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COUETTE GAS FLOW BETWEEN CYLINDERS WITH DIFFERENT TEMPERATURE*

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The stationary Couette gas flow between rotating inner cylinder and stationary outer one is studied using DSMC method and numerical solution of a continual model for relatively small (subsonic) speed. Different cases were studied by varying the temperature of the rotating cylinder and the Knudsen number. The purpose of this study is to determine the influence of the cylinder temperature difference on macro-characteristics – density, gas velocity and temperature. The results obtained by both methods are in an excellent agreement for a small Knudsen number Kn = 0.02. These results are important for applications in non-planar microfluidic problems.

1. Introduction. The cylindrical Couette flow is a fundamental problem in the rarefied gas dynamics [4, 3, 6, 10, 14]. As such, its modeling and numerical solving is of a great importance for microfluidics, which serves for theoretical background of the analysis of new emerging Micro Electro Mechanical Systems MEMS [15, 1, 8, 13].

The design of adequate mathematical models of gaseous flows in micro devices is one of the most important tasks of the studies. We consider both molecular and continuum models treating the gaseous flow by using different level of mathematical description. Both models take into account the specific microfluidic effects of gas rarefaction and slip-velocity regime at the solid boundaries [11, 12, 5, 7].

In the present paper we compare results obtained by using the molecular Direct Simulation Monte Carlo (DSMC) method with those calculated by a numerical solution of the continuum Navier-Stokes equations for compressible flow (NS). The aim of the comparison is to illustrate qualitatively the influence of temperature gradient on the macro-characteristics – ρ, V, T.

Both methods are used to model the cylindrical Couette flow for Knudsen numbers 0.02, 0.05, 0.1, subsonic velocities and different cylinder wall temperatures. The aim of the present paper is to study the gas flow for different temperatures of cylinder walls and establish the field of matching decisions in the two methods in terms of number of Knudsen and inner cylinder temperature.

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2. Formulation of the problem and methods of solution. We study a rarefied gas between two coaxial unconfined cylinders (one dimensional, axis-symmetrical problem).

2.1. Continuous model (NS) and numerical simulation. The continuous model is based on the Navier–Stokes equations for compressible fluid, completed with the equations of continuity and energy transport. The governing equations are written as follows:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{V}) = 0, \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} - \frac{v^2}{r}\right) = -\frac{\partial P}{\partial r} - \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{\tau_{\varphi\varphi}}{r}, \tag{2}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{uv}{r}\right) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_{r\varphi}\right), \tag{3}$$

$$\rho c_P \frac{DT}{Dt} = \operatorname{div}(\lambda \operatorname{grad} T) - p \operatorname{div}\mathbf{V} + \mu\Phi, \tag{4}$$

$$P = \rho R T, \tag{5}$$

where $\mathbf{V}$ is the velocity vector, u and v are the velocity components along axis r and φ. A rather standard notation is used in Eqs (1)–(5): ρ is the density and T is the temperature. ρ, P, T, $u, v = f(r,t)$. $\tau_{i,j}$ are the stress tensor components and Φ is the dissipation function [9]. For a perfect monatomic hard spheres gas, the viscosity and the coefficient heat transfer read as [8]:

$$\mu = \mu(T) = C_\mu \rho_0 l_0 V_0 \sqrt{T}, \quad C_\mu = \frac{5}{16}\sqrt{\pi}, \tag{6}$$

$$\lambda = \lambda(T) = C_\lambda \rho_0 l_0 V_0 \sqrt{T}, \quad C_\lambda = \frac{15}{32}\sqrt{\pi}. \tag{7}$$

The above written equations are normalized by using the following scales: for density, $\rho_0 = m n_0$, (m – is the molecular mass, n_0-the average number density), for velocity $V_0 = \sqrt{2RT_0} - R$ is the gas constant, for length – the distance between the cylinders $L = R_2 - R_1$, for time $t_0 = L/V_0$, for temperature $T_0 = T_{w,1}$ – the wall temperature of both cylinders. The Knudsen number is $\mathrm{Kn} = l_0/L$, where the mean free path is l_0 and $\gamma = c_P/c_V = 5/3$ (c_P and c_V. are the heat capacities at constant pressure and constant volume respectively). In this way in the dimensionless model the characteristic number Kn and the constants C_μ and C_λ take part. After the scaling, the same symbols for the dimensionless ρ, P, T, u, v are used.

