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SINGULAR SOLUTIONS OF PROTTER'S PROBLEM FOR A CLASS OF 3-D HYPERBOLIC EQUATIONS*

Nedyu Popivanov, Aleksey Nikolov

For 3-D wave equation M. Protter formulated (1952) some boundary value problems which are three-dimensional analogues of the Darboux problems on the plane. Protter studied these problems in a 3-D domain, bounded by two characteristic cones and by a plane region. Now, more than 50 years later, it is well known that, for an infinite number of smooth functions in the right-hand side, these problems do not have classical solutions and the generalized solution have a strong power type singularity at the vertex of the characteristic cone, which is isolated and does not propagate along the cone. In the present paper we consider the third boundary value problem for the wave equation involving lower order terms with a right-hand function of the form of trigonometric polynomial and give a new upper estimate of possible singularity of the solutions. It is interesting that the solutions of the considered problem have the same order of possible singularity as the solutions of the wave equation without lower order terms.

1. Statement of the problem. We denote the points in $\mathbb{R}^3$ by $(x,t) = (x_1, x_2, t)$ and consider the wave equation involving lower order terms

$$Lu \equiv u_{x_1x_1} + u_{x_2x_2} - u_{tt} + b_1u_{x_1} + b_2u_{x_2} + bu_t + cu = f \tag{1}$$

in the simply connected region

$$\Omega_0 := \{(x,t) : 0 < t < 1/2, t < \sqrt{x_1^2 + x_2^2} < 1 - t\}.$$

The region $\Omega_0 \subset \mathbb{R}^3$ is bounded by the disk

$$\Sigma_0 := \{(x,t) : t = 0, x_1^2 + x_2^2 < 1\}$$

with center at the origin $O(0,0,0)$ and the characteristic surfaces of (1):

$$\Sigma_1 := \{(x,t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2} = 1 - t\},$$

$$\Sigma_{2,0} := \{(x,t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2} = t\}.$$

We treat the following problem

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Problem P_α. Find solutions to (1) in Ω_0 that satisfy the conditions

(2) $$u|_{\Sigma_1} = 0, \quad [u_t + \alpha u]|_{\Sigma_0 \setminus O} = 0,$$

where $\alpha \in C^1(\bar{\Sigma}_0)$.

The adjoint problem to P_α is as follows:

Problem P^*_α. Find a solution of the adjoint equation

$$L^* u \equiv u_{x_1x_1} + u_{x_2x_2} - u_{tt} - (b_1 u)_{x_1} - (b_2 u)_{x_2} - (bu)_t + cu = g \quad \text{in } \Omega_0$$

with the boundary conditions:

$$u|_{\Sigma_{2,0}} = 0, \quad [u_t + (\alpha + b)u]|_{\Sigma_0} = 0.$$

2. Protter's problems for the wave equation (without lower order terms). The following problems were introduced by Protter (see [16]):

Find a solution of the wave equation

(3) $$\Box u \equiv \Delta_x u - u_{tt} \equiv u_{x_1x_1} + u_{x_2x_2} - u_{tt} = f \quad \text{in } \Omega_0$$

with one of the following boundary conditions

$$P1: \quad u|_{\Sigma_0 \cup \Sigma_1} = 0, \qquad P1^*: \quad u|_{\Sigma_0 \cup \Sigma_{2,0}} = 0\,;$$
$$P2: \quad u|_{\Sigma_1} = 0, u_t|_{\Sigma_0} = 0, \quad P2^*: \quad u|_{\Sigma_{2,0}} = 0, u_t|_{\Sigma_0} = 0\,.$$

Protter [16] formulated and investigated both Problems $P1$ and $P1^*$ in Ω_0 as multi-dimensional analogues of the Darboux problem on the plane. It is well-known that the corresponding Darboux problems on $\mathbb{R}^2$ are well posed, which is not true for the Protter's problems in $\mathbb{R}^3$. For recent results concerning the Protter's problems see Edmunds and Popivanov [5], Choi and Park [4], Cher [9], Popivanov and Popov [13, 14, 15].

We formulate the following well-known result Kwang-Chang [17], Popivanov and Schneider [10], presented here in the terms of the polar coordinates $x_1 = \varrho \cos\varphi$, $x_2 = \varrho \sin\varphi$.

