МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2012
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2012
*Proceedings of the Forty First Spring Conference
of the Union of Bulgarian Mathematicians
Borovetz, April 9–12, 2012*

VISUALIZATION OF PROBLEMS' SOLUTION*

Djurdjica Takači

We present a didactical approach to the influence of *DGS* (*Dynamic Geometry Systems*) on solving mathematical problems. The well known and very friendly software *GeoGebra* is implemented in the concept of G. Polya, for solving mathematical problems, as a cognitive tool. The visual proof of the theorem about the inscribed angle is obtained in the first step of the solving process, enabling the discussion of the extension of the given task.

Introduction. In the field of mathematical education the problem solving activities, have been investigated over more than sixty years. George Polya [6] developed the four-step solving process, namely *understanding, devising a plan, carrying out the plan, and looking back*, which is still applicable to many instances.

The proof, as the essence of mathematics, has been taught largely in the school curriculum. Infact, it is the object of investigations in many didactical studies, over more than thirty years. Mathematics teachers believe that the proof provides certainty for the mathematician and successful access to mathematical theory [1, 2].

The dynamic geometry software (DGS), as are, e.g.,*GeoGebra, Sketchpad, Mathematica*, and many others, provide a geometry model, which, besides visualization, allows the changing of parameters satisfying the conditions of the considered problem. In didactical investigations, the influence of DGS is explored, in many papers. The central question of discussions and analysis is addressed to the opportunities of dynamic environments, to make mathematical properties "easily seen" [2]. In particular, the proof of theorems can be checked easily, for different invariant conditions.

In problem solving process, after understanding the given conditions, and their implementations by using DGS, the visual proof is obtained during the application the first Polya's stage. The obtained visual solutions or proofs, are changing the problem solving process. There are a lot of argues and discussions about the next three Polya's stages in the papers considering the didactical approach to problem solving process and proof with the use of DGS. It is interesting to mention the following questions from the paper [2], pp. 1.

"The question that we are now asking is whether the introduction of dynamic geometry systems will improve the situation – or will it make the transition from informal to formal proof in mathematics even harder? ... or will computer use be seen to replace any need for proof?"

*2000 Mathematics Subject Classification:** 97C70, 00A66, 97Uxx.
Key words: Teaching-learning processes, visualization, educational technology.

The role of DGS in teaching the proof is perhaps the most important task for the teacher in problem solving activities.

In this paper we present the didactical approach to problem solving activities by using *GeoGebra*, following Polya's stages [6]. In particular, the proving of well-known theorem about the measure of inscribed angle, done with the 15 years old gifted students, is analyzed.

Problem solving. In the computer laboratory, there were 17 students who were already involved in the dynamic package *GeoGebra*. The students had two school hours, and they got the following task to prove the well-known theorem.

Theorem. *The measure of central angle is exactly twice the measure of the inscribed angle subtend the same arc on the circle.*

The students knew this theorem, but they were curious to prove it. The teacher followed the stages of G. Polya [6] by using *GeoGebra*, as a cognitive tool.

In order to apply the first stage, namely **understanding the problem**, the students were asked to draw the circle with the center O, the points A, B, C, and the angles $\sphericalangle BAC$, $\sphericalangle BOC$, inscribed and central angles, both subtends the same arc on the given circle, as on Figure 1. Actually, it holds:

$$\sphericalangle BAC = 61.79^\circ, \quad \sphericalangle BOC = 123.58^\circ = 2 \cdot 61.79^\circ.$$

Further on, the students used the possibility to change the location of the point A, the dynamic property of package *GeoGebra* and to find out that the upper relation holds independently of the place of A, satisfying the condition that both angles subtend the same arc.

Further on, the students were taking the different central angles and applying the previous moving of the point A, they got that the relation $\sphericalangle BOC = 2\sphericalangle BAC$ is satisfied.

The second stage of Polya, G, [6], ***devising a plan***, came naturally just after obtaining the visual solution. The students continued with moving the point A, on the circle, and they obtained that the measure of the angle $\sphericalangle BAC = 241.79^\circ$, is even greater than that of angle $\sphericalangle BOC$ (Figure 2).

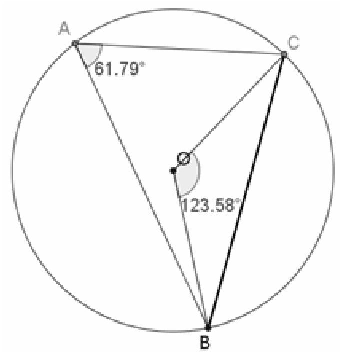

Fig. 1

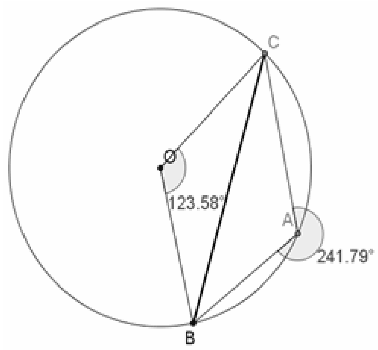

Fig. 2

In this case the students went back to the conditions of the theorem, repeated them and found out that they are not fulfilled in this case. The angle $\sphericalangle BOC$ is the central, but the angle $\sphericalangle BAC$ is not inscribed in the given circle.

