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PSEUDO-COMPACT SEMI-TOPOLOGICAL GROUPS*

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A semitopological group (topological group) is a group endowed with a topology for which multiplication is separately continuous (multiplication is jointly continuous and inversion is continuous). In this paper we give some topological conditions on a semitopological group that imply that it is a topological group. For example, we show that every separable pseudocompact group is a topological group. We also show that every locally pseudocompact group whose multiplication is jointly continuous is a topological group.

1. Introduction. In this paper we shall continue the study of which topological properties of a semitopological group ensure that it is actually a topological group. There have been many contributions to this area of research. Some of these are listed in [3, 4, 6, 9]. Our approach is based upon topological games. In fact, the present paper is essentially a sequel to the paper [6]. The main distinction between the present paper and [6] is that in this paper we have tried to incorporate the notion of pseudocompactness. There are of course limits to what can be achieved since there are examples of completely regular pseudocompact semitopological groups that are not topological groups, [7]. On the other hand, there are some positive results in the literature that show that some "nice" pseudocompact semitopological groups are topological groups, see [2, 9]. In order to describe our contribution to this area, we need some definitions.

We will say that a subset A of a topological space X is *bounded in* X if for any sequence $(W_n : n \in \mathbb{N})$ of open sets in X such that $W_{n+1} \subseteq W_n$ and $A \cap W_n \neq \varnothing$ for all $n \in \mathbb{N}$, $\bigcap_{n\in\mathbb{N}} \overline{W_n} \neq \varnothing$. When the space X is bounded in itself and completely regular we say that it is *pseudocompact*. It is well-known that such spaces are characterised by the fact that every real-valued continuous function defined on them is bounded (in fact this is the usual definition of them). In this paper we need a stronger notion. A subset A of a topological space X is said to be *strongly bounded in* X if for every infinite subset C of A there exists a separable subspace S of X such that the set $C \cap S$ is infinite and bounded in S.

Every countably compact space, as well as every separable pseudocompact space is strongly bounded in itself and it is easy to show that every strongly bounded set in X is bounded in X. The final ingredient for this paper is the following game.

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Let (X, τ) be a topological space and let D be a dense subset of X. On X we consider the $\mathcal{G}_S^*(D)$*-game* played between two players α and β. Player β goes first (always!) and chooses a nonempty open subset $B_1 \subseteq X$. Player α must then respond by choosing a nonempty open subset $A_1 \subseteq B_1$. Following this, player β must select another nonempty open subset $B_2 \subseteq A_1 \subseteq B_1$ and in turn player α must again respond by selecting a nonempty open subset $A_2 \subseteq B_2 \subseteq A_1 \subseteq B_1$. Continuing this procedure indefinitely, the players α and β produce a sequence $((A_n, B_n) : n \in \mathbb{N})$ of pairs of open sets called a *play* of the $\mathcal{G}_S^*(D)$-game. We shall declare that α *wins* a play $((A_n, B_n) : n \in \mathbb{N})$ of the $\mathcal{G}_S^*(D)$-game if:

(i) $\bigcap_{n \in \mathbb{N}} A_n \neq \varnothing$ and

(ii) for each sequence $(a_n : n \in \mathbb{N})$ with $a_n \in A_n \cap D$ for all $n \in \mathbb{N}$, $\{a_n : n \in \mathbb{N}\}$ is strongly bounded in X.

