МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2012
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2012
Proceedings of the Forty First Spring Conference
of the Union of Bulgarian Mathematicians
Borovetz, April 9–12, 2012

EVOLUTION OF SETS SYSTEMS AND HOMOTOPY GROUPS OF SPHERES*

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Evolution of systems of sets on the Euclidean sphere $\mathbb{S}^n$ is investigated. The relationship of such models with homotopy groups of spheres is established. Some combinatorial applications for polytopes are obtained.

1. Introduction. Many computer design or pure geometry models deal with continuous motion of sets, but the "continuity" of the motion may be mathematically formalized in different manners (see [1], [2]). First of all, we need a "distance" between sets. There are several natural candidates for the job, according to the desired properties of the motion model. The most general distance of this kind is the so-called "area-distance" defined in a space M with measure μ as follows: If A and B are measurable subsets of M, then set

$$d(A, B) = \mu(A\Delta B),$$

where $A\Delta B = (A\backslash B) \cup (B\backslash A)$ is the symmetric difference of A and B. This is in fact the distance we deal with in the present article. Note that formally speaking, d is not a metric, but only a pseudometric in the family $\mathcal{M}$ of measurable subsets of M, nevertheless, it becomes metric after appropriate factorization of in $\mathcal{M}$.

Another natural distance is the Hausdorff distance between sets in a metric space (M, ρ)

$$d_H(A, B) = \max\{\sup_{x\in A}\inf_{y\in B}\rho(x, y), \sup_{y\in B}\inf_{x\in A}\rho(x, y)\}.$$

If M is bounded then ρ is a metric in the family of closed subsets. There is a finer metric – the dual Hausdorff, which satisfy to a greater extent the needs of the computer and geometry design:

$$d_{Hd}(A, B) = \max\{d_H(A, B), d_H(M\backslash A, M\backslash B)\}.$$

As pointed above, we deal here with the area distance d which is in many aspects easier to investigate, on the other hand, it gives more general results in some important cases. The analogous problems for the other two metrics give rise to some hard open problems (at least their answers are not known to us).

Now, in order to formalize the concept of continuous motion of a set $A \subset M$, we say that A is moving continuously subject to the deformation A_t, $t \in [0, 1]$ if $A = A_0$, the sets A_t are measurable and the function $\phi(t) = d(A, A_t)$ is continuous in t. For example, the

*2000 Mathematics Subject Classification: Primary 55Q40, 52B11, 05B30.
Key words: area-distance, systems of sets, homotopy groups of spheres.

drying of the water spots on the asphalt is continuous with respect to the area distance, while it is discontinuous in the other two distances.

Roughly speaking, a general problem could be the description of equivalence classes of measurable finite coverings of a manifold (e.g. the Euclidean sphere $\mathbb{S}^n$), satisfying some constraint, with respect to continuous motion in the above sense. If there is no constraint on the coverings, then any two of them are obviously equivalent, so there is no problem here. The situation we deal in the article is about motion classes of finite coverings $\omega = (A_1, \dots, A_k)$ of $\mathbb{S}^n$ with the constraint such that there exists some $\alpha > 0$ so that for any ball B of radius α the intersection of B with some A_i has measure zero (so, in particular, the intersection $\cap A_i$ has measure zero). Let us call such ω "*coverings with null intersection*". For example, if A_i are closed subsets with empty intersection, then it is easily seen ω to be a covering with null intersection. We suppose here that the coverings are ordered, i.e., if we interchange two elements of ω,then we get another covering. Now, it turns out that these motion classes are in a close relation with the homotopy groups of spheres $\pi_*(\mathbb{S}^n)$ (see Theorem 1). Let us recall that the homotopy groups of spheres are not completely understood at all (see [4], [5]) and they present quite irregular and enigmatic behaviour, so the full list of $\pi_m(\mathbb{S}^n)$ is very far from being accomplished.

2. Motion classes and homotopy groups. Let ω be a covering with null intersection of $\mathbb{S}^n$ containing k sets. We call for brevity such a covering a *k-system.* Define the *width* of ω as the supremum of all $\alpha > 0$ such that for any ball B of radius α the intersection of B with some A_i has measure 0. Let us denote it by $\alpha(\omega)$.