After that the students were curious to check the value of the inscribed angle $\sphericalangle CAB$, trying with the opposite direction, and got Figure 3. But in this case the angles $\sphericalangle CAB$ and $\sphericalangle BOC$ do not subtend the same arc of the given circle.

Finally, the student drew the angles $\sphericalangle CAB$ and $\sphericalangle COB$ subtending the same arc of circle and since the conditions are fulfilled, the theorem is true (Figure 4).

After Figure 1 the students easily obtained Figure 2, and then they got a new problem considering the reasons for the obtained values of angles. Clearly, the students had to analyze the conditions carefully.

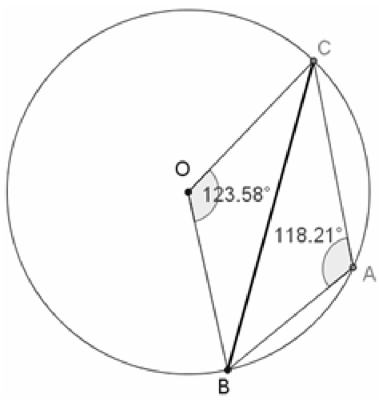

Fig. 3

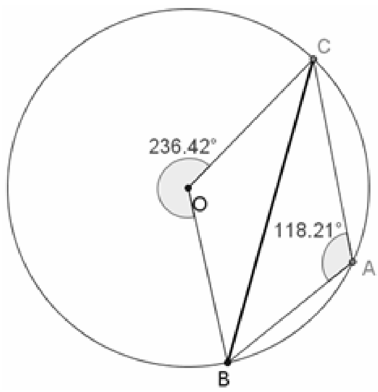

Fig. 4

Now, one of my students, Stefan, remarked that the condition of theorem:

The central and inscribed angle subtends the same arc on the circle,

should be replaced with the condition:

The central and inscribed angle are on the same side of the same chord.

In Figure 1, $\sphericalangle CAB$ and $\sphericalangle BOC$ are the inscribed and central angles on the same side of the chord BC. But in Figure 3, they are not on same side of the chord, and therefore the measures of inscribed and central angles are not connected.

The proof of this theorem, as the third stage, is also helped by dynamic properties of package.

Special case: The students were moving the point A,on the circle and came to the special case, of the theorem, when the points A, and B belonged to the diameter of circle (one chord is the diameter). In that case the proof can be done easily, because the angle $\sphericalangle BOC$ is exterior angle of the triangle ΔOAC, and it is equal to the sum of angles $\sphericalangle CAB$, $\sphericalangle ACO$, which are equal, since the triangle ΔOAC is an isosceles one (Figure 5).

Proof. After considering the special case, almost all students moved the point A, and obtained a figure similar to Figure 6, where the center of the circle is in the interior of inscribed angle.

Once the special case was proved, four of my students firstly, and soon after the others with the little help, made the deductive proof.

The second case, when the center of the circle was in the exterior of the inscribed angle, was suggested by the teacher. The students made a figure similar to Figure 7, and

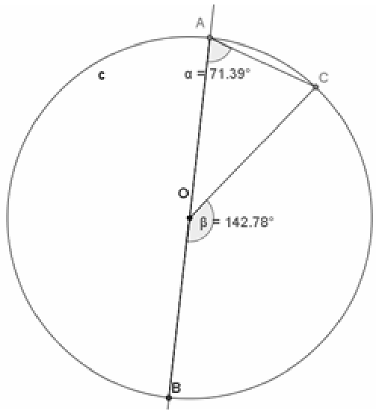

Fig. 5

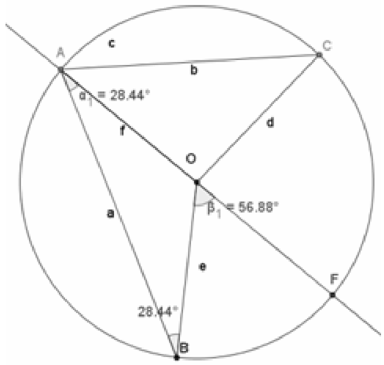

Fig. 6

together with the teacher formed the required system of equations, and thus obtained the proof of theorem. All the students were really very proud that they could finish the proof easily.

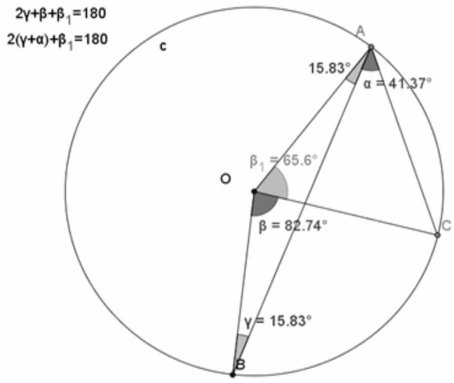

Fig. 7

In the fourth stage, ***looking back***, the obtained solution, i.e., the proof of the theorem had to be examined. Some students already, in their previous considerations, had connected this theorem and its proof with Thales' theorem:

Theorem. *The angle subtended by a diameter is always a right angle.*

The students first concluded that this is a consequence of the considered theorem and, then by using dynamic properties of GeoGebra, constructed the visual proof and repeated the deductive proof, remarking that in the case of the Thales theorem the center of the circle is always in the interior of the inscribed right angle.

