

**ASYMPTOTIC EXPANSION OF SOLUTION FOR ALMOST
 REGULAR AND WEAKLY PERTURBED SYSTEMS OF
 ORDINARY DIFFERENTIAL EQUATIONS***

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In the paper is applied the Poincare method for solving almost regular nonlinear boundary-value problems with general boundary conditions. We assume that the differential system contains an additional function, which defines the perturbation as singular. Under certain conditions we get the asymptotics of the solution.

1. Introduction. Consider the boundary-value problems

$$(1) \quad \frac{dx}{dt} = A(t)x + \varphi(t) + \varepsilon F(x, t, \varepsilon, f(t, \varepsilon)), \quad t \in [a, b],$$

$$(2) \quad l(x) = h,$$

where ε is a small positive parameter.

The coefficients of the problem (1), (2) satisfy the conditions:

(C1) $A(t)$ is $(n \times n)$ -matrix with elements of continuous functions of $t \in [a, b]$ and $\varphi(t)$ is a vector-function of the class $C([a, b])$;

(C2) The function $F(x, t, \varepsilon, f(t, \varepsilon))$ is a vector-function, having continuous partial derivatives with respect to all arguments up to $(n + 2)$ in the domain $G = D_x \times [a, b] \times [0, \bar{\varepsilon}] \times D_f$, where $D_x \subset \mathbb{R}^n$ is in some neighborhood of the solution $x^{(0)}(t)$ of the generate system ($\varepsilon = 0$)

$$\frac{dx^{(0)}}{dt} = A(t)x^{(0)} + \varphi(t), \quad l(x^{(0)}) = h, \quad t \in [a, b],$$

$D_f \subset \mathbb{R}^p$ is bounded and closed domain, $0 < \bar{\varepsilon} \ll 1$. The function $f = f(t, \varepsilon)$ is smooth in the domain $G_1 = [a, b] \times (0, \bar{\varepsilon}]$ and its values belongs to D_f .

(C3) l is linear, bounded vector functional, $l \in (x : C[a, b] \rightarrow \mathbb{R}^n, \mathbb{R}^n)$.

We assume that the function $f(t, \varepsilon)$ of (1) contains singular elements (for example $f = f(\exp(-t/\varepsilon), \sin(t/\varepsilon))$). It shows that we look at almost regular boundary value problems and almost nonlinear boundary problem. Almost regular Cauchy problems are considered in [2].

The existence and uniqueness of the solution of the problem (1) (2) were proved in the work [4] in a form of uniformly convergent power series with respect to ε . It was introduced an additional parameter in [4], thus generalizing the method of Poincare on

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the class of boundary-value problems containing singular functions. The results of [4] have been applied in [1] for systems of type (1) with integral boundary conditions. In this work, under certain conditions, we get the asymptotics of the solution, obtained in [4]. The construction of the asymptotic expansion of solution of the problem (1), (2) is based on the pseudoinverse matrices and orthogonal projections. Algorithm to finding it by using pseudoinverse matrices can be seen for example in [5], [6], [3].

If $x = (x_1, \dots, x_n)$, then as standard norm of the vector x we understand $\|x\| = \max_{i=1, n} \|x_i\|$, but as standard norm of the matrix $A = (a_{ij})$ we understand $\|A\| = \max_{i=1, n} \sum_{j=1}^n |a_{ij}|$. As norm of the linear operator l we understand $\|l(\psi)\| \leq \bar{b}\|\psi\|$, $\bar{b} > 0$.

2. Auxiliary results. Instead of boundary value problems (1) (2), we consider the problem with two parameters [2] $\varepsilon \in [0, \bar{\varepsilon}]$ and $\mu \in (0, \bar{\mu}]$

$$(3) \quad \begin{aligned} \frac{dz}{dt} &= A(t)z + \varphi(t) + \varepsilon F(z, t, \varepsilon, f(t, \mu)), \quad t \in [a, b], \\ l(z) &= h, \end{aligned}$$

