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MINIMAL SUBSPACES WITH MAXIMAL DIMENSIOANAL DIAMETERS*

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Suppose that X is a compact metric space with $\dim X = n$. Then for the $n-1$ dimensional diameter $d_{n-1}(X)$ we have $d_{n-1}(X) > 0$ and in the same time $d_n(X) = 0$. It follows now that X contains a minimal by inclusion closed subset Y for which $d_{n-1}(Y) = d_{n-1}(X)$. Under these conditions Y is a Cantor manifold [7]. In this note we prove that every such subspace Y is even a continuum V^n. Various consequences are discussed.

1. Introduction. The theory of Cantor Manifolds developed from an initial effort to give a rigorous description of a degree of connectedness of some basic objects. A typical example in this attitude is the n-dimensional cube I^n ($I = [0,1]$). In 1925, Urysohn established that the n-dimensional cube cannot be separated by any $(n-2)$-dimensional closed subset. In other words, I^n is not a sum of two proper closed sets whose intersection is at most $(n-2)$-dimensional. In 1957, Alexandroff proved that I^n is even the so-called continuum (V^n).

Later various ways in establishing properties of connectedness of I^n are proposed, namely, in 1969 by Wilkinson and in 1970 by Hadziivanov. They proved that I^n is not a union of countable many propers closed sets whose pair-wise intersections are at most $(n-2)$-dimensional. Finally we should note that there are various different results in this direction. For example, using the classical theorem of Sierpinski, Urysohn have proved that I^n is not cut by $(n-2)$-dimensional G_δ subsets. However, it is worthy of mentioning that at present, it seems that the best description of a connectedness of I^n appears in the concept of (V^n)-continua.

A mandatory condition for a "good" class of Cantor Manifolds is that every n-dimensional compact metric space X must contain a n-dimensional Cantor Manifold from the corresponding class. There is various results concerning the above mentioned classes [1], [2], [3], [6]. In [7] it is proved that if X is a compact metric finite dimensional space, then X contains a Cantor Manifold Y with additional condition $d_{n-1}(Y) = d_{n-1}(X)$. In this note we prove that X contains even a continuum V^n Y with $d_{n-1}(Y) = d_{n-1}(X)$.

2. Basic concepts and definitions. Let ϱ be the metric of the compact space X.

Definition 2.1 *For $A \subset X$ the diameter of A is the number*

$$\operatorname{diam}(A) = \sup\{\varrho(x,y) \mid x, y \in A\}.$$

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Next we recall some useful notions. Let $\mathcal{U} = \{U_1, U_2, \dots, U_m\}$ be a finite family of subsets of X. The *order* $\mathrm{ord}\mathcal{U}$ of $\mathcal{U}$ is by definition the maximal number of elements of $\mathcal{U}$, which intersection is nonempty. The *measure* $\mathrm{mesh}\mathcal{U}$ of $\mathcal{U}$ is the number $\max\{\mathrm{diam}(U_i) | \, i = 1, 2, \dots, m\}$.

We call the set $|\mathcal{U}| = \bigcup_{i=1}^{m} U_i$ the *body* of $\mathcal{U}$. If $|\mathcal{U}| = X$, then $\mathcal{U}$ is called a cover of X. If in addition $\mathcal{U}$ consists of open sets then $\mathcal{U}$ shall be named open cover.

Definition 2.2. *The closed set C is a partition in X between P and Q if $X \setminus C = U \cup V$, where U and V are open, $U \supset P$, $V \supset Q$ and $U \cap V = \emptyset$.*

Definition 2.3. *The n-dimensional diameter $d_n(X)$ of the metric space X is the number* $\inf\{\mathrm{mesh}(\mathcal{U})\}$, *where $\mathcal{U}$ runs the set of all finite open covers of X with* $\mathrm{ord}(\mathcal{U}) \leq n + 1$.

Further let us recall that X is a Cantor n-Manifold (CM) [1] if $\dim X = n$ and there is no a partition C in X with $\dim C \leq n - 2$.

X is called a Strongly Cantor Manifold (SCM) [6] if it is impossible to represent X as $\bigcup_{i=1}^{\infty} F_i$, where F_i is closed for every i and

$$\dim \bigcup_{i \neq j} (F_i \cap F_j) \leq n - 2.$$

Definition 2.4. *The subset L of X cuts X between P and Q , if for every closed subset Y of X, which connects $Y \cap P$ and $Y \cap Q$, we have $Y \cap L \neq \emptyset$.*

The space X is by definition a Mazurkiewitz Manifold (MM) if for every $L \subset X$ which cuts X one has $\dim L \geq n - 1$.

Definition 2.5. *The space X is a continuum V^n or Alexandroff Manifold (AM) if for every pair of disjoint nonempty open sets A and B there exists $\varepsilon > 0$ such that $d_{n-2}(C) \geq \varepsilon$ for every partition C between A and B.*

Sometimes we call that X is (n, ε)-connected between A and B. Note that it is well-known that $CM \subset SCM \subset MM \subset AM$ and every inclusion is strong.

3. Main theorem and corollaries.

Theorem 3.1. *Let (X, ϱ) be a compact metric space for which $\chi = d_{n-1}(X) > 0$, $d_n(X) = 0$ and for every proper closed subset Y of X one has $d_{n-1}(Y) < \chi$. Then, X is a continuum V^n.*

Proof. Choose an arbitrary disjoint pair of nonempty open sets A and B in X and put $X_A = X \setminus A$ and $X_B = X \setminus B$. In view of the fact that X_A and X_B are proper closed subsets of X we should have $d_{n-1}(X_A) < \chi$ and $d_{n-1}(X_B) < \chi$. That means one can find two finite open covers $\mathcal{U}_A$ of X_A and $\mathcal{U}_B$ of X_B respectively for which $\mathrm{ord}(\mathcal{U}_A) \leq n$; $\mathrm{ord}(\mathcal{U}_B) \leq n$ and $\mu_A = \mathrm{mesh}(\mathcal{U}_A) < \chi$; $\mu_B = \mathrm{mesh}(\mathcal{U}_B) < \chi$.