Otherwise the player β is said to have won this play. Note that if α wins a play $((A_n, B_n) : n \in \mathbb{N})$ of the $\mathcal{G}_S^*(D)$-game then for each sequence $(a_n : n \in \mathbb{N})$ with $a_n \in A_n \cap D$ for all $n \in \mathbb{N}$, $\{a_n : n \in \mathbb{N}\}$ is bounded in X. Furthermore, if in addition $\overline{A_{n+1}} \subseteq A_n$ for all $n \in \mathbb{N}$ then for any open neighbourhood W of $\bigcap_{n \in \mathbb{N}} A_n$ there exists an $n_0 \in \mathbb{N}$ such that $A_k \subseteq \overline{W}$ for all $k \geq n_0$. By a *strategy* t for the player β we mean a *'rule'* that specifies each move of the player β in every possible situation. More precisely, a strategy $t := (t_n : n \in \mathbb{N})$ for β is a sequence of τ-valued functions such that

$$\varnothing \neq t_1(\emptyset) \quad \text{and} \quad \varnothing \neq t_{n+1}(A_1, \ldots, A_n) \subseteq A_n \text{ for each } n \in \mathbb{N}.$$

The domain of t_1 is $\{\emptyset\}$, (where $\emptyset$ denotes the sequence of length 0) and the domain of t_2 is $\{(A) : A \in \tau \text{ and } \varnothing \neq A \subseteq t_1(\emptyset)\}$. For $n \geq 3$ the domain of each function t_n is precisely the set of all finite sequences $(A_1, A_2, \ldots, A_{n-1})$ of length $n-1$ in $\tau \setminus \{\varnothing\}$ such that

$$A_1 \subseteq t_1(\emptyset) \quad \text{and} \quad A_j \subseteq t_j(A_1, \ldots, A_{j-1}) \text{ for all } 2 \leq j \leq n-1.$$

Such a finite sequence $(A_1, A_2, \ldots, A_{n-1})$ or infinite sequence $(A_n : n \in \mathbb{N})$ is called a t*-sequence*. A strategy $t := (t_n : n \in \mathbb{N})$ for the player β is called a *winning strategy* if each t-sequence is won by β. We will call a topological space (X, τ) a *strongly boundedly Baire* if it is regular and there exists a dense subset D of X such that the player β does **not** have a winning strategy in the $\mathcal{G}_S^*(D)$-game played on X. It follows from [10, Theorem 1] that each strongly boundedly Baire space is in fact a Baire space and it is easy to see that each strongly boundedly Baire space has at least one q_D^*-point. Indeed, if $t := (t_n : n \in \mathbb{N})$ is any strategy for β then there is a t-sequence $(A_n : n \in \mathbb{N})$ where α wins. In this case each point of $\bigcap_{n \in \mathbb{N}} A_n$ is a q_D^*-point. Recall that a point $x \in X$ is called a q_D^*-point (with respect to some dense subset D of X) if there exists a sequence of neighbourhoods $(U_n : n \in \mathbb{N})$ of x such that for every sequence $(x_n : n \in \mathbb{N})$ with $x_n \in U_n \cap D$ for all $n \in \mathbb{N}$, $\{x_n : n \in \mathbb{N}\}$ is bounded in X.

The remainder of this paper is divided into 2 parts. In the next section we will show that every strongly boundedly Baire semitopological group is a paratopological group and then in Section 3 we will show that each strongly boundedly Baire semitopological group is in fact a topological group.

2. Paratopological groups. We begin with some definitions. Let X, Y and Z be topological spaces then we will say that a function $f : X \times Y \to Z$ is *strongly quasi-*

continuous at $(x, y) \in X \times Y$ if for each neighbourhood W of $f(x, y)$ and each product of open sets $U \times V \subseteq X \times Y$ containing (x, y) there exists a nonempty open subset $U' \subseteq U$ and a neighbourhood V' of y such that $f(U' \times V') \subseteq W$. Further, a function $f : X \times Y \to Z$ is said to be *separately continuous* on $X \times Y$ if for each $x_0 \in X$ and $y_0 \in Y$ the functions $y \mapsto f(x_0, y)$ and $x \mapsto f(x, y_0)$ are both continuous on Y and X respectively.