Problem (3) is regularly perturbed with respect to the small parameter ε and the solution can be constructed in the form of a power series:

$$(4) \quad z(t, \varepsilon, \mu) = \sum_{k=0}^{\infty} z^{(k)}(t, \mu) \varepsilon^k.$$

Then, the solution of (1), (2) has the form

$$(5) \quad x(t, \varepsilon) = \sum_{k=0}^{\infty} z^{(k)}(t, \varepsilon) \varepsilon^k.$$

By the condition (C2) the function F is analytic in G and it is possible to be presented in the form

$$(6) \quad F(z, t, \varepsilon, f(t, \mu)) = \sum_{k=0}^{\infty} B_k(t, \mu) z^{k+1} \varepsilon^k,$$

where $B_k(t, \mu)$ is $(n \times n)$ -matrix with continuous elements in the domain G_1 . We put (4) in (6) and obtain the series

$$\begin{aligned} F\left(\sum_{k=0}^{\infty} z^{(k)}(t, \mu) \varepsilon^k, t, \varepsilon, f(t, \mu)\right) \\ = \sum_{k=0}^{\infty} F_k(t, \mu, z^{(0)}(t, \mu), \dots, z^{(k)}(t, \mu)) \varepsilon^k = \sum_{k=0}^{\infty} F^k(t, \mu) \varepsilon^k, \end{aligned}$$

where

$$F^k(t, \mu) = B_0(t, \mu) z^{(k)} + g_k(t, \mu, z^{(0)}, \dots, z^{(k-1)}), \quad k \geq 0, \quad g_0 \equiv 0.$$

Let $U(t, s)$ be the Cauchy matrix for the system $\dot{z} = A(t)z$. Then, the solution of the Cauchy problem $\dot{z} = A(t)z$, $z(a) = \xi$ has the form $z(t) = U(t, a)\xi$.

We assume that for $(n \times n)$ -matrix $D = l(U(\cdot, a))$ the following condition is satisfied:

(C4) rank $D = r < n$.

Denote by D^+ the unique pseudoinverse $(n \times n)$ -matrix of the matrix D , with P_D and P_{D^*} the orthoprojectors $P_D : \mathfrak{R}^n \rightarrow \ker D$, $P_{D^*} : \mathfrak{R}^n \rightarrow \ker D^*$, $D^* = D^T$. From the conditions (C4) it follows that $\text{rank } P_D = \text{rank } P_{D^*} = n - r = p$. Then, the matrices D and D^* contain p linearly independent columns and p linearly independent rows, respectively. Let P_{D_p} be $(n \times p)$ matrix, consisting of p linearly independent columns on the matrix P_D and $P_{D_p^*}$ $(p \times n)$ -matrix, consisting of p linearly independent lines of the matrix P_{D^*} . The solution of the algebraic system $Dy = q$ under the condition (C4) has the form $y = P_{D_p}\eta + D^+q$, $\eta \in R^p$ if and only if $P_{D_p^*}q = 0$.

We introduce the notations:

$$\Phi_p(t) = U(t, a)P_{D_p} - (n \times p)\text{-matrix};$$

$$Q(\mu) = P_{D_p^*}l \left(\int_a^{(\cdot)} U(\cdot, s) B_0(s, \mu) \Phi_p(s) ds \right) - (p \times p)\text{-matrix}$$

Theorem 1 [4]. *Let the conditions (C1)–(C4), $P_{D_p^*}\bar{h} = 0$, $\bar{h} = h - l \left(\int_a^{(\cdot)} U(\cdot, s) \varphi(s) ds \right)$ and $\det Q(\mu) \neq 0 \forall \mu \in (0, \varepsilon^*]$ be satisfied. Then, in the domain G_1 there exist uniquely determined continuously differentiable functions $z^{(k)}(t, \mu)$, $k \geq 0$ with respect to $t \in [a, b]$ and continuous for $\mu \in (0, \varepsilon^*]$, which satisfy the boundary problems*

$$\begin{aligned} \dot{z}^{(0)} &= A(t)z^{(0)} + \varphi(t), \quad l(z^{(0)}) = h, \\ \dot{z}^{(k)} &= A(t)z^{(k)} + F_{k-1}(t, \mu, z^{(0)}, \dots, z^{(k-1)}), \quad k \geq 1, \end{aligned}$$

