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SOME MARGINAL DENSITIES OF THE WISHART DISTRIBUTION*

Evelina Veleva

Wishart distribution arises as the distribution of the sample covariance matrix for a sample from a multivariate normal distribution. Some marginal densities, derived by integration of the Wishart density function are obtained. Necessary and sufficient conditions for positive definiteness of a matrix are established, which give the bounds of the integration.

Wishart distribution has been considered in the literature as multivariate generalization of the χ^2-distribution. It is a basic distribution in many models of the multivariate statistical analysis. In practice it arises as the distribution of the sample covariance matrix for a sample from a multivariate normal distribution. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be n independent observations on a random vector $\mathbf{x}$ with p-variate normal distribution $N_p(\mu, \Sigma)$, $p < n$, with mean vector μ and positively definite covariance matrix Σ. Let $\mathbf{S}$ be the sample covariance matrix

$$\mathbf{S} = \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{x}_i - \vec{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})^t, \qquad \bar{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\,.$$

Then, the joint distribution of the elements of the matrix $\mathbf{S}$ is Wishart distribution $\mathrm{W}_p(n-1,\ 1/(n-1)\Sigma)$ (see [1], [4]). Hence, the joint distributions of sets of elements of the matrix $\mathbf{S}$ are marginal distributions of the Wishart distribution.

A $p \times p$ random matrix $\mathbf{W}$ with Wishart distribution $\mathrm{W}_p(n, \Sigma)$, where $p < n+1$ and Σ is a positively definite $p \times p$ matrix, has probability density function of the form

$$f(\mathrm{V}) = \frac{1}{2^{np/2}\Gamma_p\left(n/2\right)\left(\det\Sigma\right)^{n/2}}(\det \mathrm{V})^{(n-p-1)/2}e^{-tr(\mathrm{V}\Sigma^{-1})/2} \tag{1}$$

for any real $p \times p$ positively definite matrix V, where $\Gamma_p(\cdot)$ is the multivariate gamma function defined as $\Gamma_p(\gamma) = \pi^{p(p-1)/4}\prod_{j=1}^{p}\Gamma[\gamma + (1-j)/2]$ and $\det(\cdot)$, $tr(\cdot)$ denote the determinant and the trace of a matrix.

Let $\mathbf{W} = (W_{i,j})$. The marginal distribution of the Wishart distribution $\mathrm{W}_p(n, \Sigma)$, corresponding to a set of elements of the form $\{W_{i,j}, k \leq i \leq j \leq s\}$ for arbitrary integer k, s, such that $1 \leq k \leq s \leq p$, is $\mathrm{W}_{s-k+1}(n, \Sigma[\{k, \ldots, s\}])$, where $\Sigma[\{k, \ldots, s\}]$ denotes the submatrix of the matrix Σ, composed of the rows and columns with numbers from the set $\{k, \ldots, s\}$ (see [1], [4]). These marginal distributions correspond to all the elements

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of a submatrix of the random matrix $\mathbf{W}$ of the form $\mathbf{W}[\{k,\dots,s\}]$. Marginal densities for the sets of the form $\{W_{i,j}, k \le i \le j \le s\}\backslash\{W_{q,r}\}$, where $1 \le k \le q < r \le s \le p$, can be obtained by integration the density of the Wishart distribution (1) with respect to the element $v_{q,r}$ of the positively definite matrix V. The result is given by Theorem 1.

Subsequently, by $I_v(\cdot)$ we denote the modified Bessel function of the first kind (see [3], 8.445). Throughout the paper, the elements of Σ^{-1} are denoted by $\sigma^{i,j}$, $1 \le i \le j \le p$.

Let α and β be nonempty subsets of the set $\{1,\dots,p\}$. By $\mathrm{V}[\alpha,\beta]$ we denote the submatrix of V, composed of the rows with numbers from α and the columns with numbers from β. When $\beta \equiv \alpha$, $\mathrm{V}[\alpha,\alpha]$ is denoted simply by $\mathrm{V}[\alpha]$. For the complement of α in $\{1,\dots,p\}$ we use the notation α^c. For instance, $\mathrm{V}[\{q\}^c,\{r\}^c]$ denotes the submatrix, which can be obtained from V by deleting its q-th row and r-th column.

