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INTEGRAL INEQUALITIES AND APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS WITH "MAXIMA"*

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This paper deals with some nonlinear integral inequalities that involve the maximum of the unknown scalar function of two variables. The considered inequalities are generalizations of the classical integral inequality of Gronwall-Bellman. The importance of these integral inequalities is due to their wide applications in the qualitative investigations of partial differential equations with "maxima" and it is illustrated by some direct applications.

1. Introduction. In the last few decades, great attention has been paid to automatic control systems and their applications to computational mathematics and modeling. Many problems in the control theory correspond to the maximal deviation of the regulated quantity [6] and are adequately modeled by differential equations with "maxima"[2], [4]. The qualitative investigation of properties of differential equations with "maxima" requires the creation of an appropriate mathematical means. One of the main mathematical tools, employed successfully for studying existence, uniqueness, continuous dependence, comparison, perturbation, boundedness, and stability of solutions of differential and integral equations is the method of integral inequalities. This method is well studied for ordinary differential equations and delay differential equations ([1], the monograph [3] and cited therein references). At the same time there are only some partial results for integral inequalities containing the maximum value of the unknown function [5].

The purpose of this paper is to establish some new integral inequalities in the case when maximum of the unknown scalar function is involved in the integral. The direct application of the obtained results is illustrated on partial differential equations with "maxima".

2. Main results. Let $h > 0$ be a constant and $x_0,\ y_0,\ X,\ Y$ be fixed points such that $0 \le x_0 < X \le \infty$ and $0 \le y_0 < Y \le \infty$.

Definition 1. *The function* $\alpha \in C^1([x_0, X), \mathbb{R}_+)$ *is from the class* $\mathcal{F}$ *if it is a nondecreasing and* $\alpha(x) \le x$ *for* $x \in [x_0, X)$.

Let the functions $\alpha_i,\ \beta_j \in \mathcal{F}$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. Denote $J = \min\Big(\min\limits_{1\le i\le n} \alpha_i(x_0),\ \min\limits_{1\le j\le m} \beta_j(x_0)\Big)$.

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Consider the sets $G,\ \Psi,\ \Lambda$ defined by

$$\begin{aligned} G &= \big\{(x,y)\in\mathbb{R}^2 : x\in[x_0,X),\ y\in[y_0,Y)\big\},\\ \Psi &= \big\{(x,y)\in\mathbb{R}^2 : x\in[J-h,x_0],\ y\in[y_0,Y)\big\},\\ \Lambda &= \big\{(x,y)\in\mathbb{R}^2 : x\in[J-h,X),\ y\in[y_0,Y)\big\} = G\cup\Psi. \end{aligned}$$

Definition 2. *The function $\omega\in C(\mathbb{R}_+,\mathbb{R}_+)$ is said to be from the class Ω if:*

(i) $\omega(x)>0$ *for* $x>0$ *and it is a nondecreasing function;*

(ii) $\displaystyle\int^{\infty}\frac{dx}{\omega(x)}=\infty\,.$

In the case when the nonlinear functions under the integrals are from the class Ω we obtain the following result:

Theorem 1. *Let the following conditions be fulfilled:*

1. *The functions $\alpha_i,\ \beta_j\in\mathcal{F}$ for $i=1,2,\dots,n$ and $j=1,2,\dots,m$.*
2. *The functions $f_i,\ g_j\in C([J,X)\times[y_0,Y),\mathbb{R}_+)$ for $i=1,2,\dots,n$ and $j=1,2,\dots,m$.*
3. *The function $\phi\in C(\Psi,\mathbb{R}_+)$.*
4. *The function $\varphi\in C(\mathbb{R}_+,\mathbb{R}_+)$ is nondecreasing with $\lim\limits_{t\to\infty}\varphi(t)=\infty$.*
5. *The functions $\omega_i,\ \tilde\omega_j\in\Omega$, $i=1,2,\dots,n$ and $j=1,2,\dots,m$.*
6. *The function $u\in C(\Lambda,\mathbb{R}_+)$ and satisfies the inequalities*

