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OPTIMAL CONTROL OF HETEROGENEOUS SYSTEMS*

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The present paper is a survey on some results on optimal control of continuous heterogeneous systems, which were recently published in periodic journals. A dynamical system is called heterogeneous if each of its elements has specific dynamics. The heterogeneity of the systems we consider is described by a one- or two-dimensional parameter – each element of the system corresponds to a specific value of the parameter.
The heterogeneous dynamical systems are used to model processes in economics, epidemiology, biology, social security (preventing the use of illicit drugs) etc. Here we consider models of optimal investment in education at the macroeconomic level [11], of restricting the damage caused by the spread of HIV [9], of markets for emission permits [3, 4] and optimal macroeconomic growth with endogenous improvement of the cutting-edge technologies [1].

1. Introduction. The optimal control of continuous heterogeneous dynamical systems is under intensive study in the recent years. A dynamical system is called heterogeneous if each of its elements has its own dynamics. Typical representatives of the heterogeneous systems are the age-structured ones. Optimal control problems for age-structured systems are of interest for many areas of application, as harvesting, birth control, epidemic disease control and optimal vaccination, illicit drug abuse, investment economic models, demography, and for a variety of models in the social area (for a quick overview of the applications the reader is referred to the bibliography in [8]). Many of the papers on optimal control of age-structured systems present optimality conditions for particular models, usually in the form of a *maximum principle* of Pontryagin's type. A general maximum principle for nonlinear McKendrick-type systems is obtained in [5]. A number of extensions of the McKendrick and Gurtin-MacCamy [10] models appeared afterwards, where the existing optimality conditions were not applicable. A nontrivial extension of the result of Brokate, proved in [8], was able to cope with such situations and, therefore, found numerous applications, like in [6], [7], [11], [12], etc. The class of the heterogeneous systems contains also the size-structured systems, the age- and duration-structured systems, and some other types of systems. A detailed description of various classes of continuous heterogeneous systems as well as necessary optimality conditions for optimal control of such systems can be found in [13].

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In the present survey we consider models of optimal investment in education at the macroeconomic level [11], of restricting the damage caused by the spread of HIV [9], of markets for emission permits [3, 4] and optimal macroeconomic growth with endogenous improvement of the cutting-edge technologies [1].

2. Optimal education under changing labor demand and supply. The model presented in this section, as well as its analysis, is taken from [11]. It is assumed that all individuals in a given society start working at the same age $s = 0$ (which corresponds to, say, 20 years of biological age) and retire at age $s = \omega$. Here $t \in [0, T]$ denotes time, T is the end of the (presumably large) planning horizon. The heterogeneity in this model is the moment of "birth" (i.e the moment of entering the work force) $t - s$, of each worker. The workers are distinguished also by their skills, considering for simplicity two levels of qualification: low-skilled and high-skilled workers. The amount of low-skilled workers of age s at time t is denoted by $L(t, s)$, and similarly, by $H(t, s)$ – the amount of high-skilled workers resulting from the government's investment in human capital.

Upgrading of low-skilled workers into high-skilled workers takes place with a rate $l(t, s)u(t, s) + e(s)$. Here $u(t, s)$ denotes the educational rate at time t for workers of age s. The function $l(t, s)$ reflects the dependence of the learning abilities of the workers on time and age, and $e(s)$ represents the learning by doing which depends on the years spent working. At the same time, due to the technological progress or other reasons, high-skilled workers of age s may lose their skills with a rate $\delta(t, s)$.

Therefore, the equations for the dynamics of the stock of low- and high-skilled workers are

$$\begin{aligned} L_t + L_s &= \delta(t, s)H(t, s) - e(s)L(t, s) - l(t, s)u(t, s)L(t, s), \qquad L(t, 0) = L_0(t), \\ H_t + H_s &= -\delta(t, s)H(t, s) + e(s)L(t, s) + l(t, s)u(t, s)L(t, s), \quad H(t, 0) = H_0(t), \\ & \quad L(0, s),\ H(0, s) \text{ – given initial data.} \end{aligned}$$

The left hand side, $L_t + L_s = \lim_{h\to 0}(L(t + h, s + h) - L(t, s))/h$, represents the change in one unit of time of the low-skilled labor that is of age s at time t. This change is composed of downgrading high-skilled to low-skilled workers (decay of human capital) at the rate $\delta(t, s)$ and of upgrading low-skilled to high-skilled workers at the rate $e(s)$ due to costless on the job learning by doing and at the rate $l(t, s)u(t, s)$ due to costly education. Similarly, the left hand side, $H_t + H_s$ represents the change in one unit of time of the high-skilled labor. This change is composed of the same components as the change of the low-skilled labor, but with the opposite sign.