Definition 1. *Let ω_0 and ω_1 be two k-systems on $\mathbb{S}^n$: $\omega_\varepsilon = (A_1^\varepsilon, \dots, A_k^\varepsilon)$, $\varepsilon = 0, 1$. We say that ω_0 and ω_1 are **equivalent**, if they may be connected by a family of k-systems $\omega_t = (A_1^t, \dots, A_k^t)$, $t \in [0,1]$ so that the functions $d(A_i^0, A_i^t)$ are continuous in t for any $i = 1, \dots, k$ and moreover, $\inf_{0 \le t \le 1} \alpha(\omega_t) > 0$.*

We denote the equivalence classes of k-systems on $\mathbb{S}^n$ by $\mathcal{A}(n,k)$.

This definition gives rise to a list of curious problems, such as: if we interchange two elements of a k-system ω, then do we get a system equivalent to ω? (Recall, that k-systems are ordered k-tuples of sets.) We shall see that in many cases the answer is negative! Another question is whether a given k-system is *demountable*, i.e. whether it is equivalent to the *trivial* one $\omega^0 = (\mathbb{S}^n, \emptyset, \dots, \emptyset)$. It is natural to call such a system detachable, as the equivalence to ω^0 is a kind of disassembly of the original system. Well, we shall see further that the answer to this question may be easily given in terms of homotopy groups. (For elementary introduction to homotopy groups see for example [3].)

Theorem 1. *The equivalence classes of k-systems on $\mathbb{S}^n$ $\mathcal{A}(n,k)$ are in a one-to-one correspondence with the elements of the homotopy group $\pi_n(\mathbb{S}^{k-2})$.*

Moreover, a geometrical "addition" in $\mathcal{A}(n,k)$ may be defined which corresponds to the addition in $\pi_n(\mathbb{S}^{k-2})$. We shall call sometimes the elements of $\mathcal{A}(n,k)$ "motion classes".

The proof of Theorem 1 relies essentially on a technical lemma.

Definition 2. *Let ω be a k-system in $\mathbb{S}^m$ and $f_t : \mathbb{S}^n \times I \to \mathbb{S}^m$ be a (continuous) homotopy. We say that this homotopy is **regular** over ω, if for any $A \in \omega$ the function*

$d(f_0^{-1}(A), f_t^{-1}(A))$ *is continuous in* t *and, moreover,* $\inf_{0\le t\le 1} \alpha(f_t^{-1}(\omega)) > 0$.

Let us note that the term "regular homotopy" has different meaning in the smooth case with no confusion to our definition.

Now we define some basic $(n+2)$-system in $\mathbb{S}^n$ – the so-called *simplicial* system. To define it, it is convenient to consider $\mathbb{S}^n$ as a subset of $\mathbb{R}^{n+2}$, rather than in $\mathbb{R}^{n+1}$. It is clear, that the sphere $\mathbb{S}^n$ is homeomorphic to the following subset of $\mathbb{R}^{n+2}$:

$$\Sigma^n = \left\{ x \in \mathbb{R}^{n+2} \mid \sum_{i=1}^{n+2} x_i = 1, \sum_{i=1}^{n+2} x_i^2 = 1 \right\}.$$

Define now for any $i = 1, \ldots, n+2$,

$$C_i = \{x \in \Sigma^n \mid |x_i| \ge |x_j| \text{ for any } j = 1, \ldots, n+2\}.$$

This collection of sets $\Omega = (C_1, \ldots, C_{n+2})$ form a standard $(n+2)$-system in Σ^n, but as $\Sigma^n \approx \mathbb{S}^n$ canonically, we refer to Ω as a **simplicial** system on $\mathbb{S}^n$.

Lemma 1. *If two maps* $f, g : \mathbb{S}^m \to \mathbb{S}^n$ *are homotopic, then they are regularly homotopic over the simplicial system* Ω *in* $\mathbb{S}^n$.

We define now the so-called *scan map* which is crucial in the proof of Theorem 1. Let $\xi = (A_1, \ldots, A_k)$ be a k-system in some metric space M with full measure μ. Denote by $B_\alpha(x)$ the open ball of radius $\alpha > 0$ centered at x. Suppose in addition that the measure of each sphere $\partial B_\alpha(x)$ is zero (this is clearly true for manifolds). By the definition of k-system we have $\alpha(\xi) > 0$, take some fixed $\alpha > 0$ such that $\alpha < \alpha(\xi)$. Consider first the map

$$m(x) = \left\{ \frac{\mu(A_i \cap B_\alpha(x))}{\sum_{j=1}^k \mu(A_j \cap B_\alpha(x))} \right\}_{i=1}^k .$$

