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A LIMIT THEOREM FOR MULTI-TYPE SUBCRITICAL AGE-DEPENDENT BRANCHING PROCESSES WITH TWO TYPES OF IMMIGRATION*

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This work continues the study of the classical subcritical age-dependent branching process and the effect of the following two-type immigration pattern in multi-dimensional case. At a sequence of renewal epochs a random number of immigrants of different types enters the population. Each subpopulation stemming from one of these immigrants is revived by new immigrants and their offspring whenever it dies out, possibly after an additional delay period. Individuals from the same type have the same lifetime distribution and produce offspring according to the same reproduction law. This is the p-dimensional Bellman-Harris process with immigration at zero and immigration of renewal type (BHPIOR). With this paper we complete the study of the one-dimensional case with its multi-type counterpart generalizing the convergence in probability for such processes.

1. Introduction. What is the effect of immigration at recurrent random epochs on the longterm behavior of populations that would otherwise become extinct because their reproductive pattern is subcritical? This question was investigated by Alsmeyer and Slavtchova-Bojkova (2005) for some classical branching processes, namely simple Galton-Watson processes (discrete time) and Bellman-Harris processes (continuous time), and for a certain immigration pattern. Thus individuals of the considered populations have i.i.d. lifetimes (identically 1 in the discrete-time case), different for each type, and produce independent numbers of individuals of different types of offspring at their death with a common subcritical reproduction. Immigration is assumed to occur both, at an independent sequence of renewal epochs, the vectors of immigrants of different types being i.i.d., and further whenever a subpopulation stemming from one of these immigrants or one of the ancestors dies out, possibly after a delay period. The vectors of immigrants at these extinction epochs as well as the delay periods are each sequences of i.i.d. random vectors and variables, also. If only the second type of state-dependent immigration occurs, then, by subcriticality, the resulting branching process is easily seen to be a strongly regenerative process (see Thorisson (2000)) whose successive extinction times constitute regeneration epochs with finite mean. Therefore, it converges in distribution

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to a limiting vector with positive mean (s. Lemma 2.1). Because additional immigration at successive renewal epochs leads to a superposition of such processes, a linear growth behavior is to be expected, at least under some mild regularity conditions. Our main result is a confirmation of this conjecture, as it is in one-dimensional case. We state the result in the continuous-time case because corresponding results in discrete time are then obtained by almost trivial adjustments of the arguments. It again follows essentially by use of the theory of regenerative processes, renewal theory and occupation measures, that is in contrast to earlier related work using the classical analytic approach towards such processes based upon generating functions, Laplace transforms and integral equations. The described immigration patterns for Bellman-Harris or Galton-Watson processes have been discussed in a number of papers. The Galton-Watson process with immigration at 0 (Foster-Pakes model) was first studied by Foster(1971) and Pakes(1971, 1972, 1978) under varying additional assumptions. Its continuous time analog was studied by Yamazato (1975) and later by Mitov and Yanev (1985). Jagers (1968) and Pakes and Kaplan (1974) provided results for Bellman-Harris processes with immigration of the second type (at renewal epochs). Results for both immigration types appeared in Weiner (1991), but a combination of them was first investigated by Slavtchova-Bojkova and Yanev (1994) and by the author (2002). The last reference proves Theorem 2.1, below, under stronger conditions and by analytic means.

Following previous paper of the author (s. Slavtchova-Bojkova (1991)) and the above informal description, a p-dimensional Bellman-Harris process with immigration at 0 (BHPIO) $\mathbf{Z}(t) = \left(Z^{(1)}(t), Z^{(2)}(t), \dots Z^{(p)}(t)\right)$, whose l-th component $Z^{(l)}(t)$ means, that there exist $Z^{(l)}(t)$ particles of type l at the moment t, $l = 1, 2, \dots, p$ is a multi-type age-dependent branching process whose model parameters are the vector of an individual lifetime distributions $\mathbf{G}(t) = \left(G^{(1)}(t), G^{(2)}(t), \dots, G^{(p)}(t)\right)$, with $\mathbf{G}(0) = \mathbf{0}$, an offspring distributions $\{\mathbf{P}^{(l)}_{\boldsymbol{\alpha}}\}$, $\boldsymbol{\alpha} \in \mathbf{N}^p$, $l = 1, 2, \dots, p$ with multinomial p.g.f.s $f^{(l)}(\mathbf{s})$, corresponding to the offspring distribution of type l particles, a vector of immigrants distribution $\{g_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in \mathbf{N}^p}$ with p.g.f. $g(\mathbf{s})$, and finally a distribution D of the delay times elapsing after extinction epochs before new immigrants enter the population.