Now suppose that for every $\varepsilon > 0$ the space X is not (n, ε)-connected between A and B. In other words for every $\varepsilon > 0$ one can find some partition C in X between A and B with $d_{n-2}(C) < \varepsilon$.

Furthermore, denote by λ_A and λ_B the Lebesgue numbers of $\mathcal{U}_A$ and $\mathcal{U}_B$ and choose $\varepsilon > 0$ such that $2\varepsilon < \min\{\lambda_A, \lambda_B, \chi - \mu_A, \chi - \mu_B\}$.

Now let C be a partition between A and B for which $d_{n-2}(C) < \varepsilon$ and consider some open cover $\mathcal{U}_C$ of C with $\mu_C = \text{mesh}(\mathcal{U}_C) < \varepsilon$ and $\text{ord}(\mathcal{U}_C) \leq n - 2$. Next it is easy to see that one can take a refinement $\mathcal{V}_C$ of a cover $\mathcal{U}_C$ with $\text{ord}(\mathcal{V}_C) = \text{ord}(\mathcal{U}_C)$ and such that $cl|\mathcal{V}_C| \subset |\mathcal{V}_C|$ (here cl means closure).

C was a partition, hence, $X \setminus C = Y_A \cup Y_B$, where $Y_{A,B} \supset A, B$ are open disjoint sets. Then $Z_A = Y_A \setminus cl|\mathcal{V}_C|$ and $Z_B = Y_B \setminus cl|\mathcal{V}_C|$ are disjoint open subsets of X. Denote by $\mathcal{W}_B = \mathcal{V}_A|_{X_B}$ and $\mathcal{W}_A = \mathcal{V}_B|_{X_A}$ the restrictions of the corresponding covers over the sets X_B and X_A respectively. Clearly $\mathcal{P} = \mathcal{W}_A \cup \mathcal{U}_C \cup \mathcal{W}_B$ is an open cover of X for which $\text{mesh}\mathcal{P} < \min\{\varepsilon + \mu_A; \varepsilon + \mu_B\} < \chi$ and because of the choice of ε it is easy to check that $\mathcal{P}$ can be modified such that the order of $\mathcal{P}$ remains less than n. This contradicts to the minimality of X. $\square$

Corollary 3.1 ([2])**.** *Every compact metric n-dimensional space X contains a continuum V^n Y. Moreover, one can choose Y such that $d_{n-1}(Y) = d_{n-1}(X)$.*

Proof. It follows by the Zorn lemma that the set of all subcompacta Z of X with $d_{n-1}(Z) = d_{n-1}(X)$ ordered by inclusion has a minimal element. $\square$

Because V^n continua are Cantor Manifolds in any other sense from the mentioned above the results of [1], [6] and [5] can be obtained as corollaries (with some reinforcement). For example every n-dimensional compact metric space X contains SCM subspace Y with $d_{n-1}(Y) = d_{n-1}(X)$.

REFERENCES

[1] P. S. Urysohn. Memoire sur les multiplicites Cantoriennes (I). Fundam. Math., **7** (1925), 30–139.

[2] P. S. Alexandroff. Die Kontinua (V^p)-eine Verschärfung der Cantorshen Mannigfaltigkeiten. *Monatsh. Math.*, **61** (1957), H. 1, 67–76.

[3] J. B. Wilkinson. A lower bound for the dimension of certain G_δ sets in completely normal spaces. *Proc. Amer. Math. Soc.*, **20**, (1969) 175–178.

[4] N. Hadjiivanov. The n-dimensional cube can not be represented as a sum of contable many proper closed sets which pair-wise intersections are at most $(n-2)$-dimensional. *Compt. Rend. Acad. Sci. of USSR*, **195** (1970), No 1, 43–45 (in Russian).

[5] N. Hadjiivanov, V. Todorov. On non-Euclideam manifolds. *Compt. Rend. Bul. Acad. Sci.*, **33** (1980), No 4, 449–452 (in Russian).

[6] N. G. Hadziivanov. On Cantor manifolds. *Compt. Rend. Bulg. Acad. Sci.*, **31** (1978), No 7, 941–944 (in Russian).

[7] N. Hadziivanov, V. Todorov. On dimensional components of compact spaces. *Compt. Rend. Bulg. Acad. Sci.*, **33** (1980), No 11, 1433–1435.

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МИНИМАЛНИ ПОПРОСТРАНСТВА С МАКСИМАЛНИ РАЗМЕРНОСТНИ ДИАМЕТРИ

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Нека X е компактно метрично пространство с $\dim X = n$. Тогава за $n-1$-мерния диаметър $d_{n-1}(X)$ на X е изпълнено неравенството $d_{n-1}(X) > 0$, докато $d_n(X) = 0$ (да отбележим, че това е една от характеристиките на размерността на Лебег). От тук се получава, че X съдържа минимално по включване затворено подмножество Y, за което $d_{n-1}(Y) = d_{n-1}(X)$. Известен резултат е, че от това следва, че Y е Канторово Многообразие. В тази бележка доказваме, че всяко такова (минимално) подпространство Y е даже континуум V^n. Получени са също така някои следствия.