МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2011
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2011
Proceedings of the Fortieth Jubilee Spring Conference
of the Union of Bulgarian Mathematicians
Borovetz, April 5–9, 2011

MIXED NEGATIVE BINOMIAL DISTRIBUTION BY WEIGHTED GAMMA MIXING DISTRIBUTION

Pavel Stoynov

In this paper the mixed negative binomial distribution, known also as Pólya distribution is considered. We suppose that the mixing distribution is a weighted Gamma distribution. We derive the probability mass function and consider some special cases. The Panjer recursion formulas and some properties are given.

1. Introduction. One of the most popular counting distribution is the Negative Binomial distribution. The random variable ξ has a negative binomial distribution (Pólya distribution) with parameters r and $p \in (0,1)$ if the probability mass function (PMF) is

$$P(\xi = k) = \binom{r+k-1}{k} p^r (1-p)^k,\ k = 0,1,\dots \tag{1}$$

We use the notation $\xi \sim NB(r,p)$. The probability generating function (PGF) is given by

$$P_\xi(s) = Es^\xi = \left(\frac{p}{1-(1-p)s}\right)^r.$$

In many cases in practice, in financial and actuarial science for modeling a heterogeneous portfolio, we need counting distributions with some additional parameters. A common used method of obtaining an additional parameter in the distribution is by mixing ([4], [5]). In this paper we suppose that the parameter $p = e^{-\lambda}$ for $\lambda > 0$. Suppose that the parameter λ for the $NB(r, e^{-\lambda})$ distribution is a realization of the random variable Λ. The distribution of Λ is called mixing distribution and the $NB(r, e^{-\lambda})$ is interpret as the conditional distribution of N, given the outcome $\Lambda = \lambda$.

In [8] the $NB(r, e^{-\lambda})$ distribution is mixed by Lindley distribution. The resulting distribution is called Negative binomial Lindley (NB-Lindley). Here, the mixing distribution is a weighted version of the Gamma distribution. As a special case we obtain the NB-Lindley distribution [8].

2. The Mixing distribution. Let the random variable X be defined by the probability density function (PDF) $f(x)$ and $w(x)$ be a nonnegative function. Suppose that $Ew(X) < \infty$. The weighted distribution of X with weight function $w(x)$ is defined by the PDF

$$f^w(x) = \frac{w(x)f(x)}{Ew(X)}.$$

In the case of $w(x) = x$, the distribution $f^w(x)$ is called a length-biased distribution [7]. In this paper we suppose that the distribution of the mixing random variable Λ is a weighted version of the Gamma distribution. The PDF of the Gamma distributed random variable with parameters $r \geq 1$ and $\beta > 0$ is given by

$$f(\lambda) = \frac{\beta^r \lambda^{r-1} e^{-\beta\lambda}}{\Gamma(r)}, \quad \lambda > 0,$$

where $\Gamma(r)$ is the Gamma function. Suppose that for $n > 0$ and $-\infty < \gamma < \infty$, the weight function is $w(x) = (1 + \frac{x}{n})^{-\gamma}$. It is easy to find that $Ew(\Lambda) = (\beta n)^r \Psi(r, r+1-\gamma, n\beta)$, where $\Psi(a, c; z)$ is Tricomi's confluent hypergeometric function which admits the following integral representation

$$\Psi(a, c; z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1+t)^{c-a-1} dt, \quad a > 0, \ z > 0.$$

The mixing random variable Λ is defined by the weighted probability mass function, given by

$$f^w(\lambda) = \frac{n^{\gamma-r}(n+\lambda)^{-\gamma}}{\beta^r \Psi(r, r+1-\gamma, n\beta)} \times \frac{\beta^r \lambda^{r-1} e^{-\beta\lambda}}{\Gamma(r)}, \quad \lambda > 0, n > 0, -\infty < \gamma < \infty. \tag{2}$$

The distribution (2) is the mixing distribution to the Poisson random variable in [1]. It is a weighted Gamma distribution with mean value

$$E\Lambda = n \frac{\Psi(r+1, r+2-\gamma, n\beta)}{\Psi(r, r+1-\gamma, n\beta)}.$$

We use the notation $\Lambda \sim WGamma(r, \beta, \gamma, n)$.

3. Mixed Pólya distribution. The next proposition gives the unconditional distribution of the random variable ξ.

Proposition 1. *The PMF of the random variable ξ is given by*

$$P(\xi = k) = \binom{r+k-1}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j M_\Lambda(-(r+j)n), \quad k = 0, 1, \ldots, \tag{3}$$

where $M_\Lambda(s) = e^{s\Lambda}$ is the moment generating function of the random variable Λ.

