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A NOTE ON QUASI-LINDELÖF SPACES*

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The quasi-Lindelöf property was first introduced by Arhangelski in [1], as a strengthening of the weakly Lindelöf property. However, unlike Lindelöf and weakly Lindelöf spaces, very little is known about how quasi-Lindelöf spaces behave under the main topological operations, and how the property relates to separation axioms. In the present paper, we look at several properties of quasi-Lindelöf spaces. We consider several examples: a weakly Lindelöf space which is not quasi-Lindelöf, a product of Lindelöf spaces which is not even quasi-Lindelöf, and a quasi-Lindelöf space which is not ccc. At the end, we pose some open questions.

Note 0.1. All spaces are assumed Hausdorff.

1. Introduction. It is well known that any product of compact spaces is compact, and that the product of even two Lindelöf spaces need not be Lindelöf [3]. Various generalisations of Lindelöf spaces have been considered throughout years and attempts to compare their behaviour under basic topological operations with the behaviour of Lindelöf spaces have been made. One such generalisation – the notion of a weakly Lindelöf space - was introduced by Z. Frolik [4].

Definition 1.1. *A topological space X is called* weakly Lindelöf *if for any open cover $\mathcal{U}$ of X, one can find a countable subfamily $\mathcal{U}' \subseteq \mathcal{U}$ such that $X = \overline{\bigcup \mathcal{U}'}$.*

Unfortunately, unlike compactness and Lindelöfness, that property is not inherited by closed subspaces. Thus A. Arhangelski [1] considered a stronger property:

Definition 1.2. *A topological space X is called* quasi-Lindelöf *if for any closed subset $C \subseteq X$ and any family $\mathcal{U}$ of open in X sets which cover C, a countable subfamily $\mathcal{U}' \subseteq \mathcal{U}$ can be found such that $C \subset \overline{\bigcup \mathcal{U}'}$.*

In fact, Arhangelski defined a more general invariant – the *quasi-Lindelöf* number $\mathrm{qL}(X)$ of a given topological space X:

Definition 1.3. $qL(X) = \omega. \min\{\tau :$ $\forall$ closed $C \subset X, \forall \mathcal{U} \subset \tau_X$ with $C \subseteq \bigcup \mathcal{U}, \exists$ a countable $\mathcal{U}' \subseteq \mathcal{U}$ such that $C \subset \overline{\bigcup \mathcal{U}'}\}$.

He used this to generalise a theorem of Bell, Ginsburgh and Woods [2] for obtaining an upper bound of the cardinality of a topological space.

It follows directly from the definition that any Lindelöf space is quasi-Lindelöf and any quasi-Lindelöf space is weakly Lindelöf. The uncountable discrete subspace has neither

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of the above properties. We use the following proposition to give a non-trivial example of a quasi-Lindelöf space which is not Lindelöf.

Proposition 1.4. *Every separable topological space X is quasi-Lindelöf.*

Proof. Indeed, let $C \subset X$ be closed and let $\mathcal{U}$ be a family of open subsets with $C \subset \bigcup \mathcal{U}$. Take a countable dense subset $A \subset X$ and let $A_1 = A \cap (\bigcup \mathcal{U}) = \{a_1, \ldots, a_n, \ldots\}$. For every n choose $U_n \in \mathcal{U}$ such that $a_n \in U_n$. Consider $V = \bigcup \mathcal{U} \setminus \overline{A_1}$ and note that V is open and $V \cap A = \emptyset$. Hence, V must be the empty set. Therefore, $\bigcup \mathcal{U} \subset \overline{A_1}$. Furthermore, $A_1 \subset \bigcup_{n \in \mathbb{N}} U_n \subset \bigcup \mathcal{U}$. So, we get $\overline{\bigcup \mathcal{U}} = \overline{A_1}$ and, moreover, $C \subset \overline{\bigcup \mathcal{U}} = \overline{\bigcup_{n \in \mathbb{N}} U_n}$. Therefore, C is weakly Lindelöf, as required. □

Using this proposition, we can give the following two examples of quasi-Lindelöf spaces which are not Lindelöf:

Example 1.5. The Sorgenfrey plane $\mathbb{S} \times \mathbb{S}$ is not Lindelöf, but is separable and hence quasi-Lindelöf.

Example 1.6. The Niemytzki plane L is separable, hence, quasi-Lindelöf, but not Lindelöf.

The quasi-Lindelöf and weakly Lindelöf properties coincide in the case of normal spaces (see [8] for the proof). We use ideas from Mysior ([6]) and modify a construction from [9] in order to obtain the following example:

Example 1.7. There exists a weakly Lindelöf space X which is not quasi-Lindelöf (and not even Lindelöf).

