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ON SOME PROPERTIES OF BOOLEAN FUNCTIONS AND THEIR BINARY DECISION DIAGRAMS*

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Boolean functions manipulation is an essential component of computer science, including logic optimization, logic verification and logic synthesis. In this paper some initial results about dependency of the graph based presentation of the Boolean functions and the properties of their variables are obtained.

1. Introduction. The synthesis of Decision Diagrams and the theory of essential variables and separable sets of variables are two fields of the theoretical computer science which have been intensively (but separately) developed during the last decades.

Let $\mathbb{B}$ be two-element set, $\mathbb{B} = \{0,1\}$. A *Boolean function* of n variables $\{x_1, x_2, \ldots, x_n\}$ is a function $f(x) : \mathbb{B}^n \to \mathbb{B}$, where $x = [x_1, \ldots, x_n]$ denotes its arguments. The set of all Boolean functions $f(x) : \mathbb{B}^n \to \mathbb{B}$, is denoted by $F[x_1, \ldots, x_n]$, or by $F(n)$ if variables are not being considered.

The function $f(x_1, \ldots, x_n)$ can be decomposed with respect to (w.r.t.) variable x_i using Shannon expansion [7] such as:

$$f(x_1, \ldots, x_n) = \overline{x}_i f_1 \oplus x_i f_2. \tag{1}$$

The functions $f_1 := f(x_i = 0) \in F(n-1)$ and $f_2 := f(x_i = 1) \in F(n-1)$ are called *subfunctions* of $f(x_1, \ldots, x_n)$.

Various representations of Boolean functions in theory and practice are used, including truth-table, formulae, binary decision trees, etc. In practice, Ordered Binary Decision Diagrams (OBDD) are considered to be the state of the art in data structures for Boolean functions. The OBDD minimization problem is one of the deeply studied since OBDD was introduced by R. Bryant [1].

Definition 1. *Let π be a total order on the set of variables $x_1, \ldots, x_n$. An* Ordered Binary Decision Diagram (OBDD) *with respect to the variable order π (π-OBDD) is a directed acyclic graph with exactly one root which satisfies the following properties:*

- *There are exactly two nodes without outgoing edges. These two nodes are labeled by the constants* 1 *and* 0*, respectively, and are called* terminal nodes.
- *Each non-terminal node is labeled by a variable x_i, and has two outgoing edges, which are labeled by* 1 *and* 0*, respectively. These edges are called* 1-edge *(represented by solid line in the graph) and* 0-edge *(dashed line).*

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- *The order, in which the variables appear in a graph's path, is consistent with the variable order π, i.e., for each edge leading from node x_i to node x_j holds that $x_i <_\pi x_j$.*

Definition 2. *An OBDD with root node v denotes a function f_v, defined recursively as:*

(1) *If v is a terminal node, then $f_v = val(v)$, where $val(v) \in \{0, 1\}$.*

(2) *If v is a non-terminal node labeled with x_i, then*

$$f_v(x_1, \ldots, x_n) = \overline{x_i} f(x_i = 0) + x_i f(x_i = 1).$$

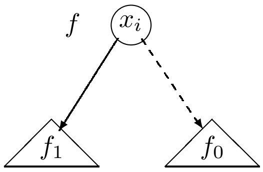

Fig. 1. OBDD for $f = \overline{x_i} f_0 + x_i f_i$

A *path* in an OBDD is a sequence of connected nodes starting from the root of the OBDD and ending with a terminal node. The arguments associated with all nodes in the path form a *product term*.

The function represented by an OBDD is the sum of the product terms associated with all paths ending with the 1 terminal node.

Because compact representation is important two reduction rules are introduced.

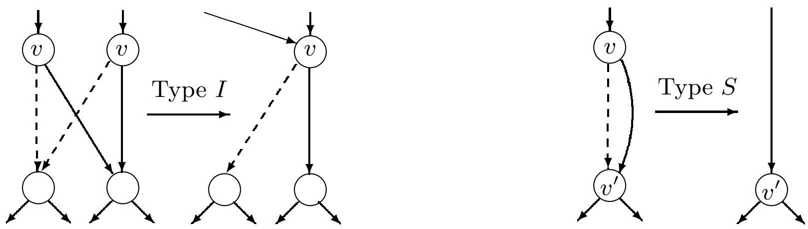

Fig. 2. Reduction Rules

OBDD with applied all possible reductions is called Reduced OBDD (ROBDD) and in [1] R. Bryant shows that ROBDD forms canonical representation of the Boolean function with respect to the given variable ordering π ($\pi-$ROBDD).