Theorem 1. *Let $\mathbf{W} = (W_{i,j})$ has Wishart distribution $\mathrm{W}_p(n,\Sigma)$ and q, r be integers, $1 \le q < r \le p$. Then, the marginal density, corresponding to the set of elements $\{W_{i,j},\ 1 \le i \le j \le p\}\backslash\{W_{q,r}\}$ has the form*

$$f_{q,r}\left(v_{i,j},\ 1 \le i \le j \le p,\ (i,j) \ne (q,r)\right) = \tag{2}$$

$$\frac{L}{2^{np/2}\Gamma_p\left(n/2\right)(\det\Sigma)^{n/2}} \frac{\left(\det \mathrm{V}_0[\{q\}^c] \det \mathrm{V}_0[\{r\}^c]\right)^{(n-p)/2}}{\left(\det \mathrm{V}_0[\{q,r\}^c]\right)^{(n-p+1)/2}} e^{-tr(\mathrm{V}_0\,\Sigma^{-1})/2},$$

where V_0 is the symmetric matrix with elements $v_{j,i} = v_{i,j}$, $1 \le i \le j \le p$, $(i,j) \ne (q,r)$ and $v_{q,r} = v_{r,q} = 0$, for all $v_{i,j}$, $1 \le i \le j \le p$, $(i,j) \ne (q,r)$ for which the matrices $V_0[\{q\}^c]$ and $V_0[\{r\}^c]$ are both positively definite. If $\sigma^{q,r} = 0$, then

$$L = \frac{\Gamma\left((n-p+1)/2\right)\ \Gamma\left(1/2\right)}{\Gamma\left((n-p+2)/2\right)}. \tag{3}$$

For $\sigma^{q,r} \ne 0$,

$$L = \Gamma\left((n-p+1)/2\right)\ \Gamma\left(1/2\right)\ e^{A}\ \left(\frac{2}{B}\right)^{(n-p)/2} I_{(n-p)/2}(B), \tag{4}$$

where

$$A = \frac{(-1)^{r-q-1}\det \mathrm{V}_0[\{q\}^c,\{r\}^c]}{\det \mathrm{V}_0[\{q,r\}^c]}\sigma^{q,r}, \quad B = \frac{-\sqrt{\det \mathrm{V}_0[\{q\}^c]\det \mathrm{V}_0[\{r\}^c]}}{\det \mathrm{V}_0[\{q,r\}^c]}\sigma^{q,r}. \tag{5}$$

Proof. The next Lemma gives the bounds of the integration of the Wishart density $f(\mathrm{V})$, given by (1) with respect to the variable $v_{q,r}$.

Lemma 1. *Let $\mathrm{V} = (v_{i,j})$ be a real $p \times p$ symmetric matrix and q, r be fixed integers, $1 \le q < r \le p$. Let V_0 be the matrix, obtained from the matrix V by replacing the elements $v_{q,r}$ and $v_{r,q}$ with zeros. Then the matrix V is positively definite if and only if the matrices $\mathrm{V}[\{q\}^c]$ and $\mathrm{V}[\{r\}^c]$ are positively definite and the element $v_{q,r}$ satisfies the inequalities*

$$a < v_{q,r} < b, \tag{6}$$

where

$$a = \frac{(-1)^{r-q}\det \mathrm{V}_0[\{q\}^c,\{r\}^c] - \sqrt{\det \mathrm{V}[\{q\}^c]\det \mathrm{V}[\{r\}^c]}}{\det \mathrm{V}[\{q,r\}^c]},$$

$$b = \frac{(-1)^{r-q} \det \mathrm{V}_0[\{q\}^c, \{r\}^c] + \sqrt{\det \mathrm{V}[\{q\}^c] \det \mathrm{V}[\{r\}^c]}}{\det \mathrm{V}[\{q, r\}^c]}.$$

For convenience of the reader, the proof of Lemma 1 is given in Appendix A. Using Lemma 1,

$$f_{q,r}\left(v_{i,j},\ 1 \le i \le j \le p,\ (i,j) \ne (q,r)\right) = \int_a^b f(\mathrm{V}) dv_{q,r}. \tag{7}$$

Let us change the variable of the integration by the substitution

$$v_{q,r} = \frac{(-1)^{r-q} \det \mathrm{V}_0[\{q\}^c, \{r\}^c] + t\sqrt{\det \mathrm{V}[\{q\}^c] \det \mathrm{V}[\{r\}^c]}}{\det \mathrm{V}[\{q, r\}^c]}. \tag{8}$$

The new variable t ranges from -1 to 1 and

$$dv_{q,r} = \frac{\sqrt{\det \mathrm{V}[\{q\}^c] \det \mathrm{V}[\{r\}^c]}}{\det \mathrm{V}[\{q, r\}^c]} dt.$$

A simple representation of $\det \mathrm{V}$ in terms of the new variable $t$is given by the next Lemma.