In order to extend the previous model by an additional immigration pattern at renewal epochs, let $\mathbf{Z}_{ij} = (\mathbf{Z}_{ij}(t))_{t \geq 0}$ for $i \geq 0$, $j \geq 1$ be independent BHPIO and with the same model parameters as $(\mathbf{Z}(t))_{t \geq 0}$. Let $(\sigma_n)_{n \geq 0}$ be a zero-delayed renewal process with increment distribution F and $(\mathbf{Y}_n)_{n \geq 1}$ be a sequence of i.i.d. integer-valued random vectors with common distribution $(h_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbf{N}^p}$ with multinomial p.g.f. $h(\mathbf{s})$. The $\mathbf{Y}_n$ are supposed to be the vectors of individuals entering the population at times σ_n. A further integer-valued random vector $\mathbf{Y}_0$ gives the number of ancestors of the considered population. It is assumed that $(\sigma_n)_{n \geq 0}$, $(\mathbf{Y}_n)_{n \geq 0}$, $\mathbf{Y}_0$ and all $\mathbf{Z}_{ij}$ are mutually independent. Then, a p-type Bellman-Harris process with immigration at zero and immigration of renewal type (BHPIOR) $(\mathbf{X}(t))_{t \geq 0}$ is obtained as

$$\mathbf{X}(t) = \sum_{i=0}^{N(t)} \mathbf{Z}_i(t - \sigma_i), t \geq 0, \tag{1}$$

where $\mathbf{Z}_i(0) = 0$, for $t < 0$, $N(t) = \sup\{n \geq 0 : \sigma_n \leq t\}$ and

$$(2) \qquad \mathbf{Z}_i(t) = \sum_{j=1}^{\mathbf{Y}_i} \mathbf{Z}_{ij}(t), t \geq 0,$$

is a BHPIO with $\mathbf{Y}_i$ ancestors.

2. Main results. In order to formulate our results some further notation is needed. Let $\mathbf{s} = (s_1, s_2, \dots s_p)$, $\mathbf{0} = (0, 0, \dots, 0)$, $\mathbf{1} = (1, 1, \dots 1)$, $\mathbf{s}^{\boldsymbol{\alpha}} = s_1^{\alpha_1} s_2^{\alpha_2} \dots s_n^{\alpha_p}$, $\boldsymbol{\delta}^{\mathbf{i}} = (\delta_1^i, \dots, \delta_j^i, \dots, \delta_p^i) = (0, \dots, 1, \dots 0)$, $\delta_j^i = \begin{cases} 0 & , \quad i \neq j \\ 1 & , \quad i = j \end{cases}$, $\sum_{\boldsymbol{\alpha}} = \sum_{\alpha_1=0}^{\infty} \sum_{\alpha_2=0}^{\infty} \cdots \sum_{\alpha_p=0}^{\infty}$.

Let $(\mathbf{Z}(t))_{t \geq 0}$ be a BHPIO as described in the Introduction. Define

$$\mathbf{m}_G = \left(\int_0^\infty t G^{(1)}(dt), \dots, \int_0^\infty t G^{(p)}(dt) \right),$$

and similarly m_F and m_D. Put $\mathbf{P}_{\boldsymbol{\alpha}}^{(l)}(t) = \mathbf{P}(\mathbf{Z}(t) = \boldsymbol{\alpha} | \mathbf{Z}(0) = \boldsymbol{\delta}^{(l)})$ and $\mathbf{P}^\star = \sum_{\boldsymbol{\alpha}, l} g_{\boldsymbol{\alpha}} \mathbf{P}_{\boldsymbol{\alpha}}^{(l)}$, so that the initial distribution of $(\mathbf{Z}(t))_{t \geq 0}$ under $\mathbf{P}^\star$ is $(g_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbf{N}}$. We simply write $\mathbf{P}$ in assertions where the distribution of $\mathbf{Z}(0)$ does not matter. Let T_1 be the first extinction epoch of $(\mathbf{Z}(t))_{t \geq 0}$ after 0, defined as

$$T_1 = \inf\{t > 0 : \mathbf{Z}(t-) > 0 \quad \text{and} \quad \mathbf{Z}(t) = 0\}.$$

Note that, under each $\mathbf{P}_{\boldsymbol{\alpha}}^{(l)}$, $\left(\widetilde{\mathbf{Z}}(t)\right)_{t \geq 0} = \left(\mathbf{Z}(t) \mathbf{I}_{\{T_1 > t\}}\right)_{t \geq 0}$ is a $p-$dimensional BHP with lifetime distribution $\mathbf{G}$, offspring distributions $\mathbf{P}_{\boldsymbol{\alpha}}^{(l)}$ and extinction time T_1, which has finite mean under every $\mathbf{P}_{\boldsymbol{\alpha}}^{(l)}$. Let $\boldsymbol{\Phi}(\mathbf{s}, t) = \left(\Phi^{(1)}(\mathbf{s}, t), \dots, \Phi^{(p)}(\mathbf{s}, t)\right)$ be the p.g.f. of $\widetilde{\mathbf{Z}}(t)$, where $\Phi^{(l)}(\mathbf{s}, t)$ is the p.g.f under $\mathbf{P}_{\boldsymbol{\alpha}}^{(l)}$.