Lemma 1. *Let X be a strongly boundedly Baire space, Y a topological space and Z a completely regular space. If $f : X \times Y \to Z$ is a separately continuous function and D is a dense subset of Y, then for each q_D^*-point $y_0 \in Y$ the mapping f is strongly quasi-continuous at each point of $X \times \{y_0\}$.*

Proof. Let D_X be any dense subset of X such that β does not have a winning strategy in the $\mathcal{G}_S^*(D_X)$-game played on X. (Note: such a dense subset is guaranteed by the fact that X is a strongly boundedly Baire space.) We need to show that f is strongly quasi-continuous at each point $(x_0, y_0) \in X \times \{y_0\}$. So in order to obtain a contradiction let us assume that f is not strongly quasi-continuous at some point $(x_0, y_0) \in X \times \{y_0\}$. Then there exist open neighbourhoods W of $f(x_0, y_0)$, U of x_0 and V of y_0 so that $f(U' \times V') \not\subseteq \overline{W}$ for each nonempty open subset U' of U and neighbourhood $V' \subseteq V$ of y_0. By the complete regularity of Z there exists a continuous function $g : Z \to [0, 1]$ such that $g(f(x_0, y_0)) = 1$ and $g(Z \setminus W) = \{0\}$. Let $W' := \{z \in Z : g(z) > 3/4\} \subseteq W$. Note that by possibly making U smaller we may assume that $f(x, y_0) \in W'$ for all $x \in U$. We will now inductively define a strategy $t := (t_n : n \in \mathbb{N})$ for the player β in the $\mathcal{G}_S^*(D_X)$-game played on X, but first we shall (a) denote by $(O_n : n \in \mathbb{N})$ any sequence of open neighbourhoods of y_0 with the property that for each sequence $(y_n : n \in \mathbb{N})$ in D with $y_n \in O_n$ for all $n \in \mathbb{N}$, $\{y_n : n \in \mathbb{N}\}$ is bounded in Y and (b) set $A_0 := U$ and $V_0 := V$.

Step 1. Select $(x_1, y_1) \in X \times Y$ and open sets V_1 and $t_1(\emptyset)$ so that:

(i) $y_0 \in V_1 := \{y \in V_0 \cap O_1 : f(x_0, y) \in W'\}$;

(ii) $(x_1, y_1) \in (A_0 \cap D_X) \times (V_1 \cap D)$ and $f(x_1, y_1) \notin \overline{W}$;

(iii) $t_1(\emptyset) := \{x \in A_0 : f(x, y_1) \notin \overline{W}\}$.

Now suppose that (x_j, y_j), V_j and t_j have been defined for each t-sequence $(A_1, A_2, \ldots, A_{j-1})$ of length $(j-1)$, $1 \le j \le n$ so that for each $1 \le j \le n$

(i) $y_0 \in V_j := \{y \in V_{j-1} \cap O_j : f(x_{j-1}, y) \in W'\}$;

(ii) $(x_j, y_j) \in (A_{j-1} \cap D_X) \times (V_j \cap D)$ and $f(x_j, y_j) \notin \overline{W}$;

(iii) $t_j(A_1, \ldots, A_{j-1}) := \{x \in A_{j-1} : f(x, y_j) \notin \overline{W}\}$.

Step $n+1$. For each t-sequence $(A_1, \ldots, A_n)$ of length n we select $(x_{n+1}, y_{n+1}) \in X \times Y$ and open sets V_{n+1} and $t_{n+1}(A_1, \ldots, A_n)$ so that:

(i) $y_0 \in V_{n+1} := \{y \in V_n \cap O_{n+1} : f(x_n, y) \in W'\}$;

(ii) $(x_{n+1}, y_{n+1}) \in (A_n \cap D_X) \times (V_{n+1} \cap D)$ and $f(x_{n+1}, y_{n+1}) \notin \overline{W}$;

(iii) $t_{n+1}(A_1, \ldots, A_n) := \{x \in A_n : f(x, y_{n+1}) \notin \overline{W}\}$.