For the problem (1)–(4), first-order slip boundary conditions are imposed at the both walls, which can be written directly in dimensionless form as follows [16, 13]:

$$v \mp 1.1466\,\mathrm{Kn}_{\mathrm{local}}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) = V_i, \tag{8}$$

$$u = 0, \tag{9}$$

$$T \pm 2.1904\,\mathrm{Kn}\frac{\partial T}{\partial r} = T_i, \tag{10}$$

at $r = R_i$, $i = 1, 2$. In Eqs (8)–(10) $V_i = v_i/V_0$ and $T_i = T_{W,i}/T_0$ are the dimensionless

wall velocity and temperature for both cylinders (v_i, $i = 1, 2$ is the dimensional wall velocity). The boundary conditions are modeled by using the local Knudsen number $\mathrm{Kn}_{\mathrm{local}}$.

$$\mathrm{Kn}_{\mathrm{local}} = \frac{l}{L} = \left(L\sqrt{2}\pi\sigma^2 \frac{\rho}{\rho_0}.n_0 \right)^{-1} = \frac{\rho_0.\,\mathrm{Kn}}{\rho}, \tag{11}$$

where l denote the local mean free path.

The equations of transfer (1)–(4), together with the boundary conditions (8)–(11), and zero initial distributions for u, v and T, formulate the initial unsteady state boundary-value problem. A second order of approximation, implicit difference scheme to solve numerically the formulated problem is used [13]. Starting from the inner cylinder wall M grid knots are introduced along the coordinate r. Thus the difference value problem, for a given time t, is reduced to the solution of 4 linearized systems of M algebraic equations. The obtained algebraic system has a diagonal and weakly filled matrix. An appropriate optimized sparse linear system solver from IMSL Fortran library is applied for solution of the algebraic system. Due to the problem non-linearity, additionally an internal iteration process is used.