Theorem 2 [4]. *There exists $\varepsilon^* > 0$, so that the series (4) is uniformly convergent in $G_2 = \{(t, \varepsilon, \mu) | a \leq t \leq b, 0 \leq \varepsilon \leq \varepsilon^*, 0 < \mu \leq \varepsilon^*\}$ and its sum is a solution of the problem (3).*

Moreover, by Theorem 2 it is proved that the functions $z^{(k)}(t, \mu)$ satisfy the inequalities $\|z^{(k)}(t, \mu)\| \leq C$, $k \geq 0$ in the set $[a, b] \times (0, \varepsilon^*]$.

3. Main results. We introduce the partial sums of series (4), (5) and the function H_n :

$$(7) \quad \begin{aligned} X_n(t, \varepsilon) &= \sum_{k=0}^n z^{(k)}(t, \varepsilon) \varepsilon^k, & Z_n(t, \varepsilon, \mu) &= \sum_{k=0}^n z^{(k)}(t, \mu) \varepsilon^k, \\ H_n(u, t, \varepsilon, \mu) &= \varepsilon F(u + Z_n, t, \varepsilon, f) - \sum_{k=1}^n F_{k-1}(t, \mu, z^{(0)}, \dots, z^{(k-1)}) \varepsilon^k. \end{aligned}$$

Lemma 3. *There exists a constant ε_1 , $0 < \varepsilon_1 \leq \bar{\varepsilon}$, so that in the domain $G_3 = \{(t, \varepsilon, \mu) | a \leq t \leq b, 0 \leq \varepsilon \leq \varepsilon_1, 0 < \mu \leq \varepsilon_1\}$ the function $H_n(0, t, \varepsilon, \mu)$ satisfies the inequality*

$$\|H_n(0, t, \varepsilon, \mu)\| \leq C\varepsilon^{n+1}, \quad C > 0.$$

Lemma 3 can be proved inductively.

In (C2) the set D_x is a neighborhood of the generated solution $z^{(0)}$. Then, the sum of the series (4) and its partial sums also belong to this neighborhood. Therefore, for $\delta > 0$ and $\|z^{(0)}\| < \delta$, we have $\|z\| \leq \rho < \delta$.

Lemma 4. *There exist $\bar{\delta}$, $0 < \bar{\delta} < \delta$ and $0 < \varepsilon_1 \leq \bar{\varepsilon}$, so that for $\|\bar{u}\| \leq \bar{\delta}$ and $\|\bar{v}\| \leq \bar{\delta}$,*

$t \in [a, b]$, $0 \leq \varepsilon \leq \varepsilon_1$, $0 < \mu \leq \varepsilon_1$ the function $H_n(u, t, \varepsilon, \mu)$ satisfies the inequality

$$\|\Delta H_n\| = \|H_n(\bar{u}, t, \varepsilon, \mu) - H_n(\bar{\bar{u}}, t, \varepsilon, \mu)\| \leq C\varepsilon\|\bar{u} - \bar{\bar{u}}\|.$$

In the proof of Lemma 4 is substantially used that the function F has continuous partial derivatives with respect to x .

Let the following conditions be fulfilled:

$$(C5) \quad P_{D_p^*} b(\varepsilon, \mu) = 0, \quad b(\varepsilon, \mu) = -l \left(\int_a^{(\cdot)} U(\cdot, s) H_n(u, s, \varepsilon, \mu) ds \right);$$

(C6) The function $\xi = \xi(\varepsilon, \mu)$ satisfies the inequality

$$\|\xi(\varepsilon, \mu)\| \leq \tilde{b}\varepsilon^{n+1}, \quad \tilde{b} > 0, \quad 0 < \varepsilon \leq \bar{\varepsilon}, \quad 0 < \mu \leq \bar{\mu}.$$

Theorem 5. *Let the conditions (C1)–(C6) be satisfied. Then, there exist positive constants ε^* and C^* such that for $t \in [a, b]$ and $\varepsilon \in (0, \varepsilon^*]$, the unique solution $x(t, \varepsilon)$ of the problem (1), (2) satisfies the inequality*

$$\|x(t, \varepsilon) - X_n(t, \varepsilon)\| \leq C^* \varepsilon^{n+1}.$$

Proof. We accomplish the change

$$(8) \quad u(t, \varepsilon, \mu) = z(t, \varepsilon, \mu) - Z_n(t, \varepsilon, \mu).$$

It suffices to show that $\|u(t, \varepsilon, \mu)\| \leq C^* \varepsilon^{n+1}$.