This completes the definition of $t := (t_n : n \in \mathbb{N})$. Now since t is not a winning strategy for the player β in the $\mathcal{G}^*_S(D_X)$-game there exists a t-sequence $(A_n : n \in \mathbb{N})$ where α wins and since $x_{n+1} \in A_n \cap D_X$ for all $n \in \mathbb{N}$ there exists a separable subspace $S \subseteq X$ and a subsequence $(x_{n_k} : k \in \mathbb{N})$ of $(x_n : n \in \mathbb{N})$ such that $\{x_{n_k} : k \in \mathbb{N}\} \subseteq S$ and $\{x_{n_k} : k \in \mathbb{N}\}$ is bounded in S. Define $\varphi : Y \to C_p(S)$ - [the continuous real-valued functions defined on S endowed with the topology of pointwise convergence on S] by, $\varphi(y)(s) := (g \circ f)(s, y)$ for all $s \in S$. Then φ is well-defined and continuous on Y. Now, since $y_n \in O_n \cap D$ for all $n \in \mathbb{N}$, $\{y_n : n \in \mathbb{N}\}$ is bounded in Y and so $\{\varphi(y_m) : m \in \mathbb{N}\}$ is bounded in $C_p(S)$. Thus, by [5, Corollary 2.3], $\overline{\{\varphi(y_m) : m \in \mathbb{N}\}}^{\tau_p}$ is a compact subspace of $C_p(S)$. Hence the sequence $(\varphi(y_n) : n \in \mathbb{N})$ has a cluster point $h \in C(S)$. Now, for each fixed $k \in \mathbb{N}$,

$$f(x_{n_k}, y_i) \in f(\{x_{n_k}\} \times V_i) \subseteq f(\{x_{n_k}\} \times V_{n_k+1}) \subseteq W'$$

for all $i > n_k$, since $y_i \in V_i$ for all $i \in \mathbb{N}$. Therefore, $\varphi(y_i)(x_{n_k}) \in (3/4, 1]$ for all $i > n_k$ and so $h(x_{n_k}) \in [3/4, 1] \subseteq (2/3, 1]$ for all $k \in \mathbb{N}$. Since h is continuous, for every $k \in \mathbb{N}$ there exists a relatively open subset U_k of S such that $x_{n_k} \in U_k \subseteq A_{n_k - 1}$ and $h(U_k) \subseteq (2/3, 1]$. Hence the set $\bigcap_{k\in\mathbb{N}} \overline{\bigcup_{i\geq k} U_i}^S$ is nonempty. [Here, $\overline{X}^S$ denotes the closure of a subset X of S with respect to the relative topology on S]. Let $x_\infty \in \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{i\geq k} U_i}^S \subseteq S$. Then $h(x_\infty) \in [2/3, 1]$. On the other hand, if we again fix $k \in \mathbb{N}$ then

$$f(U_i \times \{y_k\}) \subseteq f(A_{n_i-1} \times \{y_k\}) \subseteq f(A_{i-1} \times \{y_k\}) \subseteq f(A_k \times \{y_k\}) \subseteq Z \setminus \overline{W}$$

for all $i > k$. Therefore, $f(\overline{\bigcup_{i>k} U_i}^S \times \{y_k\}) \subseteq Z \setminus W$ for each $k \in \mathbb{N}$ and so $f(x_\infty, y_k) \in Z\setminus W$ for each $k \in \mathbb{N}$; which implies that $h(x_\infty) = 0$. This however, contradicts our earlier conclusion that $h(x_\infty) \in [2/3, 1]$. Hence f is strongly quasi-continuous at (x_0, y_0). □

We shall call any semitopological group $(G, \cdot, \tau)$ whose multiplication is jointly continuous a *paratopological group*.