Construction. Let $A = \{(a_\alpha, -1) : \alpha < \omega_1\}$ be an ω_1-long sequence in the set $\{(x, -1) : x \geq 0\} \subseteq \mathbb{R}^2$. Let $Y = \{(a_\alpha, n) : \alpha < \omega_1, n \in \omega\}$. Let $a = (-1, -1)$. Finally, let $X = Y \cup A \cup \{a\}$.

We topologize X as follows:

- all points in Y are isolated;
- for $\alpha < \omega_1$ the basic neighborhoods of $(a_\alpha, -1)$ will be of the form
$$U_n(a_\alpha, -1) = \{(a_\alpha, -1)\} \cup \{(a_\alpha, m) : m \geq n\} \text{ for } n \in \omega$$
- the basic neighborhoods of $a = (-1, -1)$ are of the form
$$U_\alpha(a) = \{a\} \cup \{(a_\beta, n) : \beta > \alpha, n \in \omega\} \text{ for } \alpha < \omega_1.$$

Let us point out that A is closed and discrete in this topology. Indeed, for any point $x \in X$ there is a basic neighborhood $U(x)$ such that $A \cap U(x)$ contains at most one point and also that $X \setminus A = \{a\} \cup Y$ is open (because $U_\alpha(a) \subset Y \cup \{a\}$. Hence, X contains an uncountable closed discrete subset and, therefore, it cannot be Lindelöf.

Note that for any open $U \ni a$ the set $X \setminus \overline{U}$ is at most countable. Indeed, for any $\alpha < \omega_1$, $\overline{U_\alpha(a)} = U_\alpha(a) \cup \{(a_\beta, -1) : \beta > \alpha\}$. Hence, $X \setminus \overline{U_\alpha(a)}$ is at most countable.

It is easily seen that X is Hausdorff. Without much effort, it can also be proved that X is Urysohn.

Let us now prove that X is weakly Lindelöf. Let $\mathcal{U}$ be an open cover of X. Then, there exists a $U(a) \in \mathcal{U}$ such that $a \in U(a)$. We can find a basic neighborhood $U_\beta(a) \subset U(a)$. Then, $\overline{U_\beta(a)} \subset \overline{U(a)}$ and hence $X \setminus \overline{U(a)}$ will also be at most countable. Hence $X \setminus \overline{U(a)}$

can be covered by (at most) countably many elements of $\mathcal{U}$, say $\mathcal{U}^*$. Set $\mathcal{U}' = U^* \cup \{U(a)\}$. Then, $X \subseteq \bigcup_{U \in \mathcal{U}'} \overline{U} \subseteq \overline{\bigcup_{U \in \mathcal{U}'} U}$. Therefore, X is weakly Lindelöf.

Now, let us prove that X is not quasi-Lindelöf. Consider the 1-neighborhood of a: $U_1(a) = \{a\} \cup \{(a_\beta, n) : \beta > 1, n \in \omega\}$. We have that $C = X \setminus U_1(a)$ is closed. We show the uncountable family of basic open sets $\mathcal{U} = \{U_0(a_\alpha, -1) : \alpha < \omega_1\}$ forms an open cover of C which has no countable subcover with dense union. Note that the sets $U_0(a_\alpha, -1)$ are closed and open. Indeed, $X \setminus U_0(a_\alpha, -1) = \bigcup \{U_0(a_\beta, -1) : \beta \neq \alpha\} \cup U_{\alpha+1}(a)$. Hence, if we remove even one of the $U_0(a_\alpha, -1)$, the point $(a_\alpha, -1)$ would remain uncovered. Therefore, X is not quasi-Lindelöf.

Lindelöfness is equivalent to requiring that every cover of basic open sets has a countable subcover. We have a similar result here:

Proposition 1.8. *Let X be a topological space. Then the following assertions are equivalent:*

1. *X is quasi-Lindelöf.*
2. *Let $\mathcal{B}$ be a fixed base for X. Then, for any closed subset $C \subset X$ and any cover $\mathcal{U}$ of C with $\mathcal{U} \subset \mathcal{B}$ there is a countable subfamily $\mathcal{U}'$ of $\mathcal{U}$ such that $C \subset \overline{\bigcup \mathcal{U}'}$.*

Proof. The direct statement is trivial.

For the converse, let $C \subset X$ be closed and let $\mathcal{U}$ be a family of open subsets of X covering C, i.e. $C \subset \bigcup \mathcal{U}$. Let $\mathcal{B}$ be any base for the topology of X. For every $U \in \mathcal{U}$ there is a family $\mathcal{V}_U \subset \mathcal{B}$ such that $U = \bigcup \mathcal{V}_U$. Then, $C \subset \bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} (\bigcup \mathcal{V}_U)$. Since X is base quasi-Lindelöf, there exists a countable

$$\mathcal{V}' = \{V_n : n \in \mathbb{N}\} \subset \bigcup_{U \in \mathcal{U}} \left(\bigcup \mathcal{V}_n \right) = \bigcup \mathcal{U}$$

such that $C \subset \overline{\bigcup \mathcal{V}'}$. As above, choose a countable $\mathcal{U}' \subseteq \mathcal{U}$ such that $\bigcup \mathcal{V}' \subset \bigcup \mathcal{U}'$. Then, $C \subset \overline{\bigcup \mathcal{U}'}$ and hence X is quasi-Lindelöf. □

Note that this is independent of the choice of basis, since if $\mathcal{B}_1$ and $\mathcal{B}_2$ are two bases, then the respective (2)-conditions are both equivalent to (1), and, hence, also equivalent to each other.