Theorem 1. *For all $n \in \mathbb{N}$, $n \geq 4$; a_n, b_n arbitrary constants, the functions*

(4) $$v_n(\varrho, \varphi, t) = t\varrho^{-n} \left(\varrho^2 - t^2\right)^{n-\frac{3}{2}} (a_n \cos n\varphi + b_n \sin n\varphi)$$

are classical solutions of the homogeneous problem $P1^*$ and the functions

(5) $$w_n(\varrho, \varphi, t) = \varrho^{-n} \left(\varrho^2 - t^2\right)^{n-\frac{1}{2}} (a_n \cos n\varphi + b_n \sin n\varphi)$$

are classical solutions of the homogeneous problem $P2^*$.

This theorem shows that for the classical solvability (see Bitsadze [3]) of the problem $P1$ (respectively, $P2$) the function f at least must be orthogonal to all smooth functions (4) (respectively, (5)). The reason of this fact has been found by Popivanov and Schneider in [10], [11], where they announced for Problems $P1$ and $P2$ that there exist singular solutions for the wave equation (3) with power type isolated singularities even for very smooth functions f. First a priori estimates for singular solutions of Protter's Problems $P1$ and $P2$, concerning the wave equation in $\mathbb{R}^3$, were obtained in [12].

3. Estimates of singular solutions of this problem for the equation involving lower order terms. For Problem P_α, i.e. (1)–(2), we refer the reader to [7] and [6]. The case $\alpha(x) \equiv 0$ has been studied by Aldashev [1], Aldashev [2]. Also, we point out that in the case of (1), with nonzero lower order terms, Karatoprakliev [8] obtained a priori estimates, but only for solutions of Problem $P1$ smooth enough in Ω_0.

To formulate known results for Problem P_α we first give the definition of generalized solutions.

Definition 2. *A function $u = u(x_1, x_2, t)$ is called a generalized solution of problem P_α in Ω_0, if*
i) $u \in C^1(\bar{\Omega}_0 \backslash O)$, $[u_t + \alpha(x)u]\big|_{\Sigma_0 \backslash O} = 0$, $u\big|_{\Sigma_1} = 0$,
ii) the equality

$$\int_{\Omega_0} [u_t v_t - u_{x_1} v_{x_1} - u_{x_2} v_{x_2} + (b_1 u_{x_1} + b_2 u_{x_2} + b u_t + cu - f)v] dx_1 dx_2 dt$$

$$= \int_{\Sigma_0} \alpha(x)(uv)(x,0) dx_1 dx_2$$

holds for all v from

$$V_0 := \{v \in C^1(\bar{\Omega}_0) : [v_t + (\alpha + b)v]\big|_{\Sigma_0} = 0,\ v = 0 \text{ in a neighborhood of } \Sigma_{2,0}\}.$$

The Definition 2 assures that generalized solutions of Problem P_α may have singularities on the cone $\Sigma_{2,0}$.

In Grammatikopoulos et al. [6] under appropriate conditions for the coefficients of the general equation (1) are derived results which ensure the existence of many singular generalized solutions of Problem P_α. More precisely, the equation (1) is considered in polar coordinates:

$$Lu = \frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2} u_{\varphi\varphi} - u_{tt} + a_1 u_\varrho + a_2 u_\varphi + b u_t + cu = f, \tag{6}$$

where

$$a_1 := b_1 cos\varphi + b_2 sin\varphi, \quad a_2 := \varrho^{-1}(b_2 cos\varphi - b_1 sin\varphi),$$

and the right-hand side function f is of the form

$$f(\varrho, \varphi, t) = f_n^{(1)}(\varrho, t) \cos n\varphi + f_n^{(2)}(\varrho, t) \sin n\varphi, n \in \mathbb{N}. \tag{7}$$

Further, the coefficients of the equation (6) and the coefficient α from the boundary condition are assumed to have a polar symmetry:

$$a_1 = a_1(|x|, t), \quad a_2 = a_2(|x|, t), \quad b = b(|x|, t), \quad c = c(|x|, t), \quad \alpha = \alpha(|x|). \tag{8}$$

Then, for Problem P_α treated in polar coordinates, i.e. equation (6) with the boundary conditions (2), where the coefficients and the right-hand side are of the special form (7)–(8), the following theorem is proved:

Theorem 3 ([6])**.** *Let $\alpha \geq 0$; a_1, b, $c \in C^1(\bar{\Omega}_0 \backslash O)$, $a_2 \equiv 0$ and*

$$a_1(|x|, t) \geq |b|(|x|, t), \quad a_1(|x|, t) \geq 2|x| c(|x|, t), \quad (x, t) \in \Omega_0.$$

Then for each function

$$f_n(x, t) = |x|^{-n}(|x|^2 - t^2)^{n-1/2} \cos n \left(\arctan \frac{x_2}{x_1}\right) \in C^{n-2}(\bar{\Omega}_0) \cap C^\infty(\Omega_0),$$

$n \in \mathbb{N}$, $n \geq 4$ the corresponding generalized solution u_n of the problem P_α belongs to $C^2(\bar{\Omega}_0 \backslash O)$ and satisfies the estimate

$$|u_n(x,t)|_{t=|x|} \geq c_0|x|^{-n} \left|\cos n \left(\arctan \frac{x_2}{x_1}\right)\right|, \quad 0 < |x| < 1/2,$$

where $c_0 = \text{const} > 0$.