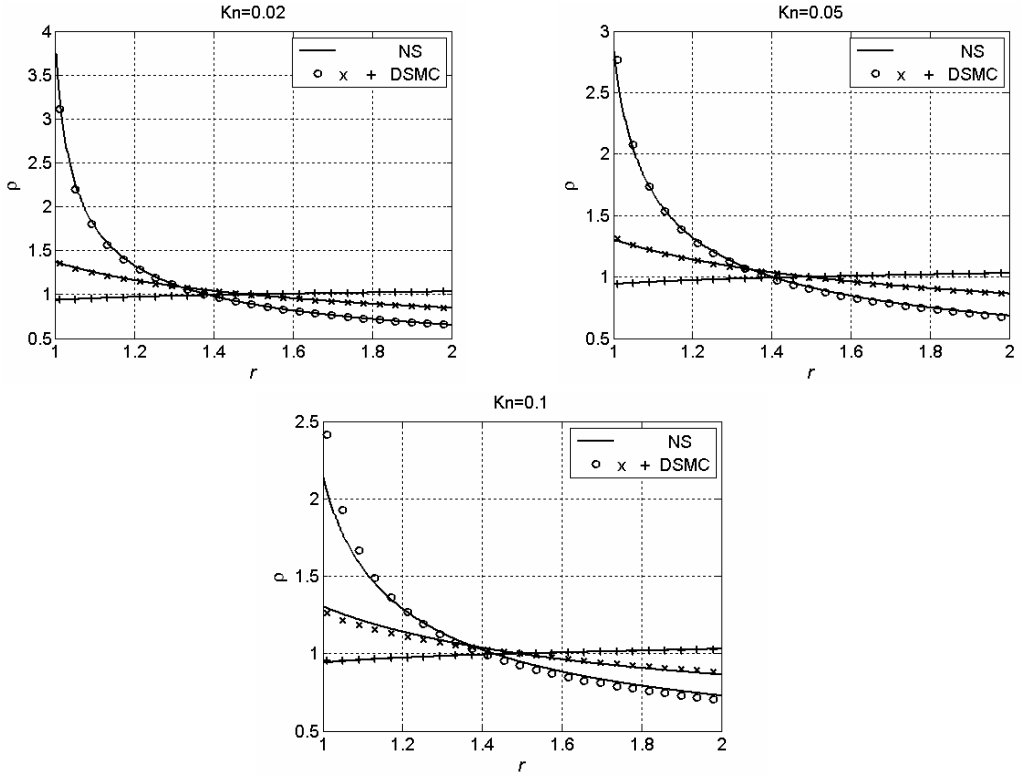

Fig 1. Density profiles "○" – $T1 = 0.1$; "×" – $T_1 = 0.5$, "+" – $T_1 = 1$

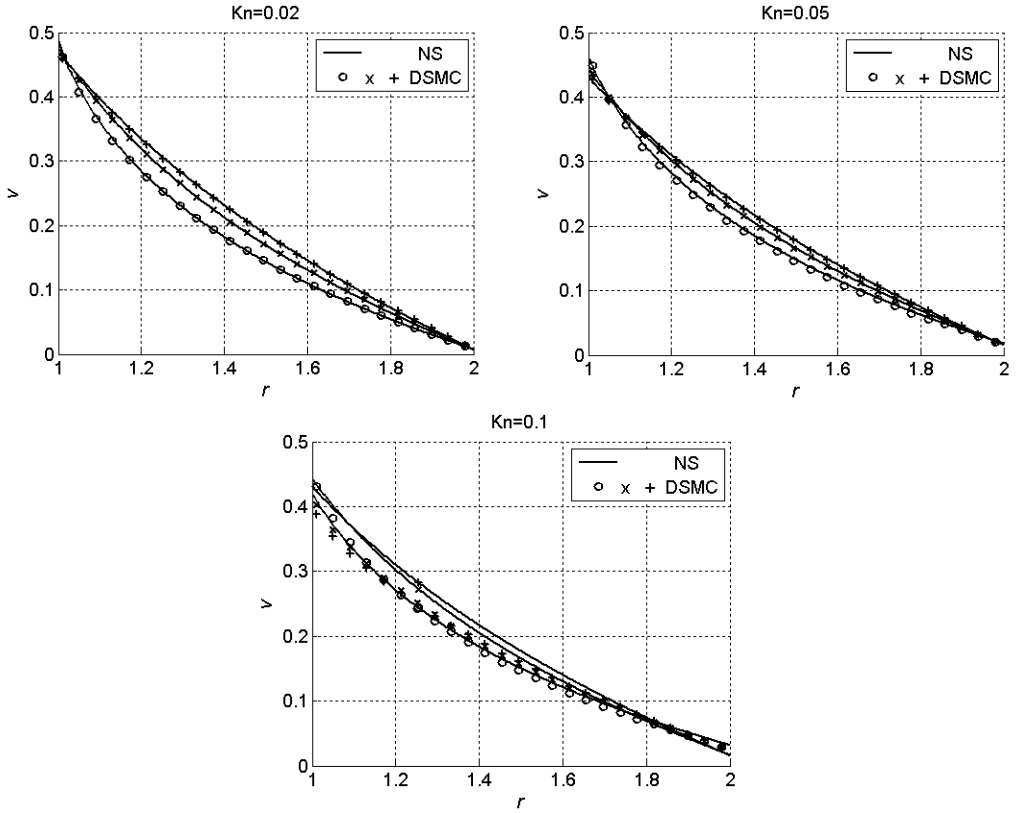

Fig 2. Velocity profiles "∘" – $T1 = 0.1$; "×" – $T_1 = 0.5$, "+" – $T_1 = 1$

2.2. Direct Simulation Monte Carlo (DSMC) Method. The gas considered is simulated as a stochastic system of N particles [2, 3]. All quantities used are non-dimensional, so that the mean free path at equilibrium is equal to 1. The basic steps of simulation are as follows:

A. The time interval $\left[0; \hat{t}\right]$ over which the solution is found, is subdivided into subintervals with step Δt.

B. The space domain is subdivided into cells with sides Δz, Δr.

C. Gas molecules are simulated in gap G using a stochastic system of N points (particles) having position $z_i(t), r_i(t)$ and velocities $\overrightarrow{\xi_i(t)}$.

D. N_m particles are located in the m-th cell at any given time. This number varies during the computer simulation by the following two stages:

Stage 1. Binary collisions in each cell are calculated, whereas particles do not move. Collision modeling is realized using Bird's scheme "no time counter".

Stage 2. Particles move with new initial velocities acquired after collisions, and no external forces act on particles. No collisions are accounted for at this stage.