Then, m is a well-defined continuous map $m : M \to L^{k-1}$ into the hyperplane $L^{k-1} = \left\{ x \in \mathbb{R}^k \mid \sum_{i=1}^k x_i = 1 \right\}$. Let $v_0 \in L^{k-1}$ denote the point with all coordinates equal to $1/k$, then v_0 is not in the image $m(M)$, since $\alpha < \alpha(\xi)$. Thus we may compose m with the radial projection with center v_0:

$$s(x) = \frac{m(x) - v_0}{\|m(x) - v_0\|}.$$

It is clear that s sends M into the sphere $\Sigma^{k-2} = \{x \in L^{k-1} \mid \|x - v_0\| = 1\}$ and since Σ^{k-2} is canonically homeomorphic to $\mathbb{S}^{k-1}$, we may consider s as a map $s : M \to \mathbb{S}^{k-1}$. The map $s(x)$ be called **scan map** associated with system ξ. As it depends on α and ξ, we denote it by $s_\alpha(x, \xi)$.

Proof of Theorem 1. Let ω_0 and ω_1 be two equivalent k-systems on $\mathbb{S}^n$ connected by a family of k-systems ω_t, $t \in [0,1]$. Since by definition $\inf_{0\le t\le 1} \alpha(\omega_t) > 0$, take α so that $0 < \alpha < \inf_{0\le t\le 1} \alpha(\omega_t)$. Then, the scan maps $s_\alpha(x, \omega_0)$ and $s_\alpha(x, \omega_1)$ are connected by a homotopy $s_\alpha(x, \omega_t)$. Therefore, if $[\omega_0]$ is the motion class of ω_0 in $\mathcal{A}(n,k)$, and $[s_\alpha(x, \omega_0)]$ is the homotopy class of the scan map $s_\alpha(x, \omega_0)$ in $\pi_n(\mathbb{S}^{k-2})$, we have defined correctly the correspondence

$$[\omega_0] \to [s_\alpha(x, \omega_0)] : \mathcal{A}(n,k) \to \pi_n(\mathbb{S}^{k-2}). \tag{1}$$

We shall prove that this correspondence is bijective. Let $\gamma \in \pi_n(\mathbb{S}^{k-2})$ and $f : \mathbb{S}^n \to$

$\mathbb{S}^{k-2}$ is such that $[f] = \gamma$. Let Ω be the standard simplicial k-system in $\mathbb{S}^{k-2}$ defined above, then $f^{-1}(\Omega)$ is a k-system in $\mathbb{S}^n$. If $[f^{-1}(\Omega)]$ denotes its class in $\mathcal{A}(n,k)$, then it is easy to see now that the correspondence $\gamma \to [f^{-1}(\Omega)]$ is the inverse to (1). It suffices to show that if f_0 is homotopic to f_1, then $f_0^{-1}(\Omega)$ and $f_1^{-1}(\Omega)$ are equivalent k-systems. But this follows immediately from Lemma 1. Indeed, then f_0 and f_1 are regularly homotopic over Ω with regular homotopy f_t and, then, the family $f_t^{-1}(\Omega)$ defines an equivalence between $f_0^{-1}(\Omega)$ and $f_1^{-1}(\Omega)$. □

3. Applications and comments. 1) Let $\Omega = (C_1, \ldots, C_{n+2})$ be the simplicial system in $\mathbb{S}^n$ and σ be a permutation of the set $\{1, 2, \ldots, n+2\}$. Consider the system $\Omega_\sigma = (C_{\sigma(1)}, \ldots, C_{\sigma(n+2)})$ (recall that our systems are *ordered* collections). It is natural to ask whether Ω and Ω_σ are equivalent $(n+2)$-systems. By means of the scan map, it is easy to see that

$$\Omega \sim \Omega_\sigma \text{ if and only if the permutation } \sigma \text{ is even.}$$

In particular, if we interchange two elements of Ω, the new system Ω' is not equivalent to Ω. So, roughly speaking, if we try to transform continuously Ω into Ω', all $(n+2)$-sets of the system intersect somewhere inevitably.

Another fact is that each Ω_σ is not *demountable*, i.e. it is not equivalent to the *trivial* system $\omega^0 = (\mathbb{S}^n, \emptyset, \ldots, \emptyset)$.

2) Since $\mathcal{A}(n,k)$ turns to be a group, it is natural to define geometrically the opposite $-\gamma$ of some $\gamma \in \mathcal{A}(n,k)$. Let $\gamma = [\omega]$ for some k-system $\omega = (A_1, \ldots, A_k)$ in $\mathbb{S}^n$. Take in $\mathbb{R}^{n+1}$ some reflection with respect to a hyperplane, say $s : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is interchanging the first two coordinates. Then, the system $s(\omega) = (s(A_1), \ldots, s(A_k))$ is representing the opposite of γ: $[s(\omega)] = -\gamma$.