Lemma 2. *Let $(G, \cdot, \tau)$ be a completely regular semitopological group whose multiplication is strongly quasi-continuous at (e, e). If there exists a dense subset D of G and a sequence of neighbourhoods $(U_n : n \in \mathbb{N})$ of e so that every sequence $(z_n : n \in \mathbb{N})$ in D with $z_n \in U_n \cdot U_n$ for all $n \in \mathbb{N}$, $\{z_n : n \in \mathbb{N}\}$ is strongly bounded in G, then $(G, \cdot, \tau)$ is a paratopological group.*

Proof. Since $(G, \cdot, \tau)$ is a semitopological group it is sufficient to show that the mapping $\pi : G \times G \to G$ defined by, $\pi(g, h) := g \cdot h$ is jointly continuous at (e, e). So in order to obtain a contradiction we will assume that π is not jointly continuous at (e, e). Therefore by the regularity of (G, τ) there exists an open neighbourhood W of e so that for every neighbourhood U of e, $U \cdot U \not\subseteq \overline{W}$. By the complete regularity of (G, τ) there exists a continuous function $f : G \to [0, 1]$ such that $f(e) = 1$ and $f(G \setminus W) = \{0\}$. Let $V := \{g \in G : f(g) > 3/4\} \subseteq W$ and let $V^* := \{g \in G : (g, e) \in \text{int}\, \pi^{-1}(V)\}$. Then by the strong quasi-continuity of π at (e, e), $e \in \overline{V^*}$. We will now inductively define sequences $(z_n : n \in \mathbb{N})$ and $(v_n : n \in \mathbb{N})$ in D and decreasing neighbourhoods $(Z_n : n \in \mathbb{N})$ and $(V_n : n \in \mathbb{N})$ of e.

Step 1. Choose $v_1 \in V^* \cap D$ and a neighbourhood Z_1 of e so that $Z_1 \subseteq U_1$ and $(v_1 \cdot Z_1) \cdot Z_1 \subseteq V$. Then choose $z_1 \in (Z_1 \cdot Z_1 \setminus \overline{W}) \cap D$ and a neighbourhood V_1 of e so that $V_1 \subseteq U_1$ and $V_1 \cdot z_1 \subseteq G\setminus\overline{W}$.

For purely notational reasons we will define $V_0 := U_0 := G$. Now suppose that v_j, $z_j \in D$ and Z_j, V_j have been defined for each $1 \leq j \leq n$ so that:

(i) $v_j \in (V^* \cap V_{j-1}) \cap D$ and $(v_j \cdot Z_j) \cdot Z_j \subseteq V$;

(ii) $z_j \in (Z_j \cdot Z_j \backslash \overline{W}) \cap D$ and $V_j \cdot z_j \subseteq G \backslash \overline{W}$;

(iii) $Z_j \subseteq Z_{j-1} \cap U_j$ and $V_j \subseteq V_{j-1} \cap U_j$.

Step $n+1$. Choose $v_{n+1} \in (V^* \cap V_n) \cap D$ and a neighbourhood Z_{n+1} of e so that $Z_{n+1} \subseteq Z_n \cap U_{n+1}$ and $(v_{n+1} \cdot Z_{n+1}) \cdot Z_{n+1} \subseteq V$. Then choose $z_{n+1} \in (Z_{n+1} \cdot Z_{n+1} \backslash \overline{W}) \cap D$ and a neighbourhood V_{n+1} of e so that $V_{n+1} \subseteq V_n \cap U_{n+1}$ and $V_{n+1} \cdot z_{n+1} \subseteq G \setminus \overline{W}$. This completes the induction. Now, since $v_{n+1} = v_{n+1} \cdot e \in V_n \cdot V_n \subseteq U_n \cdot U_n$ for each $n \in \mathbb{N}$, $\{v_n : n \in \mathbb{N}\}$ is strongly bounded in G. Therefore there exists a separable subset S of G and a subsequence $(v_{n_k} : k \in \mathbb{N})$ of $(v_n : n \in \mathbb{N})$ such that $\{v_{n_k} : k \in \mathbb{N}\} \subseteq S$ and $\{v_{n_k} : k \in \mathbb{N}\}$ is bounded in S. Define, $\varphi : G \to C_p(S)$ by, $\varphi(y)(s) := f(s \cdot y)$ for all $s \in S$. Then φ is well-defined and continuous on G. Now, since $z_n \in U_n \cdot U_n \cap D$ for each $n \in \mathbb{N}$, $\{z_n : n \in \mathbb{N}\}$ is bounded in G (in fact strongly bounded in G) and so $\{\varphi(z_n) : n \in \mathbb{N}\}$ is bounded in $C_p(S)$. Thus, by [5, Corollary 2.3], $\overline{\{\varphi(z_n) : n \in \mathbb{N}\}}^{\tau_p}$ is compact. Hence the sequence $(\varphi(z_n) : n \in \mathbb{N})$ has a cluster point $h \in C(S)$. Fix $n \in \mathbb{N}$, then