At age $s = 0$ the number of those who enter the work force at time t is $L_0(t)$ for the low-skilled and $H_0(t)$ for the high-skilled. We assume throughout the paper that $H_0(t)$ is for each t relatively small compared with $L_0(t)$. This reflects the fact that all but a few high school graduates enter the work force as low-skilled workers and have to undergo additional education/training in order to become high-skilled.

We assume also that there is no unemployment and there is no mortality at working ages, therefore the sum $L(t, s) + H(t, s)$ equals the total working age population, which is determined by the exogenously given total inflow, $N_0(t)$. Thus $L_0(t) + H_0(t) = N_0(t)$. This allows to exclude the variable $H(t, s)$ from the model and to pass to a single differential equation for $L(t, s)$. However, for the purposes of better transparency of the exposition we work with both equations for $L(\cdot, \cdot)$ and $H(\cdot, \cdot)$.

The price of the per capita education $u(t, s)$ is $p(s, u(t, s))$, where $p(\cdot, \cdot)$ is a given function of (s, u). Therefore, the total cost $P(t)$ of the educational effort for the society at time t is represented as

$$P(t) = \int_0^{\omega} p(s, u(t, s))L(t, s)\,\mathrm{d}s.$$

We allow for imperfect substitutability across age groups for both low-skilled and high-skilled labour. If $\pi_L(\cdot)$ and $\pi_H(\cdot)$ are the respective relative efficiency parameters (assumed to be fixed over time), then the two sub-aggregates of low-skilled and high-skilled labour at time t, $\tilde{L}(t)$ and $\tilde{H}(t)$, are given by the following two CES functions

$$\tilde{L}(t) = \left(\int_0^{\omega} \pi_L(s)(L(t, s))^{\lambda_L}\,\mathrm{d}s\right)^{1/\lambda_L}, \qquad \tilde{H}(t) = \left(\int_0^{\omega} \pi_H(s)(H(t, s))^{\lambda_H}\,\mathrm{d}s\right)^{1/\lambda_H}.$$

Here $\lambda_L \in (-\infty, 1]$ and $\lambda_H \in (-\infty, 1]$ give the respective partial elasticity of substitution $\dfrac{1}{1-\lambda_i}$ for $i = L, H$. In the limit case of perfect substitutability across age groups, λ_L and λ_H are equal to 1 and the aggregate of low-skilled and skilled-labor is simply a weighted sum of age-specific supply.

We assume that the production technology depends only on the labor. The aggregate output at time t is given by a CES function of the two sub-aggregates of low-skilled and high-skilled labor, in which the technological level is represented by the two efficiency parameters $\theta_L(t)$ and $\theta_H(t)$:

$$Y(t) = \left(\theta_L(t)(\tilde{L}(t))^{\rho} + \theta_H(t)(\tilde{H}(t))^{\rho}\right)^{1/\rho}.$$

Here again $\rho \in (-\infty, 1]$ gives the partial elasticity of substitution $\sigma_W = \dfrac{1}{1-\rho}$ between high-skilled and low-skilled labor. Note that the marginal product of labor for a given age-education group depends on the group's own supply of labor and the aggregate supply – of labor in the education category.

The net revenue of the society at time t is the aggregate output $Y(t)$ minus the cost of education, $P(t)$, i.e. $Y(t) - P(t)$.

The formal problem of a central planner is to maximize the accumulated discounted net revenue by choosing optimally the educational rate $u(t, s)$. Discounting the future with a rate $r \geq 0$, we come up with the following dynamic optimization problem with state variables $L(t, s)$, $H(t, s)$, $\tilde{L}(t)$, $\tilde{H}(t)$ and $P(t)$, and control variable $u(t, s)$:

$$\text{(1)} \qquad \max \int_0^T e^{-rt}\left[\left(\theta_L(t)(\tilde{L}(t))^{\rho} + \theta_H(t)(\tilde{H}(t))^{\rho}\right)^{1/\rho} - P(t)\right]\mathrm{d}t$$

subject to

$$\text{(2)} \quad L_t + L_s = \delta(t, s)H(t, s) - e(s)L(t, s) - l(t, s)u(t, s)L(t, s), \quad L(t, 0) = L_0(t),$$

$$\text{(3)} \quad H_t + H_s = -\delta(t, s)H(t, s) + e(s)L(t, s) + l(t, s)u(t, s)L(t, s), \quad H(t, 0) = H_0(t),$$

$$L(0, s),\ H(0, s) \text{ – given initial data,}$$

$$\tilde{L}(t) = \left(\int_0^\omega \pi_L(s)(L(t,s))^{\lambda_L} \, \mathrm{d}s \right)^{1/\lambda_L}, \tag{4}$$

$$\tilde{H}(t) = \left(\int_0^\omega \pi_H(s)(H(t,s))^{\lambda_H} \, \mathrm{d}s \right)^{1/\lambda_H}, \tag{5}$$

$$P(t) = \int_0^\omega p(s, u(t,s)) L(t,s) \, \mathrm{d}s, \tag{6}$$

$$u(t,s) \geq 0. \tag{7}$$

Since our model does not include the education as a separate sector, the control u is directly interpreted as on-the-job training.

