} \subset Cl(X)$ is called to be Wijsman convergence to $A \in Cl(X)$ if and only if for each $x \in X$, $\lim\limits_{k\to\infty} d(x, A_k) = d(x, A)$ (Assumption 3).

These definitions and assumptions allow us the presentation of the main theorem of this work.

Theorem 1. *If Assumptions* 1, 2 *and* 3 *hold, then:*
(a) $PO\,(X, f) = SPO\,(X, f)$.
(b) $PO\,(X, f)$ *is a retract of* X.
(c) $PF\,(X, f)$ *is a retract of* $f(X)$.
(d) $PO\,(X, f)$ *is homeomorphic to* $PF\,(X, f)$.

In order to give the proof of Theorem 1, we prove some lemmas.

Lemma 1. *If $x \in X$, $x \in PO\,(X, f)$ is equivalent to $\{x\} = \psi(x)$.*

Proof. Let $x \in PO\,(X, f)$ and assume that $\{x\} \neq \psi(x)$. From both conditions $x \in \psi(x)$ and $\{x\} \neq \psi(x)$, it follows that there exists $y \in \psi(x) \setminus \{x\}$ such that $f(y) \geqq f(x)$. As a result we get that $s(y) \geq s(x)$, but $x \in PO\,(X, f)$ implies $s(y) = s(x)$ and $f(y) = f(x)$. We assumed that f is bijective (Assumption 1), therefore $x = y$ which contradicts the condition $y \in \psi(x) \setminus \{x\}$. Thus we obtain $\{x\} = \psi(x)$.

Conversely, let $\{x\} = \psi(x)$ and assume that $x \notin PO\,(X, f)$. From the assumption $x \notin PO\,(X, f)$, it follows that there exists $y \in X \backslash \{x\}$ such that $f(y) \geq f(x)$. Thus we deduce that $y \in \psi(x)$ and $x \neq y$ which contradicts the condition $\{x\} = \psi(x)$. Therefore, we obtain $x \in PO\,(X, f)$. The lemma is proved. □

Lemma 2. *If $x \in X$, then $Arg\max(s, \psi(x)) \subset PO(X, f)$.*

Proof. Let us choose $y \in Arg\max(s, \psi(x))$ and assume that $y \notin PO(X, f)$. From the assumption $y \notin PO(X, f)$ it follows that there exists $z \in X \backslash \{y\}$ such that $f(z) \geq f(y)$. As a result we derive $z \in \psi(x)$ and $s(z) > s(y)$. This leads to a contradiction, hence $y \in PO(X, f)$. The lemma is proved. □

Lemma 3. *There exists a function $r : X \to PO(X, f)$ such that $r\,(x) = \mathrm{Argmax}\,(s, \psi(x))$ for all $x \in X$ and $r\,(X) = PO\,(X, f)$.*

Proof. Using $\mathrm{Argmax}(s, \psi(x)) \subset PO(X, f)$ (Lemma 2) and $|\mathrm{Argmax}(s, \psi(x))| = 1$ (Assumption 2), we are in a position to construct a function $r : X \to PO(X, f)$ such that $r\,(x) = Arg\max\,(s, \psi(x))$ for all $x \in X$.

From Lemmas 1 and 2 it follows that: if $x \in PO\,(X, f)$, then $x = r(x)$; if $x \in X \backslash PO\,(X, f)$, then $x \neq r(x)$. This means that $r\,(X) = PO\,(X, f)$. The lemma is proved. □

Lemma 4. *ψ is continuous on X.*

Proof. First, we prove that if $\{x_k\}_{k=1}^{\infty}$,$\{y_k\}_{k=1}^{\infty} \subset X$ is a pair of sequences such that $\lim\limits_{k\to\infty} x_k = x_0 \in X$ and $y_k \in \psi(x_k)$ for all $k \in N$, then there exists a convergent subsequence of $\{y_k\}_{k=1}^{\infty}$ whose limit belongs to $\psi(x_0)$.

The assumption $y_k \in \psi(x_k)$ for all $k \in N$ implies $f(y_k) \underline{\geq} f(x_k)$ for all $k \in N$. From the condition $\{y_k\}_{k=1}^{\infty} \subset X$ it follows that there exists a convergent subsequence $\{q_k\}_{k=1}^{\infty} \subset \{y_k\}_{k=1}^{\infty}$ such that $\lim\limits_{k\to\infty} q_k = y_0 \in X$. Hence, there exists a convergent subsequence $\{p_k\}_{k=1}^{\infty} \subset \{x_k\}_{k=1}^{\infty}$ such that $q_k \in \psi(p_k)$ and $\lim\limits_{k\to\infty} p_k = x_0$. Thus we find that $f(q_k) \underline{\geq} f(p_k)$ for all $k \in N$. Taking the limit as $k \to \infty$ we obtain $f(y_0) \geq f(x_0)$. This implies $y_0 \in \psi(x_0)$. This means that ψ is upper semi-continuous on X [12].