E. Stage 1 and Stage 2 are repeated until $t = \hat{t}$.

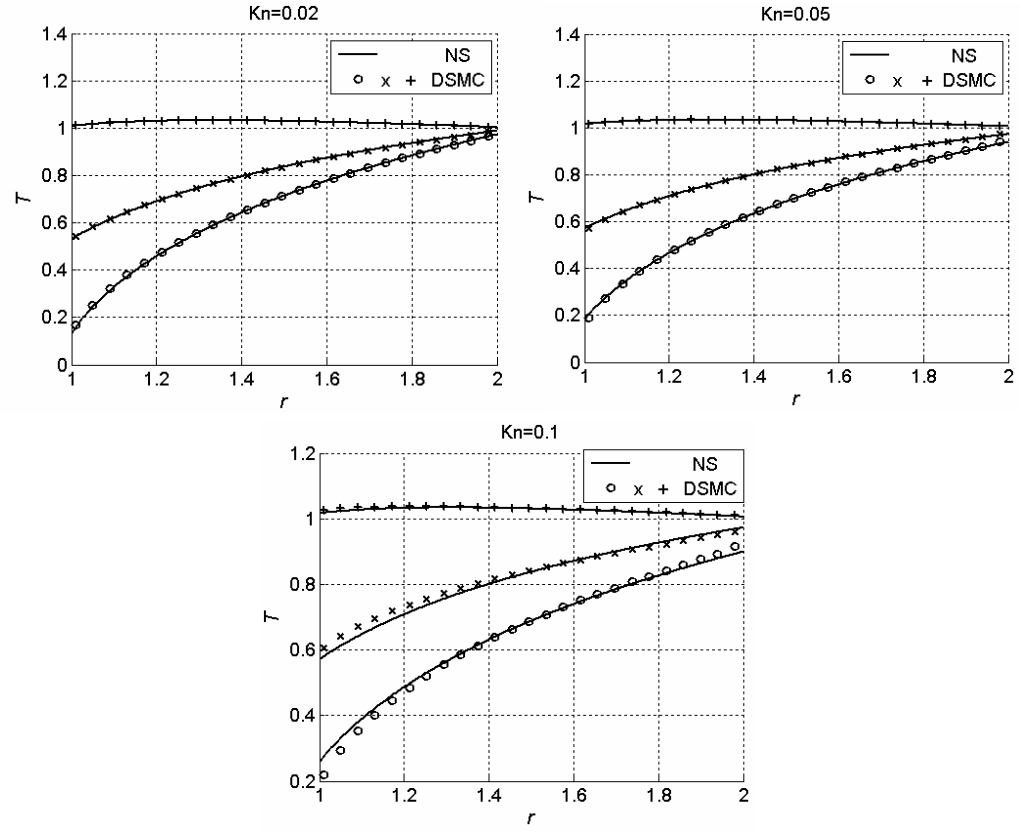

Fig 3. Temperature profiles "$\circ$" – $T1 = 0.1$; "$\times$" – $T_1 = 0.5$, "+" – $T_1 = 1$

F. Flow macro-characteristics (density, velocity, temperature) are calculated as time-averaged when steady regime is attained.

G. Boundary conditions are diffusive at the cylinder walls and periodic along axis Oy. The number of particles (simulators) used in DSMC calculations is 3200000.

3. Numerical results. In a previous research [5, 7, 6] it was found that only for $\mathrm{Kn} = 0.02$ and less both method solutions were in an excellent agreement. And here, the initial studies are for the cases with $\mathrm{Kn} = 0.02$, $V_1 = 0.5$, $V_2 = 0$. The wall temperatures are different: T_1 varies from 0.1 to 1.0 with step 0.1 and $T_2 = 1$.

We have studied the following cases of rarefied gas flow between both cylinders for $\mathrm{Kn} = 0.02$, 0.05 and 0.1:

Case 1: $V_1 = 0.5$, $V_2 = 0$, $T_1 = 1$, $T_2 = 1$, $R_1 = 1$, $R_2 = 2$;

Case 2: $V_1 = 0.5$, $V_2 = 0$, $T_1 = 0.5$, $T_2 = 1$, $R_1 = 1$, $R_2 = 2$;

Case 3: $V_1 = 0.5$, $V_2 = 0$, $T_1 = 0.1$, $T_2 = 1$, $R_1 = 1$, $R_2 = 2$.

The results obtained by both methods are: in an excellent agreement at a small

Knudsen number Kn = 0.02 and Kn = 0.05, in a satisfactory agreement at 0.1 – Figure 1, 2 and 3. By increasing the temperature difference between the cylinder walls at Kn = 0.1 the difference between flow macrocharacteristics is getting larger, while at Kn = 0.02 and Kn = 0.05 it is small. This is due to the fact that at larger Knudsen number the NS model cannot adequately describe the non-equilibrium effects in the gas flow. It is important to note that the temperature varying does not lead to significant differences in the velocity profile, as with increasing of Knudsen number these differences are decreasing.