$$v_n \cdot z_i \in v_n \cdot Z_i \cdot Z_i \subseteq v_n \cdot Z_n \cdot Z_n \subseteq V$$

for all $i \geq n$. Therefore, for each $k \in \mathbb{N}$, $\varphi(z_i)(v_{n_k}) \in (3/4, 1]$ for all $i \geq n_k$ and so $h(v_{n_k}) \in [3/4, 1]$ for all $k \in \mathbb{N}$. Since h is continuous for every $k \in \mathbb{N}$, there exists a relatively open subset N_k of S such that $v_{n_k} \in N_k \subseteq V_{n_k - 1}$ and $h(N_k) \subseteq (2/3, 1]$. Thus the set $\bigcap_{k \in \mathbb{N}} \overline{\bigcup_{i \geq k} N_i}^S$ is nonempty. [Here $\overline{X}^S$ denotes the closure of a subset X of S with respect to the relative topology on S]. Let $v_\infty \in \bigcap_{k \in \mathbb{N}} \overline{\bigcup_{i \geq k} N_i}^S \subseteq S$. Then $h(v_\infty) \in [2/3, 1]$. On the other hand, if we again fix $k \in \mathbb{N}$, then

$$N_i \cdot z_k \subseteq V_{n_i - 1} \cdot z_k \subseteq V_{i-1} \cdot z_k \subseteq V_k \cdot z_k \subseteq G \setminus \overline{W}$$

for all $i > k$. Therefore, $(\overline{\bigcup_{i > k} N_i}) \cdot z_k \subseteq G \backslash W$ for each $k \in \mathbb{N}$ and so $v_\infty \cdot z_k \in G \backslash W$ (i.e., $\varphi(z_k)(v_\infty) = 0$ for each $k \in \mathbb{N}$); which implies that $h(v_\infty) = 0$. This however, contradicts our earlier conclusion that $h(x_\infty) \in [2/3, 1]$. Hence $(G, \cdot, \tau)$ is a paratopological group. □

The proof of Theorem 1 (below) follows the proof of [6, Theorem 1] and so is not reproduced here.

Theorem 1. *Let $(G, \cdot, \tau)$ be a completely regular semitopological group. If (G, τ) is a strongly boundedly Baire space then $(G, \cdot, \tau)$ is a paratopological group.*

3. Continuity of inversion. Let X and Y be topological spaces. Then a function $f : X \to Y$ is said to be *quasi-continuous at* $x \in X$ if for each neighbourhood W of $f(x)$ and neighbourhood U of x there exists a nonempty open set $V \subseteq U$ such that $f(V) \subseteq W$. The following lemma is based upon [1, 2, 8].

Lemma 3. *Let $(G, \cdot, \tau)$ be a paratopological group. If (G, τ) is a strongly boundedly Baire space then, inversion is quasi-continuous at e.*

Proof. In order to obtain a contradiction let us assume that inversion is not quasi-continuous at $e \in G$. Then there exist neighbourhoods U and W of e such that for each nonempty open subset V of U, $V^{-1} \not\subseteq W$. Note that by possibly making U smaller (and using the fact that $(G, \cdot, \tau)$ is a paratopological group) we may assume that $\overline{U \cdot U} \subseteq W$.