Proof. The unconditional distribution of the random variable ξ follows from (1) with $p = e^{-\lambda}$ and λ defined by the PMF (2). The PMF of ξ and is given by

$$P(\xi = k) = \binom{r+k-1}{k} \int_0^\infty e^{-\lambda r} (1 - e^{-\lambda})^k f^w(\lambda) d\lambda$$

$$= \frac{\binom{r+k-1}{k}}{\Gamma(r) n^r \Psi(r, r-\gamma+1, n\beta)} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \int_0^\infty \lambda^{r-1} e^{-(r+\beta+j)\lambda} \left(1 + \frac{\lambda}{n}\right)^{-\gamma} d\lambda.$$

The change of the variable $\lambda = nv$ leads to

$$P(\xi = k) = \frac{\binom{r+k-1}{k}}{\Gamma(r)\Psi(r, r-\gamma+1, n\beta)} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \int_0^\infty v^{r-1} e^{-(r+\beta+j)nv} (1+v)^{-\gamma} dv$$

$$= \frac{\binom{r+k-1}{k}}{\Gamma(r)\Psi(r, r-\gamma+1, n\beta)} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \int_0^\infty e^{-(r+j)nv} v^{r-1} e^{-\beta nv} (1+v)^{-\gamma} dv$$

and (3). □

Remark 1. The PMF (3) can be written as

$$(4)\quad P(\xi = k) = \binom{r+k-1}{k} \frac{\sum_{j=0}^{k} \binom{k}{j} (-1)^j \Psi(r, r-\gamma+1, (r+\beta+j)n)}{\Psi(r, r-\gamma+1, n\beta)}, \quad k = 0, 1, \ldots,$$

4. Examples. In this section we consider particular cases useful in actuarial practice. We suppose that the parameters $r = n = 1$. In this case the distribution (3) is the mixed geometric distribution and the mixing distribution (2) is weighted exponential. The weight function is $w(x) = (1+x)^{-\gamma}$.

4.1. The WGamma(1, β, −2, 1) mixing distribution. Let $\gamma = -2$, $\beta > 0$ and $r = n = 1$. Then, the mixing random variable Λ has the probability mass function

$$(5)\qquad f^w(\lambda) = \frac{\beta^3 e^{-\beta\lambda}(1+\lambda)^2}{\beta^2 + 2\beta + 2}, \quad \lambda > 0.$$

Proposition 2. *The moment generating function of the random variable Λ is given by*

$$(6)\qquad M_\Lambda(s) = \frac{C(\beta)}{C(\beta - s)}, \quad s < \beta,$$

where $C(x) = \dfrac{x^3}{x^2 + 2x + 2}$.

Proof. For the moment generating function of Λ we have

$$M_\Lambda(s) = Ee^{s\Lambda} = \int_0^\infty e^{s\lambda} \frac{\beta^3 e^{-\beta\lambda}(1+\lambda)^2}{\beta^2 + 2\beta + 2} d\lambda$$

$$= \frac{\beta^3}{\beta^2 + 2\beta + 2} \int_0^\infty e^{-(\beta - s)\lambda} (1+\lambda)^2) d\lambda$$

$$= \frac{\beta^3}{\beta^2 + 2\beta + 2} \left[\int_0^\infty e^{-(\beta-s)\lambda} d\lambda + 2 \int_0^\infty \lambda e^{-(\beta-s)\lambda} d\lambda + \int_0^\infty \lambda^2 e^{-(\beta-s)\lambda} d\lambda \right]$$

$$= \frac{\beta^3}{\beta^2 + 2\beta + 2} \left[\frac{1}{\beta - s} + 2\frac{\Gamma(2)}{(\beta-s)^2} + \frac{\Gamma(3)}{(\beta-s)^3} \right]$$

$$\frac{\beta^3}{\beta^2 + 2\beta + 2} \frac{(\beta-s)^2 + 2(\beta - s) + 2}{(\beta-s)^3},$$

which is just (6). □

Remark 2. It is easy to see that the PMF of (5) is a discrete mixture of exponential distribution, $Gamma(2,\beta)$ and $Gamma(3,\beta)$ distributions, i.e.

$$f^w(\lambda) = p_1\beta e^{-\beta\lambda} + p_2\beta^2\lambda e^{-\beta\lambda} + p_3\frac{\beta^3\lambda^2 e^{-\beta\lambda}}{2!},$$

where

$$p_1 = \frac{\beta^2}{\beta^2+2\beta+2}, \quad p_2 = \frac{2\beta}{\beta^2+2\beta+2}, \quad p_3 = \frac{2}{\beta^2+2\beta+2}.$$

In the next theorem a version of the Panjer recursion [6] is given.