The following Theorem is proved in [8]:

Theorem 1.9. *If X satisfies the countable chain condition (i.e. is ccc), then X is quasi-Lindelöf.*

This shows that the ccc property implies the quasi-Lindelöf property, which in turn implies that the space is weakly-Lindelöf.

The converse, however, does not hold, as the following example shows.

Example 1.10 [7]**.** The lexicographic square is quasi-Lindelöf (in fact, it is compact), but not ccc.

As we pointed out in the beginning, products of compact spaces is compact, and a product of two Lindelöf spaces might not be Lindelöf. Such products might not even be weakly Lindelöf, as the following example from [5] shows:

Example 1.11. There is a topological space X that is not weakly Lindelöf (and, hence, not quasi-Lindelöf), but which is a product of two Lindelöf spaces.

Hence, neither the weakly Lindelöf nor the quasi-Lindelöf property is productive, i.e. both spaces have the same behaviour with respect to products as the Lindelöf property. For weakly Lindelöf spaces, we have the following result:

Proposition 1.12. *If X is weakly Lindelöf and Y is compact, then $X \times Y$ is weakly Lindelöf.*

A proof can be found in [8]. It is natural to ask whether this can be extended to quasi-Lindelöf spaces, namely:

Open question. Is the product of a quasi-Lindelöf space X and a compact space Y quasi-Lindelöf?

This question is interesting even in the particular case:

Open question. Is the product of the unit interval $[0,1]$ with a quasi-Lindelöf space, quasi-Lindelöf?

The following proposition is a very special case of the first question:

Proposition 1.13. *If $A = \{0,1,2,\dots,n\}$ is a finite discrete set and Y is a quasi-Lindelöf space, then the product $A \times Y$ is quasi-Lindelöf.*

This can be proved by induction. The key step is to prove this for a two-point discrete set:

Lemma 1.14. *If X is quasi-Lindelöf and $Y = \{0,1\}$, then $X \times Y$ is quasi-Lindelöf.*

Proof. Let $C \subset X \times Y$ be closed and $\mathcal{U}$ be an open cover of C. Consider $C_0 = \{x \in X \colon (x,0) \in C\}$ and $C_1 = \{x \in X \colon (x,1) \in C\}$. Then, both C_0 and C_1 are closed in X since C is closed. Moreover, $C \subset (C_0 \times \{0\}) \cup (C_1 \times \{1\})$. Set $\mathcal{U}_0 = \{U \in \mathcal{U} \colon U \cap (C_0 \times \{0\}) \neq \emptyset\}$. Clearly, $\mathcal{U}_0$ is an open cover of $C_0 \times \{0\}$. Since C_0 is closed and X is quasi-Lindelöf, we find a countable subfamily, say $\mathcal{U}'$, of $\mathcal{U}_0$ such that $C_0 \times \{0\} \subset \overline{\bigcup \mathcal{U}'}$ (here the identification of C_0 and $C_0 \times \{0\}$ is obvious). Likewise, we deal with C_1 and find a countable subfamily, say $\mathcal{U}^*$, of $\mathcal{U}$ such that $C_1 \subset \overline{\bigcup \mathcal{U}^*}$. Then, $\hat{\mathcal{U}} = \mathcal{U}' \cup \mathcal{U}^*$ is as it is required, i.e. $C \subset \overline{\bigcup \hat{\mathcal{U}}}$. $\square$

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In the initially submitted version of the paper, the unit interval [0,1] was mistyped as {0,1} in the second open question. Thus it was pointed out by the reviewer that that Lemma 1.14 holds. Here, we provide the proof generously offered.

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ЕДНА БЕЛЕЖКА ВЪРХУ КВАЗИ-ЛИНДЕЛЬОФОВИТЕ ПРОСТРАНСТВА

Петра Г. Стайнова

Квази-линдельофовите пространства са въведени от Архангелски като усилване на слабо-линдельофовите. В тази статия се разглеждат няколко свойства на квази-линдельофовите пространства и се правят сравнения със съответни резултати за линдельофовите и слабо-линдельофовите пространства. Дадени са няколко примера, включително на слабо-линдельофово пространство, което не е квази-линдельофово; на пространство, което е топологично произведение на две линдельофови, но не е дори квази-линдельофово, и на пространство, което е квази-линдельофово, но не Суслиново. Накрая са поставени няколко отворени въпроси.