Next, we let D be any dense subset of G such that β does not have a winning strategy in the $\mathcal{G}_S^*(D)$-game played on G. We inductively define a strategy $t := (t_n : n \in \mathbb{N})$ for β in the $\mathcal{G}_S^*(D)$-game played on G, but first we set $A_0 := U$ and $x_0 := e$.

Step 1. Choose $x_1 \in A_0$ so that $(x_0^{-1} \cdot x_1)^{-1} = x_1^{-1} \notin W$. Then choose U_1 to be any open neighbourhood of e, contained in U, such that $x_1 \cdot \overline{U_1} \subseteq A_0$. Then define $t_1(\emptyset) := x_1 \cdot U_1$.

Now, suppose that x_j, U_j and t_j have been defined for each t-sequence $(A_1, \ldots, A_{j-1})$ of length $(j-1), 1 \leq j \leq n$ so that:

(i) $x_j \in A_{j-1}$ and $(x_{j-1}^{-1} \cdot x_j)^{-1} \notin W$;

(ii) U_j is an open neighbourhood of e, contained in U, and $x_j \cdot \overline{U_j} \subseteq A_{j-1}$;

(iii) $t_j(A_1, \ldots, A_{j-1}) := x_j \cdot U_j$.

Step $n+1$. For each t-sequence $(A_1, \ldots, A_n)$ of length n choose $x_{n+1} \in A_n$ so that $(x_n^{-1} \cdot x_{n+1})^{-1} \notin W$. Note that this is possible since $x_n^{-1} \cdot A_n$ is a nonempty open set and

$$x_n^{-1} \cdot A_n \subseteq x_n^{-1} \cdot (x_n \cdot U_n) = U_n \subseteq U.$$

Then choose U_{n+1} to be any open neighbourhood of e, contained in U, such that $x_{n+1} \cdot \overline{U_{n+1}} \subseteq A_n$. Finally, define $t_{n+1}(A_1, \ldots, A_n) := x_{n+1} \cdot U_{n+1}$.

This completes the definition of $t := (t_n : n \in \mathbb{N})$. Now, since t is not a winning strategy for β, there exists a t-sequence $(A_n : n \in \mathbb{N})$ where α wins. Hence there exists a $2 \leq k \in \mathbb{N}$ such that $A_{k-1} \subseteq \overline{\left(\bigcap_{n\in\mathbb{N}} A_n\right) \cdot U}$ since $\overline{A_{n+1}} \subseteq A_n$ for all $n \in \mathbb{N}$ and $\left(\bigcap_{n\in\mathbb{N}} A_n\right) \cdot U$ is an open neighbourhood of $\bigcap_{n\in\mathbb{N}} A_n$. Thus,

$$x_k \in A_{k-1} \subseteq \overline{(\bigcap_{n\in\mathbb{N}} A_n) \cdot U} \subseteq \overline{A_{k+1} \cdot U} \subseteq \overline{x_{k+1} \cdot U_{k+1} \cdot U} \subseteq \overline{x_{k+1} \cdot U \cdot U}$$

$$= x_{k+1} \cdot \overline{U \cdot U} \subseteq x_{k+1} \cdot W.$$

Therefore, $(x_k^{-1} \cdot x_{k+1})^{-1} = x_{k+1}^{-1} \cdot x_k \in W$. However, this contradicts the way x_{k+1} was chosen. This shows that inversion is quasi-continuous at e. □

Lemma 4 ([6, Lemma 4]). *Let $(G, \cdot, \tau)$ be a paratopological group. If the inversion is quasi-continuous at e, then $(G, \cdot, \tau)$ is a topological group.*

The following theorem is now just a consequence of Theorem 1, Lemma 3 and Lemma 4.