Note that the addition in $\mathcal{A}(n,k)$ may also be described in purely geometrical terms.

3) As it follows from Theorem 1, $\mathcal{A}(n, n+2) = \pi_n(\mathbb{S}^n) = \mathbb{Z}$. Each element of $\pi_n(\mathbb{S}^n)$ is represented by some $f : \mathbb{S}^n \to \mathbb{S}^n$ and equals to the degree $\deg f$. We may obtain from these considerations some purely combinatorial facts about convex polytopes in $\mathbb{R}^{n+1}$.

Let P be a convex polytope in $\mathbb{R}^{n+1}$ with nonempty interior. We say that P is a *general position* polytope, if each vertex $v \in P$ is an intersection of exactly $(n+1)$ faces of P which are in a general position. Let P be such a polytope and let us color its faces with $(n+2)$ colors $1, 2, \ldots, n+2$. Choose an arbitrary color i and let the remaining colors be $i_1, \ldots, i_{n+1}$. Then, to any vertex v, where all the colors $i_1, \ldots, i_{n+1}$ meet, we assign +1 or -1, according to the orientation defined by the coloring in v. More precisely, let P_{i_j} be the face containing v and colored i_j, then $v = \cap_{j=1}^{n+1} P_{i_j}$. Take an arbitrary point a_j lying in the interior of P_{i_j} and consider the vector $e_j = (v, a_j)$. Then, $e_1, \ldots, e_{n+1}$ is a base of $\mathbb{R}^{n+1}$ that defines either positive, or negative orientation of $\mathbb{R}^{n+1}$. Set $\sigma(v) = 1$ in the first case and $\sigma(v) = -1$ in the second case. Finally, define

$$\sigma = \sum_{v \in P^0} \sigma(v),$$

where P^0 denotes the set of all vertices of P.

Then, it turns out that σ is an invariant which does not depend on the initial choice of color i (the omitted one). Furthermore, consider the system $\omega = (A_1, \ldots, A_{n+2})$, where A_j is the union of all faces of P colored with j. Then, ω is demountable on $\partial P \approx \mathbb{S}^n$ if and only if $\sigma = 0$. More generally, two colorings of P with invariants σ and σ', respectively, define equivalent $(n+2)$-systems if and only if $\sigma = \sigma'$. So, it is not difficult to obtain non equivalent systems in a purely geometrical way.

4) It is natural to try to carry out these investigations for the Hausdorff distance d_H, instead of the area-distance d. Then, one obtains motion classes $\mathcal{H}(n,k)$ instead of $\mathcal{A}(n,k)$, but their examination seems to be quite more complicated. It is easy to see that

$$\mathcal{A}(n,k) \subset \mathcal{H}(n,k),$$

while it is not difficult to see that $\mathcal{A}(1,k) \neq \mathcal{H}(1,k)$. On the other hand, it is possible that $\mathcal{A}(n,k) = \mathcal{H}(n,k)$ for $n \geq 2$. The answer is not known to us. Another issue is to consider motion classes $\mathcal{H}d(n,k)$ with respect to the dual Hausdorff distance d_{Hd}. Anyway, the motion properties of k-systems with respect to the Hausdorff metrics are not covered adequately by the approach presented here.

REFERENCES

[1] L. Joskowicz, E. Sacks. Computational kinematics. *Artificial Intelligence*, **51** (1991), 381–416.

[2] D. Mumford. Mathematical theories of shape: Do they model perception? In: Proc. Geometric methods in computer vision Conference, SPIE, vol. **1570** (1991), 2–10.

[3] W.S. Massey. Algebraic Topology: An Introduction. Graduate texts in mathematics, vol. **56**, Springer-Verlag, 1977.

[4] J. Strom. Modern Classical Homotopy Theory. AMS Graduate studies in mathematics, vol. **127**, 2011.

[5] S. O. Kochman. Stable Homotopy Groups of Spheres. Springer Verlag, 1990.

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ЕВОЛЮЦИЯ НА СИСТЕМИ ОТ МНОЖЕСТВА И ХОМОТОПИЧНИ ГРУПИ НА СФЕРИТЕ

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В работата е изследвана еволюцията на системи от множества върху n-мерната евклидова сфера S^n. Установена е връзката на такива системи с хомотопичните групи на сферите. Получени са някои комбинаторни приложения за многостени.