The teacher continued with asking for the next consequences of this theorem and two students mentioned the following:

Theorem. *Opposite angles of cyclic quadrilaterals sum to* 180°*, conversely, any quadrilateral for which this is true can be inscribed in a circle.*

Further on, we mentioned theorems related to power of points of the circle.

Interesting visual proofs. In the following we show the visual proof of the theorem, constructed in *GeoGebra*. This is really very easy, but useful for students mathematical education. In that sense in the paper [6], the author shows how difficult contents can be taught by using *GeoGebra*.

Theorem. *Let the right triangle be inscribed in a conic section. If the vertex with right angle is fixed, then all lines containing hypotenuses have a common point.*

The visual proof can be obtained easily by using the dynamic properties of package *GeoGebra.* In Figure 8, the triangle ΔABC, with the right angle $\sphericalangle CAB$ is inscribed in an ellipse. By changing the vertex B, the hypotenuse is changing. All lines containing hypotenuses, presented with the trace included, have the common point D.

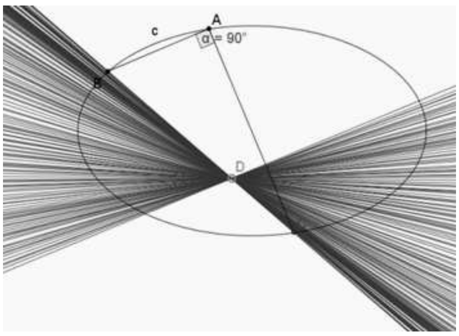

Fig. 8

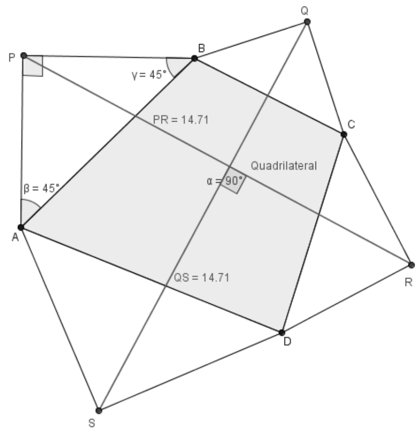

Fig. 9

Analogously, the visual proofs for hyperbola and parabola can be obtained.

In the paper [5] the visual solution is obtained and by using *GeoGebra* of Napoleon's theorem is obtained (Figure 9).

Theorem. *Let* A*,* B*,* C*,* D *be a convex quadrilateral. The points* P*,* Q*,* R*,* S *are external points of the given quadrilateral, constructed such that the triangle* ABP*,* BCQ*,* CDR *and* DAS *are right-isosceles triangles (at* P*,* Q*,* R *and* S*, respectively). Show that* $PR = QS$ *and* $PR \perp QS$*.*

REFERENCES

[1] C. Christou, N. Mousoulides, M. Pittalis, D. Pitta-Pantazi. Proofs through Exploration in dynamic geometry Environments. Proceed. 28th Conf. Intern. Gr. Psych. Math. Educ., 2004, 215–222.

[2] C. Hoyles, K. Jones. In: Proof in Dynamic Geometry Contexts, Perspectives on the Teaching of Geometry for the 21st Century. Dordrecht, Kluwer, 1998, 121–128.

[3] K. Jones, A. Gutiérrez, M. A. Mariotti. Proof in Dynamic Geometry Environments. *Educational Studies in Mathematics*, **44** (2000), 1–3.

[4] C. Laborde. Dynamic Geometry Environments as a Source of Rich Learning Contexts for the Complex Activity of Proving. *Educational Studies in Mathematics*, **44** (2000), 151–161.

[5] D. Păunescu, Dj. Takači. On the role of GeoGebra in the proof of theorems. Proc. of *GeoGebra* Conference, Novi Sad, 2011, 7–18.

[6] G. Polya. How to Solve It: A New Aspect of Mathematical Method. Princeton University Press, 1973.

[7] A. Vassileva. The Right Way to Look at a Complicated Problem. Proc. of *GeoGebra* Conference, Novi Sad, 2011, 78–89.

Djurdjica Takači
Department of Mathematics and Informatics
Faculty of Sciences
University of Novi Sad
Novi Sad, Serbia
e-mail: djtak@dmi.uns.ac.rs

ОНАГЛЕДЯВАНЕ НА РЕШЕНИЯТА НА ЗАДАЧИ

Джурджица Такачи

В доклада се разглеждат дидактически подходи за решаване на задачи, упражнения и доказване на теореми с използване на динамичен софтуер, по-специално – с вече широко разпространената система *GeoGebra*. Въз основа на концепцията на Пойа се анализира използването на *GeoGebra* като когнитивно средство за решаване на задачи и за обсъждане на техни възможни обобщения.