$$(1)\qquad \begin{aligned} u(x,y)\le k &+\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{y_0}^{y} f_i(s,t)\varphi\Big(u(s,t)\Big)\omega_i\Big(u(s,t)\Big)dt\,ds\\ &+\sum_{j=1}^{m}\int_{\beta_j(x_0)}^{\beta_j(x)}\int_{y_0}^{y} g_j(s,t)\varphi\Big(u(s,t)\Big)\tilde\omega_j\Big(\max_{\xi\in[s-h,\,s]}u(\xi,t)\Big)dt\,ds \end{aligned}\qquad \text{for } (x,y)\in G,$$

$$(2)\qquad u(x,y)\le\phi(x,y)\qquad \text{for } (x,y)\in\Psi,$$

where $k=\mathrm{const}>0$.

Then for $(x,y)\in G_1$ the inequality

$$(3)\qquad u(x,y)\le W^{-1}\Big(W\big(M\big)+A(x,y)\Big)$$

holds, where W^{-1} is the inverse function of

$$(4)\qquad W(r)=\int_{r_0}^{r}\frac{ds}{\varphi(s)q(s)}\,,\quad 0<r_0<M$$

$$(5)\qquad q(t)=\max\Big(\max_{1\le i\le n}\omega_i(t),\ \max_{1\le j\le m}\tilde\omega_j(t)\Big),\quad t\in\mathbb{R}_+$$

$$(6)\qquad M=\max\Big(k,\ \max_{s\in[J-h,\,x_0]}\phi(s,y)\Big),\quad y\in[y_0,Y)$$

$$(7)\qquad A(x,y)=\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{y_0}^{y}f_i(s,t)dt\,ds+\sum_{j=1}^{m}\int_{\beta_j(x_0)}^{\beta_j(x)}\int_{y_0}^{y}g_j(s,t)dt\,ds,$$

$$G_1 = \sup\Big\{(x,y)\in G:\quad W(M)+A(x,y)\ \in Dom\big(W^{-1}\big)\Big\}.$$

Proof. Define a function $z:\Lambda\to(0,\infty)$ by the equalities

$$z(x,y)=\begin{cases} M+\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{y_0}^{y} f_i(s,t)\varphi(u(s,t))\omega_i(u(s,t))dt\,ds \\ \quad+\sum_{j=1}^{m}\int_{\beta_j(x_0)}^{\beta_j(x)}\int_{y_0}^{y} g_j(s,t)\varphi(u(s,t))\tilde{\omega}_j\Big(\max_{\xi\in[s-h,\,s]}u(\xi,t)\Big)dt\,ds, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,y)\in G \\ M, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,y)\in\Psi\end{cases}$$

where the constant M is defined by (6).

The function $z(x,y)$ is nondecreasing with respect its both arguments, $z(x_0,y)=M$ for $y\in[y_0,Y)$ and the following inequalities are valid

$$u(x,y)\le z(x,y),\qquad (x,y)\in\Lambda, \tag{8}$$

$$\max_{\xi\in[s-h,\,s]}u(\xi,y)\le \max_{\xi\in[s-h,\,s]}z(\xi,y)=z(s,y),\ \ s\in[J,X),\ \ y\in[y_0,Y). \tag{9}$$

From (1), (8), (9) and the definition of the function $q(t)$ we get for $(x,y)\in G$

$$\begin{aligned} z(x,y)\le M&+\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{y_0}^{y} f_i(s,t)\varphi\big(z(s,t)\big)q\Big(z(s,t)\Big)dt\,ds \\ &+\sum_{j=1}^{m}\int_{\beta_j(x_0)}^{\beta_j(x)}\int_{y_0}^{y} g_j(s,t)\varphi\big(z(s,t)\big)q\Big(z(s,t)\Big)dt\,ds. \end{aligned} \tag{10}$$

