МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2012
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2012
Proceedings of the Forty First Spring Conference
of the Union of Bulgarian Mathematicians
Borovetz, April 9–12, 2012

ON A SUMMABILITY METHOD DEFINED BY MEANS OF HERMITE POLYNOMIALS*

Georgi S. Boychev

A summability method, defined by mean of the Hermite polynomials, is proposed. For this summation method Tauberian theorems are given

The classical Hermite polynomials $\{H_n(z)\}_{n=0}^{+\infty}$ are uniquely defined by the equalities

$$\int_{-\infty}^{+\infty} \exp(-x^2) H_m(x) H_n(x) dx = \sqrt{\pi} n! 2^n \delta_{m\,n}, \quad m, n = 0, 1, 2, \dots,$$

and the requirement that the coefficient of x^n in the n-th polynomial to be positive [1, 5.5, (5.5.1)].

Let us introduce the functions $\lambda(z) = \sqrt{2}\exp(z^2/2)$ and $c_n(z) = (2n/e)^{n/2}\cos[(2n+1)^{1/2}z - n\pi/2]$. Hermite polynomials have the representation ($n \geq 1$) [2]

$$H_n(z) = \lambda(z) c_n(z)\{1 + h_n(z)\}, \tag{1}$$

where $\{h_n(z)\}_{n=1}^{+\infty}$ are holomorphic functions in $G = \mathbf{C}\backslash(-\infty, +\infty)$ and

$$h_n(z) = O(n^{-1/2}) \qquad (n \to +\infty)$$

uniformly on every compact subset of G.

A series of kind

$$\sum_{n=0}^{+\infty} a_n H_n(z) \tag{2}$$

we call Hermite series.

Let $0 < \tau < +\infty$. We introduce the denotations $S(\tau) = \{z \in \mathbf{C} : \ |\operatorname{Im} z| < \tau\}$ and $S^*(\tau) = \mathbf{C}\backslash\overline{S(\tau)}$. Obviously, $S(\tau)$ is the infinite strip bounded by the lines $\operatorname{Im} z = \pm\tau$. We assume that $S(\infty) = \mathbf{C}$, $S(0) = \emptyset$, $S^*(0) = G$ and $S^*(\infty) = \emptyset$.

Theorem 1 [3, (**IV.3.1**)]. (*a*) *If the series* (2) *converges at a point* $z_0 \in G$, *then it is uniformly convergent on every compact subset of the strip* $S(\tau_0)$ *with* $\tau_0 = |Imz_0|$.

(*b*) *If*

$$\tau_0 = \max\left[0, -\limsup_{n\to+\infty} (2n+1)^{-1/2} \log|(2n/e)^{n/2} a_n|^{\frac{1}{n}}\right], \tag{3}$$

then the series (2) *is uniformly convergent on every compact subset of the strip* $S(\tau_0)$ *and diverges in* $S^*(\tau_0)$.

*2000 Mathematics Subject Classification: 33C45, 40G05.

Key words: Hermite polynomials, Hermite series, summability.

Remarks. (1) The equality (3) can be regarded as a formula of Cauchy-Hadamard type for the Hermite series of the kind (2).

(2) In the proofs of (a) and (b) it is used the asymptotic formula (1).

An important property of the series (2) is given by the following

Theorem 2 [3, (IV.4.6)]. *If the series* (2) *is convergent at a point* $z_0 \in G$, *then*

$$\lim_{z \to z_0} \sum_{n=0}^{+\infty} a_n H_n(z) = \sum_{n=0}^{+\infty} a_n H_n(z_0),$$

when $z \in S(\tau_0)$ $(\tau_0 = |\operatorname{Im} z_0|)$ *and* $|z - z_0| = O(|\operatorname{Im}(z - z_0)|)$.

This proposition is called Abel's theorem for the series of the kind (2).

Let $z_0 \in G$, $\tau_0 = |\operatorname{Im} z_0|$,

$$H_n(z, z_0) = \frac{H_n(z)}{H_n(z_0)}, \qquad n = 0, 1, 2, \dots,$$

and

$$D(z_0) = \{S(\tau_0) \backslash (-\infty, +\infty)\} \cap \{z \in \mathbf{C} : \operatorname{Re} z = z_0\}.$$

A series

$$\sum_{n=0}^{+\infty} a_n \tag{4}$$

is called $H(z, z_0)$ *summable* (or *Hermite summable at the point* z_0) if the series

$$\sum_{n=0}^{+\infty} a_n H_n(z, z_0) \neq \infty$$

is convergent in strip $S(\tau_0)$ and there exists

$$\lim_{z \to z_0} \sum_{n=0}^{+\infty} a_n H_n(z, z_0) \neq \infty,$$

when $z \in D(z_0)$.

Every $H(z, z_0)$-summation is regular and this property is a corollary of Theorem 2.

Our aim here is to prove a Tauberian theorem of Litlewood type for the Hermite summation, namely

Theorem 3. *Let* $z_0 \in G$. *If the series* (4) *is* $H(z, z_0)$*-summable and*

$$a_n = O(n^{-1}) \quad (n \to \infty) \tag{5}$$

then it is convergent.