Here we consider the simplest case, in which problem (1)–(7) becomes linear with respect to the state variables: the case of perfectly substitutable labor across ages and across qualifications, $\rho = \lambda_L = \lambda_H = 1$. In this case we have

Proposition 1 ([11], Proposition 2)**.** *The optimal educational rate $u(t,s)$ is independent of the initial data $L(0,s)$, $H(0,s)$, and of the low- and high-skilled workers inflows $L_0(t)$ and $H_0(t)$. If all data are time-invariant* (*except for $L_0(t)$ and $H_0(t)$*)*, then $u(t,s) = u(s)$ is also time-invariant in the time-interval* $[0, T-\omega]$.

The proposition implies, in particular, that a demographic change (represented by $L_0(t)$ and $H_0(t)$) would not have any influence on the optimal educational rate. Moreover, in the stationary case the end of the time-horizon T may influence the optimal control no longer than one generation time before the end of the horizon, that is, on $[T-\omega, T]$ only.

Several papers on optimal education and human capital formation (cf. [2], [14] for a review of the theoretical literature on optimal life-cycle human capital investment) assume or conclude that the educational efforts are optimally allocated in youngest ages and decrease with age. It was established numerically that this is not always true in the present model, namely, the optimal educational rate $u(s)$ may strictly increase at certain ages. Below it is analyzed mathematically what is the reason for this effect and some economic explanations are given. The analysis is presented in the case of time-invariant data and a quadratic cost function $p(u)$.

Proposition 2 ([11], Proposition 3)**.** *Assume that all data* (*except for $L_0(t)$ and $H_0(t)$*) *are time-invariant and continuous in s, $l(s) = l$, $p(u) = bu + \frac{c}{2}u^2$. Let there exist some s for which $u(s) > 0$ and such that one of the following conditions holds:*

$$\text{(i)} \qquad d(s) - \frac{b}{c} l \geq 0 \qquad \& \qquad \theta_H \pi_H(s) - \theta_L \pi_L(s) < \frac{b^2}{2c};$$

$$\text{or} \qquad \text{(ii)} \qquad d(s) - \frac{b}{c} l < 0 \qquad \& \qquad \theta_H \pi_H(s) - \theta_L \pi_L(s) < \frac{b}{l} d(s) - \frac{c}{2l^2}(d(s))^2,$$

where $d(s) = r + \delta(s) + e(s)$. Then there exists $s_0 \in (0, \omega)$ where u is differentiable and $u'(s_0) > 0$.

The sufficient conditions for non-monotonic behaviour of the optimal educational rate are fulfilled if

(i) $d(s) = r + \delta(s) + e(s)$ is sufficiently large for some s (larger than bl/c) and $f(s) = \theta_H \pi_H(s) - \theta_L \pi_L(s)$ is sufficiently small for this s (smaller than $b^2/2c$).
(ii) $d(s)$ is not so large, but $f(s)$ is small enough, now depending on the value of $d(s)$.

In economic terms, an increase of the educational rate with age would happen if for some age the sum of depreciation (r), dequalification ($\delta(s)$) and "learning by doing" ($e(s)$) rates is large relative to the productivity differential f. If in a certain age interval the productivity differential is small, this may lead to postponement of learning since in the short run the returns to education are small. The return to education may be reduced by the fact that people can lose their costly qualification (with rate $\delta(s)$), and can make use of the costless "learning by doing" (with rate $e(s)$). In addition, higher depreciation rate diminishes the role of the length of the time-interval in which the worker exercises his qualification and therefore increases the chances for non-monotone learning. We mention that a higher value of δ can be associated with a higher rate of technological progress. Then, higher technological progress may lead to postponement of learning to older ages in order to take advantage of the more advanced knowledge.

3. Age- and duration-structured systems. Optimal prevention and treatment of HIV. The model of HIV spread, where prevention and medical treatment are controls appeared in [9]. Here $t \in [0,T]$ is the time, $a \in [0,\omega]$ is the age of an individual, and if the individual is infected, then $b \in [0,a]$ is the duration of infection – the time elapsed since becoming infected (T is the time-horizon (could be $+\infty$), ω is the maximal length of life). In the present model "mortality due to AIDS" always means "becoming sexually and economically inactive due to AIDS", which usually happens somewhat later than the first symptoms of AIDS. The heterogeneity in this model is described by two parameters – the moment $t-b$ at which an individual is infected and the age $a-b$ at which he/she is infected.