Define a function $K:G\to[M,\infty)$ by the right hand-side of inequality (10). Note that $K(x,y)$ is a nondecreasing function, $K(x_0,y)=M$ for $y\in[y_0,Y)$ and the inequality $z(x,y)\le K(x,y)$ holds for $(x,y)\in G$. Differentiate the function $K(x,y)$ with respect to x and y, use its monotonicity, condition 1 and obtain

$$\begin{aligned} K_{xy}(x,y)\le&\sum_{i=1}^{n} f_i\big(\alpha_i(x),y\big)\varphi\big(K\big(\alpha_i(x),y\big)\big)q\Big(K(\alpha_i(x),y)\Big)\big(\alpha_i(x)\big)' \\ &+\sum_{j=1}^{m} g_j\big(\beta_j(x),y\big)\varphi\big(K\big(\beta_j(x),y\big)\big)q\Big(K(\beta_j(x),y)\Big)\big(\beta_j(x)\big)' \\ \le&\ \varphi\big(K(x,y)\big)q\big(K(x,y)\big) \\ &\times\left(\sum_{i=1}^{n} f_i\big(\alpha_i(x),y\big)(\alpha_i(x))'+\sum_{j=1}^{m} g_j\big(\beta_j(x),y\big)(\beta_j(x))'\right). \end{aligned} \tag{11}$$

From inequality (11) we have

$$\frac{\partial}{\partial y}\left(\frac{K_x(x,y)}{\varphi(K(x,y))q(K(x,y))}\right)\ \le\sum_{i=1}^{n} f_i(\alpha_i(x),y)\alpha_i'(x)+\sum_{j=1}^{m} g_j(\beta_j(x),y)\beta_j'(x). \tag{12}$$

Integrate inequality (12) with respect to y $\big(y \in [y_0, Y)\big)$, use (4), and obtain

$$\frac{\partial}{\partial x} W\big(K(x,y)\big) = \frac{K_x(x,y)}{\varphi\big(K(x,y)\big)q\big(K(x,y)\big)} \tag{13}$$
$$\leq \sum_{i=1}^{n} \int_{y_0}^{y} f_i\big(\alpha_i(x),t\big)\alpha_i'(x)dt + \sum_{j=1}^{m} \int_{y_0}^{y} g_j\big(\beta_j(x),t\big)\beta_j'(x)dt.$$

Integrate (13) with respect to x $\big(x \in [x_0, X)\big)$ and obtain for $(x,y) \in G_1$

$$W\big(K(x,y)\big) \leq W(M) + A(x,y), \tag{14}$$

where the function $A(x,y)$ is defined by equality (7).

Since W^{-1} is an increasing function, from (8), (14) and $z(x,y) \leq K(x,y)$ we obtain the required inequality (3). $\square$

In the case when the function $\varphi(x) \equiv x^p$ where $p = \text{const}$ $(0 < p < 1)$, we obtain the following result:

Corollary 1. *Let the following conditions be fulfilled:*

1. *The conditions* 1, 2, 3 *and* 5 *of Theorem* 1 *are satisfied.*
2. *The function* $u \in C(\Lambda, \mathbb{R}_+)$ *and satisfies the inequalities*

$$u(x,y) \leq k + \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{y_0}^{y} f_i(s,t)u^p(s,t)\omega_i\Big(u(s,t)\Big)dt\,ds \tag{15}$$
$$+ \sum_{j=1}^{m} \int\limits_{\beta_j(x_0)}^{\beta_j(x)} \int\limits_{y_0}^{y} g_j(s,t)u^p(s,t)\tilde{\omega}_j\Big(\max_{\xi \in [s-h,\, s]} u(\xi,t)\Big)dt\,ds, \ (x,y) \in G,$$

$$u(x,y) \leq \phi(x,y) \qquad (x,y) \in \Psi, \tag{16}$$

where $k = \text{const} > 0$ *and* $p = \text{const}$ *such that* $0 < p < 1$.