Theorem 2. *Let $(G, \cdot, \tau)$ be a completely regular semitopological group. If (G, τ) is a strongly boundedly Baire then $(G, \cdot, \tau)$ is a topological group.*

Remark. In the proof of Lemma 3 the only consequence of the assumption that (G, τ) was strongly boundedly Baire was that for the given strategy t there existed a t-sequence $(A_n : n \in \mathbb{N})$ with the properties that (i) $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$ and (ii) for each open neighbourhood W of $\bigcap_{n\in\mathbb{N}} A_n$ there existed a $k \in \mathbb{N}$ such that $A_k \subseteq \overline{W}$. Therefore, if $(G, \cdot, \tau)$ is locally pseudocompact and regular, then α has an obvious way of making this happen. Namely, for their first move they choose $\varnothing \neq A_1 \subseteq t_1(\emptyset)$ so that $\overline{A_1}$ lies in a pseudocompact subset of G. Thus, every regular locally pseudocompact paratopological group is a topological group, [3, Question 2.4.5].

Corollary 1 [6, Theorem 2]**.** *Let $(G, \cdot, \tau)$ be a completely regular semitopological group. If (G, τ) is a strongly Baire, then $(G, \cdot, \tau)$ is a topological group.*

Corollary 2 [9, Corollary 2.7 part (c)]**.** *Every completely regular separable pseudocompact semitopological group is a topological group.*

REFERENCES

[1] A. V. Arhangel'skii, M. M. Choban, P. S. Kenderov. Topological games and continuity of group operations. *Topology Appl.*, **157** (2010), 2542–2552.

[2] A. V. Arhangel'skii, E. A. Reznichenko. Paratopological and semitopological groups versus topological groups. *Topology Appl.*, **151** (2005), 107–119.

[3] A. V. Arhangel'skii, M. G. Tkachenko. Topological Groups and Related Structures. Atlantis Press, Amsterdam/Paris, 2008.

[4] A. Bouziad. Every Cech-analytic Baire semi-topological group is a topological group. *Proc. Amer. Math. Soc.*, **124** (1996), 953–959.

[5] Mitrofan M. Choban, Petar S. Kenderov, Warren B. Moors. Eberlein Theorem and norm continuity of pointwise continuous mappings, preprint.

[6] P. S. Kenderov, I. Kortezov, W. B. Moors. Topological games and topological groups. *Topology Appl.*, **109** (2001), 157–165.

[7] A. V. Korovin. Continuous actions of pseudocompact groups and axioms of topological groups. *Comment. Math. Univ. Carolin.*, **33** (1992), 335–343.

[8] H. Pfister. Continuity of the inverse. *Proc. Amer. Math. Soc.*, **95** (1985), 312–314.

[9] E. A. Reznichenko. Extension of functions defined on products of pseudocompact spaces and continuity of the inverse is pseudocompact groups. *Topology Appl.*, **59** (1994), 233–244.

[10] J. Saint-Raymond. Jeux topologiques et espaces de Namioka. *Proc. Amer. Math. Soc.*, **87** (1983), 499–504.

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ПСЕВДОКОМПАКТНИ ПОЛУ-ТОПОЛОГИЧНИ ГРУПИ

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Полу-топологична група (съответно, топологична група) е група, снабдена с топология, относно която груповата оперция произведение е частично непрекъсната по всяка от променливите (съответно, непрекъсната по съвкупност от променливите и обратната операция е също непрекъсната). В настоящата работа ние даваме условия, от топологичен характер, една полу-топологична група да е всъщност топологична група. Например, ние показваме, че всяка сепарабелна псевдокомпактна полу-топологична група е топологична група. Показваме също, че всяка локално псевдокомпактна полу-топологична група, чиято групова операция е непрекъсната по съвкупност от променливите е топологична група.