The state variables in the model are:
$S(t,a)$ – number of susceptible individuals of age a at time t;
$I(t,a,b)$ – number of individuals of age a at time t, infected b years ago, but still active;
$K(t)$ – weighted number of individuals at time t with weight $m(a)$ (see (12) below);
$J(t)$ – total infectivity at time t. Depends on the part of the infected individuals who participate in risky interactions, on the intensity, $m(a)$, of their participation, and on the variable infectivity $\lambda(\cdot)$ (see (13) below);
$B(t)$ – the inflow of noninfected individuals of age zero (see (11) below);
$N(t)$ – the total number of sexually active individuals (see (14) below).

The equations below hold for $t \in [0,T]$, $a \in [0,\omega]$, $b \in [0,a]$.

$$S_t + S_a = -\rho\left(\frac{J(t)}{K(t)}\right)\psi_S\left(a, \frac{u(t,a)}{(N(t))^\sigma}\right)m(a)S(t,a) - \mu(a)S(t,a), \quad (8)$$

$$S(0,a) = S_0(a), \quad S(t,0) = B(t),$$

$$I_t + I_a + I_b = -\mu(a)I(t,a,b) - (\delta(a,b) - \varepsilon(a,b)\kappa(a,v(t)))\alpha(b)I(t,a,b), \quad (9)$$

$$I(0,a,b) = I_0(a,b),$$

$$I(t,a,0) = \rho\left(\frac{J(t)}{K(t)}\right)\psi_S\left(a, \frac{u(t,a)}{(N(t))^\sigma}\right)m(a)S(t,a), \quad (10)$$

$$(11) \qquad B(t) = \int_0^\omega \beta(a)S(t,a)\,\mathrm{d}a,$$

$$(12) \qquad K(t) = \int_0^\omega m(a)\left[S(t,a) + \int_0^a I(t,a,b)\,\mathrm{d}b\right]\mathrm{d}a,$$

$$(13) \qquad J(t) = \int_0^\omega \int_0^a \psi_I\left(a, \frac{u(t,a)}{(N(t))^\sigma}\right) m(a)\lambda(b)I(t,a,b)\,\mathrm{d}b\,\mathrm{d}a,$$

$$(14) \qquad N(t) = \int_0^\omega \left[S(t,a) + \int_0^a I(t,a,b)\,\mathrm{d}b\right]\mathrm{d}a.$$

The following variables are considered as controls:

$u(t,a)$ – money spent for prevention among a-years old individuals at time t;

$v(t)$ – price of medical treatment per person and unit of time at t;

Moreover, the following data are involved:

$\rho(z)$ – incidence rate. Depends on the weighted prevalence $z = J/K$;

$m(a)$ – age-dependent intensity of participation in risky interactions;

$\psi_S(a,u)$, $\psi_I(a,u)$ – effect of the prevention control u on the risky behavior of the susceptible (infected, respectively) individuals: decreasing functions such that $\psi(a,0) = 1$, $\psi(a,+\infty) = \psi_0 \geq 0$;

$\lambda(b)$ – infectivity of an individual, infected b years ago;

$\beta(a)$ – birth rate of the a-years old non-infected individuals;

σ – a non-negative real number that reflects the "swamping effect": $\sigma = 1$ corresponds to prevention programs where the per capita effect is proportional to the per capita expenditure; $\sigma = 0$ if the per capita effect is independent of the number of individuals and depends only on the aggregated expenditure. Mixture of the two types of programs lead to intermediate values of σ;

$\mu(a)$ – natural mortality rate of the non-infected individuals;

$\delta(a,b)$ – the "mortality" rate due to AIDS;

$\varepsilon(a,b)$ – reduction of the "mortality" rate due to AIDS for treated individuals, $\varepsilon(a,b) \leq \delta(a,b)$;

$\alpha(b)$ – the fraction of the b-years ago infected individuals who are aware that they need treatment;

$\kappa(a,v)$ – the fraction of the individuals of age a who can afford treatment, if the price of the medicine is v;

S_0, I_0 – initial conditions;

The momentary value of an individual of age a is assumed equivalent to the current net contribution of the individual to the Gross National Product. Therefore, the objective function to be minimized has the form

$$\int_0^T e^{-rt} \int_0^\omega \left[-g_S(a)S(t,a) - g_I(a)\int_0^a I(t,a,b)\,\mathrm{d}b + u(t,a)\right]\mathrm{d}a\,\mathrm{d}t$$
$$+ \int_0^T e^{-rt} \int_0^\omega \int_0^a (c^*(t) - v(t))\alpha(b)\kappa(a,v(t))I(t,a,b)\,\mathrm{d}b\,\mathrm{d}a\,\mathrm{d}t,$$

where

$g_S(a)$, $g_I(a)$ – the instantaneous "value" of a single non-infected (infected, respectively) individual of age a;

$c^*(t)$ – market price of the medicine per unit of time of consumption.