Then, for $(x,y) \in G_2$ *the inequality*

$$u(x,y) \leq W^{-1}\Big(W\big(M\big) + A(x,y)\Big) \tag{17}$$

holds, where $q(t)$, M *and* $A(x,y)$ *are defined by* (5), (6) *and* (7), *respectively,* W^{-1} *is the inverse function of* $W(r) = \displaystyle\int_{r_1}^{r} \frac{ds}{s^p\, q(s)}$, $0 < r_1 < M$,

$$G_2 = \sup\Big\{(x,y) \in G : W\big(M\big) + A(x,y) \in Dom\left(W^{-1}\right)\Big\}.$$

3. Application. We apply some of the above solved inequalities to obtain some qualitative properties of partial differential equations with “maxima“.

Example 1. *Consider the following partial differential equations with “maxima”*

$$v_{x\,y} = F\Big(x,\ y,\ v(x,y), \max_{s \in [\sigma(x),\, \tau(x)]} v(s,y)\Big) \quad \textit{for} \quad x \geq x_0, \quad y \geq y_0 \tag{18}$$

with the initial conditions

$$v(x_0,y) = \varphi_1(y) \quad \textit{for } y \in [y_0, Y), \quad v(x,y_0) = \varphi_2(x) \quad \textit{for } x \in [x_0, X), \tag{19}$$
$$v(x,y) = \psi(x,y) \quad \textit{for } (x,y) \in [\tau(x_0) - h, x_0] \times [y_0, Y),$$

where $v \in \mathbb{R}^2$, $F : [x_0, X) \times [y_0, Y) \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, $h > 0$ *is a constant.*

Theorem 2 (Upper bound)**.** *Let the following conditions be fulfilled:*

1. *The functions* $\sigma,\ \tau \in \mathcal{F}$ *and* $0 < \tau(x) - \sigma(x) \le h$ *for* $x \ge x_0$.

2. *The function* $F \in C([x_0, X) \times [y_0, Y) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *and satisfies for* $(x, y) \in G$, $v \in \mathbb{R}^2$, *the condition* $\big|F\big(x, y, v(x, y), \max_{s\in[\sigma(x),\,\tau(x)]} v(s, y)\big)\big| \le |v(x, y)|^p \Big[R(x, y)|v(x, y)|^{\tilde q} + Q(x, y)\big|\max_{s\in[\sigma(x),\,\tau(x)]} v(s, y)\big|^{\tilde q}\Big]$, *where the functions* $R,\ Q \in C(G, \mathbb{R}_+)$ *and* $p,\ \tilde q$ *are constants such that* $0 < \tilde q < 1$, $0 < p < 1$ *and* $1 - p - \tilde q > 0$.

3. *The function* $\psi \in C([\tau(x_0) - h, x_0] \times [y_0, Y), \mathbb{R})$.

4. *The functions* $\varphi_1 \in C([y_0, Y), \mathbb{R})$, $\varphi_2 \in C([x_0, X), \mathbb{R})$ *and the equalities* $\varphi_1(y_0) = \varphi_2(x_0)$, $\varphi_1(y) = \psi(x_0, y)$, $y \in [y_0, Y)$ *hold.*

5. *The initial value problem* (18), (19) *has at least one solution, defined for* $(x, y) \in [\tau(x_0) - h, X) \times [y_0, Y)$.

Then, for $(x, y) \in G$ *the solution of the initial value problem* (18), (19) *satisfies the inequality*

$$|v(x, y)| \le \sqrt[1-p-\tilde q]{M^{1-p-\tilde q} + (1 - p - \tilde q)\int_{x_0}^{x}\int_{y_0}^{y}\Big[R(s, t) + Q(s, t)\Big]dt\,ds}, \tag{20}$$

where the constant M *is defined by*

$$M = \max\Big(|\varphi_1(y)| + |\varphi_2(x)| - |\varphi_2(x_0)|,\ \max_{s\in[\tau(x_0)-h,\,x_0]} \psi(s, y)\Big). \tag{21}$$