Theorem 1. *The PMF of* (3) *with mixing distribution* (5) *satisfies the recurrence relations:*

$$p_k = p_{k-1} - \int_0^\infty e^{-\lambda} P(\xi = k-1|\Lambda = \lambda) f^w(\lambda) d\lambda, \quad k = 1, 2, \ldots, \tag{7}$$

and

$$p_0 = \frac{\beta^3(\beta^2+4\beta+5)}{(\beta+1)^3(\beta^2+2\beta+2)}.$$

Proof. Follows from well known Panjer recursion formula for the negative binomial distribution

$$p_k = (1-p)\left(1 + \frac{r-1}{k}\right) p_{k-1}, \; k = 1, 2, \ldots.$$

For $p = e^{-\lambda}$, $r = 1$ and the mixed version of λ we obtain (7). □

4.2. Pólya–Lindley distribution. Let $\gamma = -1$, $\beta > 0$ and $r = n = 1$. The random variable Λ has the $WGamma(1,\beta,-1,1)$ distribution, known as Lindley distribution with parameter $\beta > 0$ and density function

$$f^w(\lambda) = \frac{\beta^2}{1+\beta} e^{-\beta\lambda}(1+\lambda), \; \lambda > 0.$$

The Lindley distribution is introduced by Lindley [3] and is a mixture of exponential and gamma distributions, i.e.

$$f^w(\lambda) = \frac{\beta}{1+\beta}\beta e^{-\beta\lambda} + \frac{1}{1+\beta}\beta^2\lambda e^{-\beta\lambda}, \; \lambda > 0.$$

Definition 1. *Mixed Pólya distribution by Lindley mixing distribution is called Pólya–Lindley distribution.*

The Pólya–Lindley distribution coincides with the NB-Lindley distribution, defined in [8].

Theorem 2. *For the Pólya–Lindley distribution, PMF satisfies the recurrence relations:*

$$p_k = p_{k-1} - \int_0^\infty e^{-\lambda} P(\xi = k-1|\Lambda = \lambda) f^w(\lambda) d\lambda \quad k = 1, 2, \ldots,$$

and

$$p_0 = \frac{\beta^2(\beta+2)}{(\beta+1)^3}.$$

Proof. The proof is similar to that of Theorem 1. □

REFERENCES

[1] M. E. Ghitany, S. A. Al-Awadhi. A Unified Approach to Some Mixed Poisson Distributions. *Tamsui Oxford Journal of Mathematical Sciences*, **17** (2001), No 2, 147–161.

[2] N. L. Johnson, S. Kotz, A. W. Kemp. Univariate Discrete Distributions. Wiley Series in Probability and Mathematical Statistics, third edition, 2005.

[3] D. V. Lindley. Fiducial distributions and Bayes' theorem. *Journal of the Royal Statistical Society*, **20** (1958), 102–107.

[4] L. D. Minkova. Mixed Pólya-Aeppli process. *C. R. Acad. Bulgare Sci.***54** (2001), No 8, 9–12.

[5] L. D. Minkova. A Generalization of the Classical Discrete Distributions. *Commun. Statist. – Theory and Methods*, **31** (2002), No 6, 871–888.

[6] H. H. Panjer. Recursive evaluation of a family of compound distributions. *ASTIN Bulletin*, **12** (1981), 22–26.

[7] C. P. Patil, C. R. Rao. Weighted distributions and size biased sampling with applications to wild-life populations and human families. *Biometrics*, **34** (1978), 179–189.

[8] H. Zamani, N. Ismail. Negative binomial–Lindley distribution and its applications. *Journal of Mathematics and Statistics*, **6** (2010), No1, 4–9.

Pavel T. Stoynov
Faculty of Economics and Business Administration
Sofia University
e-mail: todorov@feb.uni-sofia.bg

СМЕСЕНО ОТРИЦАТЕЛНО БИНОМНО РАЗПРЕДЕЛЕНИЕ С ПРЕТЕГЛЕНО ГАМА СМЕСВАЩО РАЗПРЕДЕЛЕНИЕ

Павел Т. Стойнов

В тази работа се разглежда отрицателно биномното разпределение, известно още като разпределение на Пойа. Предполагаме, че смесващото разпределение е претеглено гама разпределение. Изведени са вероятностите в някои частни случаи. Дадени са рекурентните формули на Панжер.