We put (8) in (3) and obtain that the remainder term of the series (4) is a solution of the following boundary-value problem

$$(9) \quad \begin{aligned} \frac{du}{dt} &= A(t)u + H_n(u, t, \varepsilon, \mu), \\ l(u) &= 0, \end{aligned}$$

where $H_n(u, t, \varepsilon, \mu)$ is defined by (7).

The differential system (9) is equivalent to the equation

$$(10) \quad u(t, \varepsilon, \mu) = U(t, a)\eta + \int_a^t U(t, s)H_n(u, s, \varepsilon, \mu) ds, \quad \eta \in R^n$$

We put (10) into the boundary conditions from (9) and to determine the constant vector η , we obtain the system

$$(11) \quad D\eta = b(\varepsilon, \mu),$$

where $b(\varepsilon, \mu) = -l \left(\int_a^{(\cdot)} U(\cdot, s)H_n(u, s, \varepsilon, \mu) ds \right)$.

According to (C4), the system (11) has the solution

$$(12) \quad \eta = P_{D_p} \xi(\varepsilon, \mu) + D^+ b(\varepsilon, \mu),$$

where $\xi(\varepsilon, \mu)$ is an arbitrary vector function at $0 < \varepsilon \leq \bar{\varepsilon}$ and $0 < \mu \leq \bar{\mu}$, if and only if the condition (C5) is satisfied. We assume that the function $\xi(\varepsilon, \mu)$ satisfies the condition

(C6). We put (12) in (10) and obtain the integral equation

$$(13) \quad u(t, \varepsilon, \mu) = U(t, a)P_{D_p}\xi(\varepsilon, \mu) + U(t, a)D^+b(u, \varepsilon, \mu) + \int_a^t U(t, s)H_n(u(s, \varepsilon, \mu), s, \varepsilon, \mu) ds.$$

We point out that $b(u, \varepsilon, \mu)$ depends on u through H_n . We apply the method of successive approximations to the integral equations (13)

$$(14) \quad \begin{aligned} u_0(t, \varepsilon, \mu, \xi) &= 0, \\ u_k(t, \varepsilon, \mu, \xi) &= U(t, a)\xi(\varepsilon, \mu) + U(t, a)D^+b(u_{k-1}, \varepsilon, \mu) \\ &\quad + \int_a^t U(t, s)H_n(u_{k-1}(s, \varepsilon, \mu), s, \varepsilon, \mu)ds, \quad k \geq 1 \end{aligned}$$

One can find positive constants b_i , $i = 1, 2, 3$, such that $\|\Phi_p(t)\| \leq b_1$, $\|U(t, s)\| \leq b_2$, $\|D^+\| \leq b_3$, at $t \in [a, b]$, $s \in [a, t]$.

From the properties of the function $H_n(u, t, \varepsilon, \mu)$ (Lemma 3 and Lemma 4) we get

$$\begin{aligned} \|u_1 - u_0\| &= \|\Phi_p(t)\xi(\varepsilon, \mu) + U(t, a)D^+b(u_0, \varepsilon, \mu) + \int_a^t U(t, s)H_n(u_0(s, \varepsilon, \mu), s, \varepsilon, \mu)ds\| \\ &\leq \|\Phi_p(t)\| \|\xi(\varepsilon, \mu)\| + \|U(t, a)\| \|D^+\| \|b(0, \varepsilon, \mu)\| + \int_a^t \|U(t, s)H_n(0, s, \varepsilon, \mu)\| ds \\ &\leq b_1\tilde{b}\varepsilon^{n+1} + b_2b_3\bar{b}b_2\|H_n(0, s, \varepsilon, \mu)\|(b-a) + b_2C\varepsilon^{n+1}(b-a) \\ &\leq b_1\tilde{b}\varepsilon^{n+1} + b_2^2b_3\bar{b}(b-a)C\varepsilon^{n+1} + b_2(b-a)C\varepsilon^{n+1} = (C_1 + C_2 + C_3)\varepsilon^{n+1} = \frac{\nu}{2}, \end{aligned}$$