Proof. From condition 2 of Theorem 2 for the norm of the solution $v(x, y)$ of the initial value problem (18), (19) it follows

$$\begin{aligned}|v(x, y)| \le{}& |\varphi_1(y)| + |\varphi_2(x)| - |\varphi_2(x_0)| + \int_{x_0}^{x}\int_{y_0}^{y} R(s, t)|v(s, t)|^p\big(|v(s, t)|\big)^{\tilde q}dt\,ds\\ &+\int_{x_0}^{x}\int_{y_0}^{y} Q(s, t)\big|v(s, t)\big|^p\Big(\max_{\xi\in[\sigma(s),\,\tau(s)]}\big|v(\xi, t)\big|\Big)^{\tilde q}dt\,ds, \quad (x, y) \in G\end{aligned} \tag{22}$$

$$|v(x, y)| \le \psi(x, y), \qquad (x, y) \in [\tau(x_0) - h, x_0] \times [y_0, Y). \tag{23}$$

Set $|v(x, y)| = V(x, y)$ for $(x, y) \in [\tau(x_0) - h, X) \times [y_0, Y)$, change the variable, use $\max_{\xi\in[\sigma(s),\,\tau(s)]} |v(\xi, y)| \le \max_{\xi\in[\tau(s)-h,\,\tau(s)]} |v(\xi, y)|$ for $y \in [y_0, Y)$ and $s \in [x_0, X)$ and obtain

$$\begin{aligned}V(x, y) \le{}& |\varphi_1(y)| + |\varphi_2(x)| - |\varphi_2(x_0)| + \int_{x_0}^{x}\int_{y_0}^{y} R(s, t)V^p(s, t)(V(s, t))^{\tilde q}dt\,ds\\ &+\int_{\tau(x_0)}^{\tau(x)}\int_{y_0}^{y} Q(\tau^{-1}(\eta), t)(\tau^{-1}(\eta))'V^p(\tau^{-1}(\eta), t)\big(\max_{\xi\in[\eta-h,\,\eta]} V(\xi, t)\big)^{\tilde q}dt\,d\eta,\\ &\qquad (x, y) \in G\end{aligned} \tag{24}$$

$$V(x, y) \le \psi(x, y) \qquad (x, y) \in [\tau(x_0) - h, x_0] \times [y_0, Y). \tag{25}$$

According to Corollary 1 from (24), (25) for $u(x,y) = V(x,y)$, $\phi(x,y) \equiv \psi(x,y)$, $k \equiv |\varphi_2(x)| - |\varphi_2(x_0)| + |\varphi_1(y)|$, $n = 1$, $\alpha(x) \equiv x$, $m = 1$, $\beta(x) \equiv \tau(x)$, $f(x,y) \equiv R(x,y)$, $g(s,t) \equiv R(\tau^{-1}(\eta), t)(\tau^{-1}(\eta))'$, $s \in [\tau(x_0), T)$, $t \in [y_0, Y)$, $\omega(V) \equiv \tilde{\omega}(V) = V^{\tilde{p}}$, $W(r) = \int_0^r \frac{ds}{s^{p+\tilde{q}}} = \frac{r^{1-p-\tilde{q}}}{1-p-\tilde{q}}$, $W^{-1}(r) = \big((1-p-\tilde{q})r\big)^{\frac{1}{1-p-\tilde{q}}}$, $Dom(W^{-1}) = \mathbb{R}_+$ we obtain for $(x,y) \in G$

$$(26) \qquad V(x,y) \le \sqrt[1-p-\tilde{q}]{M^{1-p-\tilde{q}} + (1-p-\tilde{q}) \int_{x_0}^{x} \int_{y_0}^{y} \Big[R(s,t) + Q(s,t)\Big] dt\, ds}\ .$$

From inequality (26) and the definition of the function $V(x,y)$ we obtain the required inequality (20). □

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ИНТЕГРАЛНИ НЕРАВЕНСТВА И ПРИЛОЖЕНИЯ В ЧАСТНИТЕ ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ С "МАКСИМУМИ"

Кремена В. Стефанова

В тази статия са разрешени някои нелинейни интегрални неравенства, които включват максимума на неизвестната функция на две променливи. Разгледаните неравенства представляват обобщения на класическото неравенство на Гронуол-Белман. Значението на тези интегрални неравенства се определя от широките им приложения в качествените изследвания на частните диференциални уравнения с "максимуми" и е илюстрирано чрез някои директни приложения.