The right-hand side of (8) represents the outflow of susceptible individuals: due to infection, or due to natural mortality. The intensity of participation of the susceptible individuals in risky interactions is decreased by a factor of $\psi_S(a, u/N^\sigma) \leq 1$, where u is the money allocated for prevention. The inflow $B(t)$ of individuals of age zero aggregates all births by non-infected parents. It is assumed that all individuals are non-infected at birth, and that only non-infected individuals may have children.

The right-hand side of (9) represents the outflow of infected individuals due to natural mortality and due to AIDS. The explanation of the second term is as follows. It is assumed that before someone dies of AIDS they are aware that they need treatment. Out of αI such individuals, $\kappa\alpha I$ have mortality rate $\delta - \varepsilon$ (those who are medicated), and $(1-\kappa)\alpha I$ have mortality rate δ. The resulting common mortality rate for all the αI individuals is then $(\delta - \varepsilon)\kappa + \delta(1 - \kappa) = \delta - \varepsilon\kappa$.

The inflow of newly infected individuals equals the outflow due to infection from the group of susceptible individuals (10).

The prevention may have an effect also on the already infected individuals, decreasing the intensity with which they participate in risky interactions. Therefore, ψ_I is presumably smaller than one in the expression for $J(t)$, in (13).

In the objective function we aggregate the discounted economic value of the active individuals and subtract the control costs: u for prevention, the last term – for treatment. The latter equals the difference between the market price and the subsidized price of the medicine, times the consumption.

A necessary optimality condition in the form of Pontryagin maximum principle for general age- and duration-structured systems is obtained in [9] and is used for analysing the HIV spread model. The conclusions are that the treatment (subsidy of the medicine) alone may be counterproductive. The subsidy is productive if the initial prevalence is high, and/or the optimization point of view is myopic: short time horizon or large discount rate. Also, reduction in the prevention program should lead to reduction of the optimal subsidized treatment. Increase in the treatment should necessarily be accompanied by increase in the prevention.

4. Markets for emission permits. The topic of this section is taken from [3] and [4]. Imagine a firm that is composed of machines of different vintages (technologies) τ: $K(\tau, s)$ denotes the capital stock of vintage τ and of age s. That is, $K(\tau, t - \tau)$ is the stock of vintage τ that exists at time $t \geq \tau$. The maximal life-time of machines of each technology is denoted by ω, and the depreciation rate of each technology – by δ. Both are assumed independent on the vintage just for notational convenience. At any time $t > 0$ the firm may invest with intensity $I(\tau, s)$ in machines of vintage τ that are of age s at time t (so that $s = t - \tau$). In the model discussed here the heterogeneity is represented by the vintage parameter τ.

The planning horizon of the firm is $[0, \infty)$. Therefore, the stock of machines of vintage $\tau \in [-\omega, 0]$ the firm has at time $t = 0$ is considered as exogenous and is denoted by $K_0(\tau)$. These machines have age $-\tau$ at time $t = 0$ and may be in use untill they reach age ω. Machines of vintage $\tau > 0$ may be in use for ages $s \in [0, \omega]$, and their stock at age zero equals zero. Therefore, $K_0(\tau)$ will be defined as zero for $\tau > 0$. The ages of possible use of any vintage can be written as $[s_0(\tau), \omega]$, where $s_0(\tau) = \max\{0, -\tau\}$.

Summarizing, the dynamics of each vintage $\tau \in [-\omega, \infty)$ is given by the equation

$$\dot{K}(\tau,s) = -\delta K(\tau,s) + I(\tau,s), \qquad K(\tau,s_0(\tau)) = K_0(\tau), \qquad s \in [s_0(\tau),\omega], \tag{15}$$

where "dot" means everywhere the derivative with respect to s (the argument representing the age).

The productivity of machines of vintage τ is denoted by $f(\tau)$, while $g(\tau)$ denotes the emission per machine of vintage τ. The firm faces at time t costs due to emissions at a price $v(t)$ per unit of emissions. From the point of view of a firm $v(t)$ represents an exogenous tax on pollution set up by a regulator. From market's (regulator's) point of view the price $v(\cdot)$ is endogenized by auctioned emission permits. Due to this cost the firm may decide to (possibly temporarily) switch off a part of the machines. Let us denote by $W(\tau,s) \in [0,1]$ the fraction of the machines of vintage τ that operate at age s.

The cost of investment I in s years old machines of any vintage is denoted by $C(s,I)$.