2. Main result. The complexity of the Boolean functions is affected by properties of their *essential* (supporting) variables. Some main results about that properties were obtained by J. Breitbart [6], K. Chimev [8], Sl. Shtrakov [3] and others. The properties of essential variables were studied also for Universal algebra by Shtrakov and Denecke [4] and for Tree automata by Damyanov and Shtrakov [2].

Definition 3. *The function* $f(x_1, \dots, x_n) \in F(n)$ depends essentially on x_i $(1 \le i \le n)$, *if two* $n-$*tuples*

$$(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_n) \quad \text{and} \quad (a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$$

exist, such that

$$f(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_n) \ne f(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n).$$

In other words, the variable x_i *is essential for* $f \in F(n)$ *if* $f(x_i = 1) \ne f(x_i = 0)$.

When the variable x_i is not essential, it is called *fictive.*

The set of all essential for f variables is denoted by $Ess(f)$ and the set of all fictive variables – $Fic(f)$.

Proposition 1. *For the Boolean function* $f \in F(n)$ *if* $x_i \in Ess(f)$ *then there exists at least one node labeled with* x_i *in it ROBDD regardless to the given variable ordering* π.

If the variable is fictive for the function f*, then there is no any node labeled with it in ROBDD regardless to the given variable ordering.*

Definition 4. *If* $f \in F(n)$, $n \ge 1$ *and* $\emptyset \ne M \subseteq Ess(f)$, $N \subseteq Ess(f)$, $M \cap N = \emptyset$, *then we say that* the set N is separable for f w.r.t. M, *i.e. if for the variables from* M *there exist values such that when replacing them by constants, then the new subfunction* g *obtained from* f *satisfies* $N \subseteq Ess(g)$. *This is denoted by* $N \in Sep(f, M)$.

Definition 5. *For the Boolean function* f *we say that* the set M $(M \subseteq Ess(f))$ is separable for f, *if* M *is separable for* f *w.r.t.* $Ess(f) \setminus M$. *We denote this by* $M \in Sep(f)$.

By *separable pair* we mean separable set with cardinality equal to 2.

Definition 6. *Let* M *and* N *be two non-empty sets of essential variables for the Boolean function* $f \in F(n)$ *and* $N = \{x_{i_1}, \dots, x_{i_s}\}$. *Then the set* N *is* distributor *for variables in* M *if for any boolean values* $c_{i_1}, \dots, c_{i_s}$ *of variables in* N

$$M \not\subseteq Ess(f(x_{i_1} = c_{i_1}, \dots, x_{i_s} = c_{i_s}))$$

holds true and N *is minimal w.r.t. this property.*

We denote the set of all distributors of M for f with $Ann(M, f)$.

Theorem 1. *Let* $f \in F(n)$, $|Ess(f)| = n$ *and* $N \in Ann(M, f)$, *where* $N \subset Ess(f)$, $M \subset Ess(f)$ *and* $N \cap M = \emptyset$. *For any variable ordering* π, *where* $\forall x_i \in N, \forall x_j \in M$ $x_i <_\pi x_j$ *holds that in any of* π*-ROBDD there is no path containing all variables from* M.

Proof. From Definition 6 it follows that $M \cap Fic(f(x_{s_1} = c_{s_1}, \dots, x_{s_k} = c_{s_k})) \ne \emptyset$.

Let us assume that there is a path from the root of π-ROBDD to the terminal node containing nodes labeled with all variables from the set M.

Since the path actually represents a given assignment of variables at some point, we reach node labeled with x_{s_j}, where $x_{s_j} \in M \cap Fic(f(x_{s_1} = c_{s_1}, \dots, x_{s_k} = x_{s_k}))$. Then outgoing edges from node labeled with x_{s_j} point one and the same next node. This means that we are able to apply reduction rule S which contradicts to the assumption that the π-OBDD is reduced. $\square$

Corollary 1. *If for the Boolean function $f \in F(n), |Ess(f)| = n$ and the variables x_i and x_j do not form separable pair, then there exists variable ordering π, such that for π-ROBDD of f there is no path containing both variables x_i and x_j.*

Corollary 2. *If $f \in F(3), |Ess(f)| = n$ and $\{x_1\} = Ann(\{x_2, x_3\}, f)$, then the minimal ROBDD is obtained for variable ordering π such that $x_1 <_\pi x_2$ and $x_1 <_\pi x_3$.*

The number of different subfunctions that continue to depend essentially on variables reflects on the OBDD complexity. Decision Diagrams complexity is expressed by the number of non-terminal nodes. Connection between number of subfunctions that depend essentially on a given variable and non-terminal nodes labeled with that variable was shown by D. Sieling and I. Wegener in [5].