Second, we prove that if $\{x_k\}_{k=1}^{\infty} \subset X$ is a convergent sequence to $x_0 \in X$ and $y_0 \in \psi(x_0)$, then there exists a sequence $\{y_k\}_{k=1}^{\infty} \subset X$ such that $y_k \in \psi(x_k)$ for all $k \in N$ and $\lim\limits_{k\to\infty} y_k = y_0$.

As usually, the distance between the point $y_0 \in X$ and the set $\psi(x_k) \subset X$ is denoted by $d_k = \inf\{dis(y_0, x) \,|x \in \psi(x_k)\}$. Clearly, $\psi(x_k)$ is a nonempty compact; therefore, if $y_0 \notin \psi(x_k)$, then there exists $\hat{y} \in \psi(x_k)$ such that $d_k = d(\hat{y}, y_k)$. There are two cases as follows: if $y_0 \in \psi(x_k)$, then $d_k = 0$ and set $y_k = y_0$; if $y_0 \notin \psi(x_k)$, then $d_k > 0$ and set $y_k = \hat{y}$. So that, we get a sequence $\{d_k\}_{k=1}^{\infty} \subset R_+$ and a sequence $\{y_k\}_{k=1}^{\infty} \subset X$ such that $y_k \in \psi(x_k)$ for all $k \in N$ and $d_k = d(y_0, y_k)$. Since $\lim\limits_{k\to\infty} x_k = x_0$ (Assumption 3) the sequence $\{d_k\}_{k=1}^{\infty}$ is convergent and $\lim\limits_{k\to\infty} d_k = 0$. Finally, we obtain $\lim\limits_{k\to\infty} y_k = y_0$. This means that ψ is lower semi-continuous on X [12]. Hence, ψ is continuous on X. The lemma is proved. □

Lemma 5 [21, Theorem 9.14]**.** *Let $S \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^m$, $h : S \times \Theta \to R$ be a continuous function, and $D : \Theta \rightrightarrows S$ be a compact-valued and continuous multifunction. Then, the function $h^* : \Theta \to R$ defined by $h^*(\theta) = \max\{h(x, \theta) \,|x \in D(\theta)\}$ is continuous on Θ, and the multifunction $D* : \Theta \rightrightarrows S$ defined by $D*(\theta) = \{x \in D(\theta) \,|h(x, \theta) = h*(\theta)\}$ is compact-valued and upper semi-continuous on Θ.*

Lemma 6. *r is continuous on X.*

Proof. The multifunction ψ is compact-valued and continuous on X. Now, applying Lemma 5 for $S = \Theta = X$ and $D = \psi$, we deduce that r is an upper semi-continuous multifunction on X. Obviously, an upper semi-continuous multifunction is continuous when viewed as a function. This shows that r is continuous on X. The lemma is proved. □

Proof of Theorem 1. (a) Applying Lemma 1 we get that $PO(X, f) = SPO(X, f)$.

(b) From Lemmas 3 and 6 it follows that there exists a continuous function $r : X \to PO(X, f)$ such that $r(X) = PO(X, f)$ and $r(x) = \text{Argmax}(s, \psi(x))$ for all $x \in X$. This means that $PO(X, f)$ is a retract of X.

(c) We already know that f is homeomorphism. It is easy to prove that $f \circ r \circ f^{-1} : f(X) \to PF(X, f)$ is retraction. This means that $PF(X, f)$ is a retract of $f(X)$.

(d) It is well-known that every continuous image of the compact set is compact. In fact, the set X is compact and the function r is continuous on X. Hence, the set $PO(X, f) = r(X)$ is compact. Recalling that the function $f : X \to \mathbb{R}^n$ is continuous, we deduce that the restriction $h : PO(X, f) \to PF(X, f)$ of f is continuous too. We know that the function h is bijective. Consider the inverse function $h^{-1} : PF(X, f) \to PO(X, f)$ of h. As it was proved before, the set $PO\,(X, f)$ is compact, therefore h^{-1} is continuous too [23]. As a result we find that the function h is homeomorphism. The theorem is proved. □

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ВЪРХУ ПАРЕТОВСКИТЕ МНОЖЕСТВА В МНОГОКРИТЕРИАЛНАТА ОПТИМИЗАЦИЯ

Здравко Д. Славов

В тази работа се разглеждат Паретовските решения в непрекъсната многокритериална оптимизация. Обсъжда се ролята на някои предположения, които влияят на характеристиките на Паретовските множества. Авторът се е опитал да премахне предположенията за вдлъбнатост на целевите функции и изпъкналост на допустимата област, които обикновено се използват в многокритериалната оптимизация. Резултатите са на базата на конструирането на ретракция от допустимата област върху Парето-оптималното множество.