The present value (at time $t=0$) of the total production of machines of vintage τ, discounted with a rate r, is

$$e^{-r\tau}\int_{s_0(\tau)}^{\omega} e^{-rs} f(\tau)K(\tau,s)W(\tau,s)\,\mathrm{d}s,$$

the cost of emission is

$$e^{-r\tau}\int_{s_0(\tau)}^{\omega} e^{-rs} v(\tau+s)g(\tau)K(\tau,s)W(\tau,s)\,\mathrm{d}s,$$

and the investment costs are

$$e^{-r\tau}\int_{s_0(\tau)}^{\omega} e^{-rs} C(s,I(\tau,s))\,\mathrm{d}s.$$

The firm maximizes the aggregated over time discounted net revenue, that is, solves the problem

$$\max_{I\geq 0,\, W\in[0,1]} \int_{-\omega}^{\infty} e^{-r\tau}\int_{s_0(\tau)}^{\omega} e^{-rs}\left[(f(\tau)-v(\tau+s)g(\tau))K(\tau,s)W(\tau,s) - C(s,I(\tau,s))\right]\,\mathrm{d}s\,\mathrm{d}\tau$$

subject to (15).

For a given $v(\cdot)$ and on the optimal trajectory (K^*,W^*,I^*) the emission of the firm at time $t>0$ is given by the expression

$$E^*[v](t) = \int_{t-\omega}^{t} g(\tau)K(\tau,t-\tau)W(\tau,t-\tau)\,\mathrm{d}\tau.$$

Consider an economy consisting of n identical firms described by the above introduced model. Let $\hat{E}(t)$ be a cap for the emission permits of the economy for $t \geq \hat{t}$. We assume that the regulation (here, the emission cap) is known by the firms at time $t=0$ for all the future. The cap takes effect at time $\hat{t} \geq \omega$.

The emission permits are auctioned at time $\hat{t}$. The question is: does the auction (the primary market) really determine the price $v(t)$? The answer is "NO", and the primary market behavior and the appearance of market failure was analyzed theoretically and numerically .

The equation for the auction price of emission permits, $v(t)$, is

$$E^*[v](t) = \hat{E}(t)/n, \quad t \geq \hat{t}. \tag{16}$$

The market would determine the price of permits if:

(i) equation (16) has a solution $v(\cdot) \in \mathcal{V}$;

(ii) the solution is unique;

(iii) the solution is positive for all t.

In general, no one of the above requirements is fulfilled. The first needs assumptions for the data, and these assumptions are difficult to specify due to the complexity of equation (16). The second requirement is obviously not fulfilled if, for example, $\hat{E}(t) \equiv 0$, since if v is a solution of (16), then every $\tilde{v}$ with $\tilde{v}(t) > v(t)$ would also be a solution. Even if the above two possibilities could be classified as "academic", then it was established in [3] that a failure of the third requirement could be rather realistic.

Regularization methods for *ill-posed* equations of the kind we face in (16), are developed first by A. Tikhonov in the 60-th years of the last century. The regularization that was employed in [3] is different and has the advantage that it has a clear economic meaning and is implementable in the real market: the emission constraint (16) was formulated period-wise, rather than at any time instant. This has a clear policy implementation. Namely, the discrete version of the emission mapping E^* was introduced as follows:

$$\hat{E}_k = \frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} \hat{E}(t)\,\mathrm{d}t$$

is the cap for the emission intensity in the time-period $[t_{k-1}, t_k]$. For simplicity we assume that all time periods have the same length $h > 0$, thus $t_k = kh$, and also that $\omega = mh$, $\hat{t} = \hat{k}h$ for appropriate natural numbers m and $\hat{k}$. Correspondingly, the price of emission is a constant, v_k, on each interval $[t_{k-1}, t_k)$ and zero for $k \le \hat{k}$.

In [3] experimental (numerical) results for the auction price of permits were obtained assuming that the firms participating in the auction are identical (thus the price is determined by equation (16), if a positive solution exists). In particular, the role of the aggregation step h was investigated. Following the terminology used under the Kyoto protocol, this step is called a *commitment period*, and h is its length. The existence and the regularity of the auction price for emission permits was scrutinized for different h.

Two basic phenomena were observed in the numerical experiments:

– if the imposed emission cap is below the level (at the time $\hat{t}$ of the imposition of the cap) of the unrestricted emissions, the firm begins to reduce the emissions much earlier than $\hat{t}$ – this is the so called anticipation effect.

– larger commitment periods regularise the permits market: decrease the volatility of the market and remove market failures (time periods in which the market "price" obtained from (16) is negative).

5. Macroeconomic growth with endogenous technological progress. The necessary optimality conditions for the model presented in this section are obtained in [1]. Let $[0, T]$ be a fixed time-interval and let $[0, \bar{Q}]$ be an interval where the parameter of heterogeneity q takes its values (here $\bar{Q} > 0$ could be $+\infty$, in which case the interval should be interpreted as $[0, \infty)$). Denote $D = [0, T] \times [0, \bar{Q}]$. The state variables in the model below is the functions

$$x : D \mapsto \mathbf{R}^n, \quad Q : [0, T] \mapsto [0, \bar{Q}], \quad y : [0, T] \mapsto \mathbf{R}^m,$$

while $u : D \mapsto U \subset \mathbf{R}^r$ is a control function. The optimal control problem we consider reads as follows:

$$\max_u \int_0^T \int_0^{Q(t)} L(t, q, x(t, q), Q(t), y(t), u(t, q))\, \mathrm{d}q\, \mathrm{d}t, \tag{17}$$

subject to the equations

$$\dot{Q}(t) = g(t, Q(t), y(t)), \quad Q(0) = Q^0 \geq 0, \;\; t \in [0, T], \tag{18}$$

$$y(t) = \int_0^{Q(t)} h(t, q, u(t, q))\, \mathrm{d}q, \tag{19}$$

$$\dot{x}(t, q) = f(t, q, x(t, q), Q(t), y(t), u(t, q)), \quad x(0, q) = x^0(q), \;\; q \in [0, Q^0], \quad x(t, Q(t)) = x^b(t), \;\; t \in [0, T], \tag{20}$$

$$u(t, q) \in U. \tag{21}$$

Here

$$L : D \times \mathbf{R}^n \times [0, \bar{Q}] \times \mathbf{R}^m \times U \mapsto \mathbf{R},$$

$$f : D \times \mathbf{R}^n \times [0, \bar{Q}] \times \mathbf{R}^m \times U \mapsto \mathbf{R}^n, \quad g : D \times \mathbf{R}^m \mapsto \mathbf{R}, \quad h : D \times U \mapsto \mathbf{R}^m,$$

$\dot{x}(t, q)$ is the derivative with respect to t.

The informal meaning is as follows. Given a control function u with values in U, equations (18) and (19) define the interval $[0, Q(t)]$ in which the parameter q takes values at time t. The state $y(t)$ represents an aggregated (over the domain of heterogeneity $[0, Q(t)]$) quantity. Equation (20) with the respective bounary conditions defines the distributed state x. Then, the objective functional (17) is to be maximized with respect to the control u.

In the notations below we systematically skip the arguments of all functions which are at their respective optimal values, e.g.

$$L(t, Q(t)) := L(t, Q(t), x(t, Q(t)), Q(t), y(t), u(t, Q(t)))$$

if $x(\cdot, \cdot), Q(\cdot), y(\cdot)$ and $u(\cdot, \cdot)$ are optimal.

Let $\lambda(\cdot, \cdot)$, $\mu(\cdot)$ and $\nu(\cdot)$ be the adjoint variables, corresponding to (20), (18) and (19) respectively, and let

$$\eta(t) := \int_0^{Q(t)} (L_y(t, q) + \lambda(t, q) f_y(t, q))\, \mathrm{d}q.$$

Under assumptions on the model's data which are standard for the control literature (the involved functions are of Caratheodory type) necessary optimality conditions were obtained in [1] in which the adjoint variable $\mu(\cdot)$ satisfies a differential inclusion instead of differential equation (let $R(t)$ denote its reachable set at time t) and as a result of this the maximization of the Hamiltonian takes the form of "min-max". Namely, defining $\tilde{H} : D \times \mathbf{R}^{n+1+m+r+n+m} \mapsto \mathbf{R}$ as

$$\tilde{H}(t, q, x, Q, y, u, \lambda, \nu) = L(t, q, x, Q, y, u) + \lambda\, f(t, q, x, Q, y, u) + \nu\, h(t, q, u).$$

one obtains

Theorem 1 ([1], Theorem 1 on p. 295)**.** *Let $u \in L_\infty(D)$ be an optimal control in the problem* (17)–(21) *and let $z := (x, Q, y)$ be the corresponding trajectory. Let λ be the adjoint variable, corresponding to* (20). *Then, for almost every $(t, q) \in D(u) := \{(t, q) :$*

$t \in [0,T)$, $q \in [0,Q(t))\}$

$$\max_{v\in U} \min_{\nu\in R(t)g_y(t)+\eta(t)} \Big(\tilde H(t,q,z(t,q),v,\lambda(t,q),\nu) - \tilde H(t,q,z(t,q),u(t,q),\lambda(t,q),\nu)\Big) \le 0.$$

If one strengthens the assumptions on the model's data and imposes an *a priory* assumption on the optimal control, then the usual Pontryagin type maximum principle can be obtained. Impose the following

Assumption: The functions L, L_x, f, f_x and h are continuous with respect to q (uniformly in the rest of the variables); the optimal control u is (equivalent to) a function which is continuous from the left with respect to q at $q = Q(t)$ for a.e. $t \in [0,T]$.