Proof. We assume that $a_0 = 0$, which is not an essential restriction. Let $\varepsilon \in [0,\ \tau)$ and for definiteness assume that $\operatorname{Im} z_0 = \tau$. Then, using the asymptotic formula (1), we obtain that

$$H_n(\operatorname{Re} z_0 + i(\tau - \varepsilon), z_0) = Q(\tau, \varepsilon) \exp(-\varepsilon\sqrt{2n+1})\{1 + q_n(\tau, \varepsilon)\},$$

where $Q(\tau, \varepsilon) \neq 0$, $\lim_{\varepsilon \to 0} Q(\tau, \varepsilon) = 1$, $q_n(\tau, 0) = 0$ and

$$q_n(\tau, \varepsilon) = O(n^{-1/2}) \quad (n \to \infty)$$

uniformly with respect to ε on any interval $[0, \omega\tau]$ with $\omega \in (0, 1)$.

Suppose that $\omega = 1/2$ and $\varepsilon \in (0, \tau/2]$. We define

$$f_1(\tau, \varepsilon) = Q(\tau, \varepsilon) \sum_{n=1}^{\infty} a_n \exp(-\varepsilon\sqrt{2n+1}), \tag{6}$$

$$f_2(\tau, \varepsilon) = Q(\tau, \varepsilon) \sum_{n=1}^{\infty} a_n \exp(-\varepsilon\sqrt{2n+1}) q_n(\tau, \varepsilon) \tag{7}$$

and

$$f(\tau, \varepsilon) = f_1(\tau, \varepsilon) + f_2(\tau, \varepsilon). \tag{8}$$

The assumption that the series (4) is Hermite summable at the point z_0 implies the existence of

$$\lim_{\varepsilon \to 0} f(\tau, \varepsilon) \neq \infty. \tag{9}$$

It is easy to prove that the series in the right hand of (7) is uniformly convergent with respect to $\varepsilon \in [0, \tau/2]$. Hence, there exists

$$\lim_{\varepsilon \to 0} f_2(\tau, \varepsilon) \neq \infty. \tag{10}$$

Then (9), (10), and (8) imply $\lim_{\varepsilon \to 0} f_1(\tau, \varepsilon) \neq \infty$. Further, using that $Q(\tau, \varepsilon) \neq 0$, $\lim_{\varepsilon \to 0} Q(\tau, \varepsilon) = 1$, and (6), we obtain

$$\lim_{\varepsilon \to 0} \sum_{n=1}^{\infty} a_n \exp(-\varepsilon\sqrt{2n+1}) \neq \infty.$$

But the equality (5) implies

$$a_n = O\left(\frac{\sqrt{2n+1} - \sqrt{2n-1}}{\sqrt{2n+1}}\right) \quad (n \to \infty).$$

By Theorem 104 from [4] it follows that the series (4) is convergent. □

From Theorem 3 it follows Tauberian's theorem for the Hermite summation, namely

Theorem 4. *Let $z_0 \in G$. If the series (4) is $H(z, z_0)$-summable and $a_n = o(n^{-1})$ $(n \to \infty)$, then it is convergent.*

Let us note that the following assertion holds:

Theorem 5. *Let $z_0 \in G$, $0 < \delta < 1$ and*

$$-K_1 n^{-1} < a_n \leq K_2 n^{-\delta} (n = 1, 2, \dots),$$

where K_1 and K_2 are a positive contstants. If the series (4) is $H(z, z_0)$-summable, then it is convergent.

The proof of this theorem is similar to that of Theorem 3, but by using Theorem 103 from [4].

A simple corollary of Theorem 5 is:

Theorem 6. *Let $z_0 \in G$, $0 < \delta < 1$ and $0 \leq a_n \leq K n^{-\delta}$ $(n = 1, 2, \dots)$, where K is a positive constant. If the series (4) is $H(z, z_0)$-summable, then it is convergent.*

We call this statement Tauberian theorem of Landau's type for the $H(z, z_0)$-summation.

Finally, we note that the following simple assertion holds:

Theorem 7. *Let τ be a real number with $\tau \neq 0$. The series* (4) *is $H(z, i\tau)$-summable if and only if it is $H(z, -i\tau)$-summable.*

Theorem 7 gives rise of the following hypothesis:

If the series (4) *is $H(z, z_0)$-summable, then it is $H(z, zeta)$-summable for every ζ with* $|\operatorname{Im}\zeta| = |\operatorname{Im} z_0|$.

REFERENCES

[1] G. Szego. Orthogonal Polynomials, AMS Colloquium Publications, **23**, 1939.

[2] G. Boychev. On Hermite series in the complex domain. *C. R. Acad. Bulgare Sci.*, **61** (2008), No 6, 689–694.

[3] P. Rusev. Classical Orthogonal Polynomials and Their Associated Functions in Complex Plane. Sofia, 2005.

[4] G. Hardy. Divergent series, Oxford, 1949

Georgi S. Boychev
6000 Stara Zagora, Bulgaria
e-mail: GBoychev@hotmail.com

ВЪРХУ ЕДИН МЕТОД НА СУМИРАНЕ, ДЕФИНИРАН ЧРЕЗ ПОЛИНОМИТЕ НА ЕРМИТ

Георги С. Бойчев

В статията се разглежда метод за сумиране на редове, дефиниран чрез полиномите на Ермит. За този метод на сумиране са дадени някои Тауберови теореми.