In the same paper one can find a proof of the uniqueness of the treated problem. Analogously to Problems $P1$ and $P2$, we see, the generalized solutions in this theorem have singularities at the vertex O of the cone $\Sigma_{2,0}$ and these singularities do not propagate in the direction of the bicharacteristics. Actually, the zero lower terms and $\alpha \equiv 0$ satisfy the conditions of this theorem, so in this particular case we have Problem $P2$.

On the other hand, T. Hristov, N. Popivanov and M. Schneider in [7] (see Theorem 4.4 there in) obtained some upper bounds for all the solutions of this problem assuming that the coefficients and the right-hand side are continuous functions in $\bar{\Omega}_0$ and are of the form (7)–(8). These upper bounds can be written of the form:

$$|u(x,t)| \leq c_0|x|^{-n-\psi(K)},$$

where c_0 is a positive constant,

$$K := \max\left\{\sup_{\bar{\Omega}_0}|b_1|, \sup_{\bar{\Omega}_0}|b_2|, \sup_{\bar{\Omega}_0}|b|, \sup_{\bar{\Omega}_0}|c|, \sup_{0\leq|x|\leq 1}|\alpha(|x|)|\right\}$$

and $\psi(K)$ is a positive function which grows up as K grows up.

Here we state that this estimate can be improved:

$$|u(x,t)| \leq c_\sigma|x|^{-n-\sigma},$$

where σ is an arbitrary positive number and c_σ is a positive constant depending on σ. That means the maximal possible order of singularity of the solution does not depend on the lower order terms of the equation.

As a result we formulate the following theorem:

Theorem 4. *Let the right-hand function* $f(\varrho,\varphi,t)$ *of the equation* (6) *is a trigonometric polynomial*

$$f = \sum_{n=0}^{l}\{f_n^{(1)}(\varrho,t)\cos n\varphi + f_n^{(2)}(\varrho,t)\sin n\varphi\}, \quad l \in \mathbb{N},$$

and the coefficients are of the form (8). *Let* $f_n^{(i)}, i = 1, 2$, a_1, a_2, b *and* c *are continuous in* $\bar{\Omega}_0$. *Then, the unique generalized solution of Problem* P_α *belongs to* $C^1(\bar{\Omega}_0\backslash O)$ *and*

$$|u(x,t)| \leq c_\sigma|x|^{-l-\sigma}$$

in Ω_0, *where* $\sigma > 0$ *is arbitrary number and* $c_\sigma > 0$ *depends on* σ.

Remark. The smoothness of α is set in the definition of P_α, that is $C^1(\bar{\Sigma}_0)$.

From Theorem 3 we know that there exist generalized solutions achieving this maximal possible order of singularity. Theorems 3 and 4 determine their exact asymptotic behavior.

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Nedyu I. Popivanov, Aleksey Y. Nikolov
Department of Mathematics and Informatics
University of Sofia
1164 Sofia, Bulgaria
e-mail: `nedyu@fmi.uni-sofia.bg`
`lio6kata@yahoo.com`

СИНГУЛЯРНИ РЕШЕНИЯ НА ЗАДАЧАТА НА ПРОТЪР ЗА КЛАС ОТ ТРИМЕРНИ ХИПЕРБОЛИЧНИ УРАВНЕНИЯ

Недю Иванов Попиванов, Алексей Йорданов Николов

През 1952 г. М. Протър формулира нови гранични задачи за вълновото уравнение, които са тримерни аналози на задачите на Дарбу в равнината. Задачите са разгледани в тримерна област, ограничена от две характеристични конуса и равнина. Сега, след като са минали повече от 50 години, е добре известно, че за безброй гладки функции в дясната страна на уравнението тези задачи нямат класически решения, а обобщеното решение има силна степенна особеност във върха на характеристичния конус, която е изолирана и не се разпространява по конуса. Тук ние разглеждаме трета гранична задача за вълновото уравнение с младши членове и дясна страна във формата на тригонометричен полином. Дадена е по-нова от досега известната априорна оценка за максимално възможната особеност на решенията на тази задача. Оказва се, че при по-общото уравнение с младши членове възможната сингулярност е от същия ред като при чисто вълновото уравнение.