Introduce the function (having in many respects the traditional meaning of "hamiltonian") $H : D \times \mathbf{R}^{n+1+m+r+n+1+m} \mapsto \mathbf{R}$ as

$$H(t,q,x,Q,y,u,\lambda,\mu,\nu) = L(t,q,x,Q,y,u) + \lambda\, f(t,q,x,Q,y,u) + \mu\, g(t,Q,y) + \nu\, h(t,q,u),$$

where the arguments in the l.-h. side should be inserted in the r.-h. side wherever appropriate. Then, Theorem 1 implies

Theorem 2 ([1], Theorem 2 on p. 297)**.** *Denote by $u \in L_\infty(D)$ an optimal control in the problem* (17)–(21) *and by $z := (x,Q,y)$ be the corresponding trajectory. Then the adjoint system has a unique solution (λ,μ,ν) and for a.e. $t \in [0,T]$ and a.e. $q \in [0,Q(t)]$*

$$H(t,q,z(t,q),u(t,q),\pi(t,q)) = \max_{u\in U} H(t,q,z(t,q),u,\pi(t,q)).$$

In what follows a stylized economic model of endogenous economic growth is presented, to which the above results can be applied.

Consider a finite time horizon $[0,T]$ (presumably rather large, so that T is an "approximation" of the infinity) and a large corporation producing at time t diverse goods labeled by the real number $q \in [0,Q(t)]$. Here $Q(t)$ is the newest good (technology) available at time t. Each of the goods q is produced by a separate firm that at time t has physical capital stock $x(t,q)$. The q-th firm ($q \in [0,Q(t)]$) invests at time t an amount $u(t,q)$ that is split in two parts: $\alpha u(t,q)$, $0 \le \alpha \le 1$, is allocated to increase the capital stock, while $(1-\alpha)u(t,q)$ is the contribution of the q-th firm to the R&D activity of the corporation which results in development of new technologies (goods) and, hence, in increase of $Q(t)$.

The model reads as follows:

$$\begin{aligned} \dot x(t,q) &= -\delta x(t,q) + \alpha u(t,q), \quad x(0,q) = x^0(q) \ \text{ for } \ q \in [0,Q^0], \\ & \qquad\qquad\qquad\qquad\qquad\quad\; x(t,Q(t)) = 0 \ \text{ for } \ t > 0, \\ \dot Q(t) &= (1-\alpha)y(t), \quad Q(0) = Q^0, \\ y(t) &= \int_0^{Q(t)} u(t,q)\,\mathrm{d}q. \end{aligned}$$

Here $y(t)$ is the total investment in R&D, $\delta \ge 0$ is the depreciation rate of the physical capital, $x^0(q)$ is the initially available capital stock for producing goods $q \in [0,Q^0]$, $Q^0 > 0$ is the newest technology available at time $t = 0$. The objective function to be maximized is

$$\int_0^T e^{-rt} \int_0^{Q(t)} \big[p(q,Q(t))x(t,q) - bu(t,q) - cu^2(t,q)\big]\,\mathrm{d}q\,\mathrm{d}t,$$

subject to the control constraint $u(t,q) \geq 0$. In this model $r \geq 0$ is the discount rate, $p(q,Q)$ is the market price of the good $q \in [0,Q]$, given that goods up to level Q are available, $bu + cu^2$, $b \geq 0$, $c > 0$ is the cost of the investments u. The dependence of the price p on q and Q reflects the fact that the market price of any available good decreases when newer products emerge (that is, when Q increases). For the present illustrative purpose we chose the specification $p(q,Q) = e^{-\gamma(Q-q)}$ with $\gamma \geq 0$. So the price of the newest product is normalized to one (which is supported by the data for personal computers and mobile telephones, where the price of the new products does not substantially change with time, while the quality increases).

Assume that $u(t,q)$ is an optimal control that is continuous from the left at $q = Q[u](t)$. Then, Theorem 2 implies that

$$u(t,q) = \max\{0, (\alpha\tilde{\lambda}(t,q) + \tilde{\nu}(t) - b)/2c\}.$$

Further analysis of the model can be found in [1].

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ОПТИМАЛНО УПРАВЛЕНИЕ НА ХЕТЕРОГЕННИ СИСТЕМИ

Цветомир Цачев

В настоящия доклад се прави преглед на някои резултати от областта на оптималното управление на непрекъснатите хетерогенни системи, публикувани в периодичната научна литература в последните години. Една динамична система се нарича хетерогенна, ако всеки от нейните елементи има собствена динамика. Тук разглеждаме оптимално управление на системи, чиято хетерогенност се описва с едномерен или двумерен параметър – на всяка стойност на параметъра отговаря съответен елемент на системата.
Хетерогенните динамични системи се използват за моделиране на процеси в икономиката, епидемиологията, биологията, опазване на обществената сигурност (ограничаване на използването на наркотици) и др. Тук разглеждаме модел на оптимално инвестиране в образование на макроикономическо ниво [11], на ограничаване на последствията от разпространението на СПИН [9], на пазар на права за въглеродни емисии [3, 4] и на оптимален макроикономически растеж при повишаване на нивото на върховите технологии [1].

Ключови думи: оптимално управление, непрекъснати хетерогенни динамични системи, приложения в икономиката и епидемиолегията