REFERENCES

[1] K. Aoki, H. Yoshida, T. Nakanishi, A. Garcia. Inverted velocity profile in the cylindrical Couette flow of a rarefied gas. *Phys. Rev. E*, **68** (2003), 016302:1–11.

[2] G. A. Bird. Molecular gas dynamics and the direct simulation of gas flows. Oxford University Press, Oxford, (1994).

[3] C. Cercignani. Rarefied gas dynamics, from basic concepts to actual calculations. Cambridge University Press, Cambridge, 2000.

[4] C. Cercignani, F. Sernagiotto. Cylindrical Couette flow of a rarefied gas. *Physics of Fluids*, **10** (1967), 1200–1205

[5] D. Y. Dankov, V. M. Roussinov. Modelling and applications of the cylindrical Couette flow of a rarefied gas. *AIP Conf. Proc.*, **1067** (2008), 239–243.

[6] P. Gospodinov, D. Dankov, V. Roussinov, S. Stefanov. Modelling of cylindrival Couette flow of rarefied gas between rotating cylinders. *Math. and Education in Math.*, **39** (2010), 184–189.

[7] P. Gospodinov, D. Dankov, V. Roussinov, S. Stefanov. Modelling of cylindrical Couette flow of rarefied gas. The Case of Rotating Outer Cylinder. 1AMiTaNS'09 *AIP Conf. Proc.*, **1186** (2009), 200–206.

[8] M. Malik, J. Dey, M. Alam. Linear stability, transient energy growth and the role of viscosity stratification in compressible plane Couette flow. *Phys. Rev. E*, **77** (2008), 036322:1–15.

[9] A. Manela, I. Frankel. On the compressible Taylor–Couette problem. *J. Fluid Mech.*, **588** (2007), 59–74.

[10] G. P. Neitzel. Marginal stability of impulsively initiated Couette and spin decay. *Journal of Fluid Mechanics*, **123** (1982), 226–232.

[11] S. K. Stefanov, V. M. Roussinov, C. Cercignani. Rayleigh-Benard flow of a rarefied gas and its attractors. Part 1: Convection regime. *Physics of Fluids*, **14** (2002), 2255–2269.

[12] S. K. Stefanov, V. M. Roussinov, C. Cercignani. Rayleigh-Benard flow of a rarefied gas and its attractors. Part 2: Chaotic and periodic convective regimes. *Physics of Fluids*, **14** (2002), 2270–2288.

[13] S. Stefanov, P. Gospodinov, C. Cercignani. Monte Carlo simulation and Navier-Stokes finite differnce solution of rarefied gas flow problems. *Physics of Fluids*, **10** (1998) Issue 1, 289–300.

[14] G. H. Tang, X. J. Gu, R. W. Barber, D. R. Emerson, Y. H. Zhang. Lattice Boltzmann simulation of nonequilibrium effects in oscillatory gas flow. *Phys. Rev. E*, **78** (2008), 026706:1–9.

[15] K. W. Tibbs, F. Baras, A. L. Garcia. Anomalous flow profile due to the curvature effect on slip length. *Phys, Rev. E*, **56** (1997), 2282–2283.

[16] H. Yoshida, K. Aoki. Linear stability of the cylindrical Couette flow of a rarefied gas. *Phys. Rev. E*, **73** (2006), 021201:1–18.

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ТЕЧЕНИЕ НА КУЕТ МЕЖДУ ЦИЛИНДРИ С РАЗЛИЧНИ ТЕМПЕРАТУРИ

Петър Господинов, Добри Данков, Владимир Русинов, Стефан Стефанов

Изследвано е стационарно течение на Кует на разреден газ в случая на въртене на вътрешния цилиндър и неподвижен външен цилиндър чрез използване на DSMC метод и числено решение на уравненията на Навие–Стокс за относително малка (дозвукова) скорост на въртене. Изследвани са различни случаи при промяна на температурата на въртящият се цилиндър и числото на Кнудсен. Целта на изследването е да се установи влиянието на малки скорости на въртене върху макрохарактеристиките – плътността, скоростта и температурата на газа. Установено е добро съвпадение на резултатите получени по двата метода за Kn = 0.02. Получените резултати са важни при решаването на неравнинни, задачи от микрофлуидиката с отчитане на ефектите на кривината.
Ключови думи: механика на флуидите, кинетична теория, разреден газ, DSMC.