Lemma 2.1. *Let $(\mathbf{Z}(t))_{t \geq 0}$ be a subcritical BHPIO with arbitrary ancestor distribution, $g'(\mathbf{1}) < \infty$, and $\mathbf{m}_G < \infty$. Suppose also $m_D < \infty$, and that the convolution $G * D$ is non-arithmetic. Then, $\mathbf{Z}(t) \xrightarrow{d} \mathbf{Z}(\infty)$, $t \to \infty$, for an integer-valued random vector $\mathbf{Z}(\infty)$ satisfying*

$$(3) \qquad \mathbf{P}(\mathbf{Z}(\infty) = \boldsymbol{\alpha}) = \begin{cases} \dfrac{m_D}{\beta}, & \text{if} \quad \boldsymbol{\alpha} = \mathbf{0}, \\ \dfrac{1}{\beta} \displaystyle\int_0^\infty \boldsymbol{P}^\star(\widetilde{\mathbf{Z}}(t) = \boldsymbol{\alpha}) dt, & \text{if} \quad \boldsymbol{\alpha} \geq \mathbf{1}, \end{cases}$$

where $\beta = \mathbf{E}^\star T_1 + m_D$ is finite. $\mathbf{Z}(\infty)$ has p.g.f.

$$(4) \qquad \boldsymbol{\Phi}(\mathbf{s}, \infty) = \frac{m_D}{\beta} + \frac{1}{\beta} \int_0^\infty (g(\boldsymbol{\Phi}(\mathbf{s}, t)) - g(\boldsymbol{\Phi}(\boldsymbol{0}, t))) dt.$$

Proof. Let $T_1, T_2, \dots$ and $X_1, X_2, \dots$ denote the successive extinction epochs and delay times of $(\mathbf{Z}(t))_{t \geq 0}$. Obviously, $(\mathbf{Z}(t))_{t \geq 0}$ is a classical regenerative process (s. Asmussen (1987), Chpt. 5) with regeneration times $T_n + X_n$, $n \geq 1$. So, cycles start (and end) at successive immigration epochs (the first one at 0). They are independent and for $n \geq 2$ also identically distributed with mean β. Because, by assumption, $G * D$ and thus the distribution of $T_1 + X_1$ are non-arithmetic, the ergodic theorem for regenerative processes (s. Asmussen (1987), Theorem V.1.2) gives $\mathbf{Z}(t) \xrightarrow{d} \mathbf{Z}(\infty)$ with

$$\text{(5)}\quad \mathbf{P}(\mathbf{Z}(\infty) = (\alpha_1, \alpha_2, \ldots, \alpha_p)) = \frac{1}{\beta}\mathbf{E}^\star\left(\int_0^{T_1+X_1} \mathbf{I}_{\{Z^{(1)}(t)=\alpha_1, Z^{(2)}(t)=\alpha_2, \ldots, Z^{(p)}(t)=\alpha_p\}} dt\right).$$

It follows directly from (5) that

$$\mathbf{P}(\mathbf{Z}(\infty) = \mathbf{0}) = \frac{m_D}{\beta}.$$

For $\boldsymbol{\alpha} \geq \mathbf{1}$, we obtain

$$\begin{aligned}
\mathbf{P}(\mathbf{Z}(\infty) = (\alpha_1, \alpha_2, \ldots, \alpha_p)) &= \frac{1}{\beta}\mathbf{E}^\star\left(\int_0^{T_1} \mathbf{I}_{\{Z^{(1)}(t)=\alpha_1, Z^{(2)}(t)=\alpha_2, \ldots, Z^{(p)}(t)=\alpha_p\}} dt\right) \\
&= \frac{1}{\beta}\int_0^\infty \mathbf{P}^\star(\widetilde{\mathbf{Z}}(t) = \boldsymbol{\alpha}) dt
\end{aligned}$$

completing the proof of (3).