where $\nu = 2(C_1 + C_2 + C_3)\varepsilon^{n+1}$, $C_1 = b_1\tilde{b}$, $C_2 = b_2^2b_3\bar{b}C(b-a)$, $C_3 = b_2C(b-a)$.

Using Lemma 4, we obtain

$$\|u_2 - u_1\| \leq C_2\varepsilon\|u_1 - u_0\| + C_3\varepsilon\|u_1 - u_0\| \leq \bar{C}\varepsilon\|u_1 - u_0\| \leq \bar{C}\varepsilon\frac{\nu}{2} \leq \frac{1}{2} \cdot \frac{\nu}{2},$$

if $\varepsilon \leq \varepsilon_2 = \frac{1}{2\bar{C}}$. Inductive approach shows that

$$\|u_k - u_{k-1}\| \leq \frac{1}{2^{k-1}} \cdot \frac{\nu}{2} \quad \forall k \geq 1, \forall t \in [a, b], \varepsilon \in (0, \varepsilon_2], \mu \in (0, \varepsilon_2], \|u_k\| \leq \delta, \|u_{k-1}\| \leq \delta.$$

Then,

$$\begin{aligned} \|u_k(t, \varepsilon, \mu)\| &\leq \|u_k - u_{k-1}\| + \|u_{k-1} - u_{k-2}\| + \cdots + \|u_2 - u_1\| + \|u_1 - u_0\| \\ &\leq \frac{1}{2^{k-1}} \frac{\nu}{2} + \frac{1}{2^{k-2}} \frac{\nu}{2} + \cdots + \frac{1}{2} \frac{\nu}{2} + \frac{\nu}{2} = \frac{\nu}{2} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{k-1}} \right) \\ &\leq 2 \cdot \frac{\nu}{2} = \nu, \end{aligned}$$

i.e. $\|u_k(t, \varepsilon, \mu)\| \leq \nu = C^*\varepsilon^{n+1}$ at $t \in [a, b]$, $\varepsilon \in (0, \varepsilon^*]$, $\mu \in (0, \varepsilon^*]$ and $C^* = 2(C_1 + C_2 + C_3)$.

The above shows that the sequence of successive approximations (14) converges uniformly to the solution $u(t, \varepsilon, \mu)$ for the problems (9), i.e. a solution of the problem (9)

exists, it is uniquely determines and satisfies the inequality

$$(15) \quad \|u(t, \varepsilon, \mu)\| \leq C^* \varepsilon^{n+1}, \quad t \in [a, b], \quad \varepsilon \in (0, \varepsilon^*], \quad \mu \in (0, \varepsilon^*],$$

where $\varepsilon^* \leq \min(\bar{\varepsilon}, \varepsilon_1, \varepsilon_2)$.

From (8) and (15) it follows that

$$\|z(t, \varepsilon, \mu) - Z_n(t, \varepsilon, \mu)\| \leq C^* \varepsilon^{n+1},$$

which shows that

$$\|x(t, \varepsilon) - X_n(t, \varepsilon)\| \leq C^* \varepsilon^{n+1}.$$

The theorem is proved. \square

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АСИМПТОТИЧНО РЕШЕНИЕ НА ПОЧТИ РЕГУЛЯРНИ И СЛАБО СМУТЕНИ СИСТЕМИ ЗА ОБИКНОВЕНИ ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ

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В работата се прилага методът на Поанкаре за решаване на почти регулярни нелинейни гранични задачи при общи гранични условия. Предполага се, че диференциалната система съдържа сингулярна функция по отношение на малкия параметър. При определени условия се доказва асимптотичност на решението на поставената задача.