Lemma 1 ([5])**.** *Let S_i be the set of subfunctions $f(x_1 = c_1, \ldots, x_{i-1} = c_{i-1})$ that essentially depend on x_i and where $c_1, \ldots, c_{i-1} \in \{0, 1\}$. A π-ROBDD for f according to the variable ordering $x_1 <_\pi x_2 <_\pi \cdots <_\pi x_n$ contains exactly $|S_i|$ nodes labeled by x_i.*

In a search of connection between distributor sets and complexity the following proposition can be stated.

Proposition 2. *Let $f \in F(n)$, $|Ess(f)| = n$ and $\{x_1\} = Ann(\{x_2, \ldots, x_n\}, f)$. If S be the set of all subfunctions of f with respect to the variable $x_k \in \{x_2, \ldots, x_n\}$ that depend essentially on x_1, then it holds that $|S| > 1$.*

Proof. Subfunctions $f(x_{i_1} = c_{j_1}, x_{i_2} = c_{j_2}, \ldots, x_{i_k} = c_{j_k})$ we denote by $f^{c_{j_1}c_{j_2}\ldots c_{j_k}}_{i_1 i_2 \ldots i_k}$. W.l.o.g. let we get $k = 2$, then $f = \overline{x_2} f^0_2 + x_2 f^1_2$. Since x_2 is essential for f, it follows that $f^0_2 \neq f^1_2$.

Let we assume that one of the subfunctions (f^0_2) does not depend essentially on x_1. Then, $f^0_2 = x_1 f^{01}_{21} + \overline{x_1} f^{00}_{21}$ and $f^{01}_{21} = f^{00}_{21} = f^{0-}_{21}$, i.e. $f^0_2 = f^{0-}_{21}$.

It follows that f can be represented as $f = \overline{x_2} f^{0-}_{21} + x_2 \overline{x_1} f^{10}_{21} + x_2 x_1 f^{11}_{21}$.

Consequently, if we represent f as decomposition on x_1, then

$$f = \overline{x_1}(x_2 f^{10}_{21} + \overline{x_2} f^{0-}_{21}) + x_1(x_2 f^{11}_{21} + \overline{x_2} f^{0-}_{21}) = \overline{x_1} f^0_1 + x_1 f^1_1.$$

Since $\{x_1\} = Ann(\{x_2, \ldots, x_n\}, f)$, one of the subfunctions f^0_1, f^1_1 does not depend on x_2. Without loose of generality we can assume that this is f^1_1.

Then, from $f^1_1 = x_1(x_2 f^{11}_{21} + \overline{x_2} f^{0-}_{21})$ and $f^{11}_{21} = f^{0-}_{21}$ it follows that

$$f = \overline{x_1}(x_2 f^{10}_{21} + \overline{x_2} f^{11}_{21}) + x_1(f^{11}_{21}) = \overline{x_1} f^0_1 + x_1 f^1_1.$$

Since x_1 is distributor it must "distribute" variables between subfunctions. This means if $x_2 \in Ess(f^1_1)$, then $x_2 \notin Ess(f^0_1)$. From here it follows that $f^{10}_{21} = f^{11}_{21}$, i.e. $f = \overline{x_2} f^{0-}_{21} + x_2 f^{10}_{21}$ which means that $x_1 \notin Ess(f)$. $\square$

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ВЪРХУ НЯКОИ СВОЙСТВА НА БУЛЕВИТЕ ФУНКЦИИ И ТЕХНИТЕ ДИАГРАМИ ЗА ДВОИЧНО РЕШАВАНЕ

Иво Й. Дамянов

Манипулирането на булеви функции е основно за теоретичната информатика, в това число логическата оптимизация, валидирането и синтеза на схеми. В тази статия се разглеждат някои първоначални резултати относно връзката между граф-базираното представяне на булевите функции и свойствата на техните променливи.