Now,

$$\begin{aligned}
\boldsymbol{\Phi}(\mathbf{s}, \infty) &= \frac{m_D}{\beta} + \frac{1}{\beta} \sum_{\boldsymbol{\alpha} \in \mathbf{N}^p, \boldsymbol{\alpha} \geq \mathbf{1}} \int_0^\infty \mathbf{s}^{\boldsymbol{\alpha}} \mathbf{P}^\star(\widetilde{\mathbf{Z}}(t) = \boldsymbol{\alpha}) dt \\
&= \frac{m_D}{\beta} + \frac{1}{\beta} \int_0^\infty \sum_{\boldsymbol{\alpha} \geq \mathbf{0}, l} g_{\boldsymbol{\alpha}} \sum_{\boldsymbol{\alpha} \in \mathbf{N}^p, \boldsymbol{\alpha} \geq \mathbf{1}} \mathbf{s}^{\boldsymbol{\alpha}} \mathbf{P}^{(l)}_{\boldsymbol{\alpha}}(\widetilde{\mathbf{Z}}(t) = \boldsymbol{\alpha}) dt \\
&= \frac{m_D}{\beta} + \frac{1}{\beta} \int_0^\infty \sum_{\boldsymbol{\alpha} \geq \mathbf{0}, l} g_{\boldsymbol{\alpha}} \left(\boldsymbol{\Phi}(\mathbf{s}, t)^{\boldsymbol{\alpha}} - \boldsymbol{\Phi}(\mathbf{0}, t)^{\boldsymbol{\alpha}}\right) dt \\
&= \frac{m_D}{\beta} + \frac{1}{\beta} \int_0^\infty \left(g(\boldsymbol{\Phi}(\mathbf{s}, t)) - g(\boldsymbol{\Phi}(\mathbf{0}, t))\right) dt,
\end{aligned}$$

completing the proof of (4). □

Theorem 2.1. *Let* $(\mathbf{X}(t))_{t \geq 0}$ *be a subcritical BHPIOR with arbitrary ancestor distribution,* $g'(\mathbf{1}) < \infty$, $h'(\mathbf{1}) < \infty$, *and* $\mathbf{m}_G < \infty$. *Suppose also* $m_D < \infty$, *and* $G * D$ *is non arithmetic. Then,*

$$\text{(6)}\qquad \frac{\mathbf{X}(t)}{t} \xrightarrow{P} \mathbf{X}(\infty), \qquad t \to \infty,$$

Proof. It suffices to prove (6) with $\mathbf{P} = \mathbf{P}^\star$, because only $\mathbf{Z}_0(t)$ in (1) depends on the initial distribution and clearly satisfies $t^{-1}\mathbf{Z}_0(t) \xrightarrow{P} \mathbf{0}$ regardless of that distribution (choice of $\mathbf{P}$). Thus, fixing $\mathbf{P} = \mathbf{P}^\star$, Lemma 2.1 and (2) give $\mathbf{Z}_i(t) \xrightarrow{d} \mathbf{Z}^\star(\infty)$ for each $i \geq 0$. Because the $(\mathbf{Z}_i(t))_{t \geq 0}$ are cadlag and independent of $(\sigma_n)_{n \geq 0}$, the Skorohod-Dudley coupling theorem (s. Kallenberg, Theorem 3.30) ensures the existence of processes $\left(\widehat{\mathbf{Z}}_i(t)\right)_{t \geq 0}$ and random variables $\widehat{\mathbf{Z}}_i(\infty)$, $i \geq 0$ such that

(1) $\widehat{\mathbf{Z}}_i(t) \stackrel{d}{=} \mathbf{Z}_i(t)$ for all $t \in [0, \infty)$ and $i \geq 0$;
(2) $\widehat{\mathbf{Z}}_0(\infty) \stackrel{d}{=} \widehat{\mathbf{Z}}_1(\infty) \stackrel{d}{=} \cdots \stackrel{d}{=} \mathbf{Z}^*(\infty)$;
(3) $\widehat{\mathbf{Z}}_i(t) \to \widehat{\mathbf{Z}}_i(\infty)$ a.s.;

(4) the $\left(\widehat{\mathbf{Z}}_i(t)\right)_{t\in[0,\infty]}$, are mutually independent and also independent of $(\sigma_n)_{n\geq 0}$.

As an immediate consequence, we get

$$\widehat{\mathbf{X}}(t) \stackrel{def}{=} \sum_{i=0}^{N(t)} \widehat{\mathbf{Z}}_i(t-\sigma_i) \stackrel{d}{=} \mathbf{X}(t), \qquad t \geq 0,$$

The rest of the proof follows by the same arguments as in Alsmeyer and Slavtchova-Bojkova (2005) and for technical reasons we omit it.

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ГРАНИЧНА ТЕОРЕМА ЗА ДОКРИТИЧЕН МНОГОМЕРЕН РАЗКЛОНЯВАЩ СЕ ПРОЦЕС, ЗАВИСЕЩ ОТ ВЪЗРАСТТА С ДВА ТИПА ИМИГРАЦИЯ

Марусия Н. Славчова-Божкова

В настоящата работа се обобщава една гранична теорема за докритичен многомерен разклоняващ се процес, зависещ от възрастта на частиците с два типа имиграция. Целта е да се обобщи аналогичен резултат в едномерния случай като се прилагат "coupling" метода, теория на възстановяването и регенериращи процеси.