Lemma 2. *Let t be defined by equality (8). Then,*

$$\det \mathrm{V} = \frac{\det \mathrm{V}[\{q\}^c] \det \mathrm{V}[\{r\}^c]}{\det \mathrm{V}[\{q, r\}^c]}(1 - t^2). \tag{9}$$

The proof of Lemma 2 is given in Appendix B.

Since the matrices V and Σ^{-1} are symmetric, we have the representation

$$tr(\mathrm{V}\,\Sigma^{-1}) = \sum_{i=1}^{p} v_{i,i}\sigma^{i,i} + 2\sum_{i<j} v_{i,j}\sigma^{i,j} = tr(\mathrm{V}_0\,\Sigma^{-1}) + 2v_{q,r}\sigma^{q,r}.$$

Hence, changing the variable of the integration, the marginal density (7) takes the form (2) with

$$L = e^A \int_{-1}^{1} (1 - t^2)^{(n-p-1)/2} e^{Bt} dt, \tag{10}$$

where A and B are given by (5). If $\sigma^{q,r} = 0$, then $A = B = 0$. Now, using the equalities 3.196 3 and 8.384 4 in [3], we obtain (3).

When $\sigma^{q,r} \ne 0$, then using 8.431 in [3], (10) can be written in the form (4).

As an immediate consequence of Theorem 1, we get the next Corollary.

Corollary 1. *Let $\mathbf{W} = (W_{i,j})$ has Wishart distribution $\mathrm{W}_p(n, \Sigma)$ and q, r be integers, $1 \le q < r \le p$. Then, the conditional density of $W_{q,r}$ given that $W_{i,j} = v_{i,j}$, $1 \le i \le j \le p$, $(i,j) \ne (q,r)$ has the form*

$$g\left(v_{q,r} \mid v_{i,j},\ 1 \le i \le j \le p,\ (i,j) \ne (q,r)\right) = \frac{\left(\det \mathrm{V}[\{q,r\}^c]\right)^{(n-p+1)/2} \left(\det \mathrm{V}\right)^{(n-p-1)/2}}{\left(\det \mathrm{V}[\{q\}^c] \det \mathrm{V}[\{r\}^c]\right)^{(n-p)/2} L} e^{-v_{q,r}\sigma^{q,r}},$$

where L is given by the equalities (3)–(5), for all $v_{i,j}$, $1 \le i \le j \le p$ for which the matrix $\mathrm{V} = (v_{i,j})$ is positively definite.

Appendix A.

Proof of Lemma 1. If α is a nonempty subset of the set $\{1, \dots, p\}$ and the matrix $\mathrm{V}[\alpha]$ is invertible, then the Schur complement $\mathrm{V} / \mathrm{V}[\alpha]$ of $\mathrm{V}[\alpha]$ in V is defined as (see [6])

$$\mathrm{V} / \mathrm{V}[\alpha] = \mathrm{V}[\alpha^c] - \mathrm{V}[\alpha^c, \alpha](\mathrm{V}[\alpha])^{-1} \mathrm{V}[\alpha, \alpha^c].$$

An important property of the Schur complement is that (see [6])

$$\det(\mathrm{V} / \mathrm{V}[\alpha]) = \frac{\det \mathrm{V}}{\det \mathrm{V}[\alpha]}.$$

The Schur complement of $\mathrm{V}[\{q, r\}^c]$ in the matrix V is the 2×2 matrix

$$\mathrm{V} / \mathrm{V}[\{q, r\}^c] = \begin{pmatrix} v_{q,q} & v_{q,r} \\ v_{r,q} & v_{r,r} \end{pmatrix} - \begin{pmatrix} \mathrm{V}_q^t \\ \mathrm{V}_r^t \end{pmatrix} (\mathrm{V}[\{q, r\}^c])^{-1} \begin{pmatrix} \mathrm{V}_q & \mathrm{V}_r \end{pmatrix},$$

where V_q and V_r are the vectors $\mathrm{V}_q = \mathrm{V}[\{q, r\}^c, \{q\}]$, $\mathrm{V}_r = \mathrm{V}[\{q, r\}^c, \{r\}]$. Since V is a symmetric matrix,

$$\mathrm{V} / \mathrm{V}[\{q, r\}^c] = \begin{pmatrix} v_{q,q} - \mathrm{V}_q^t(\mathrm{V}[\{q, r\}^c])^{-1} \mathrm{V}_q & v_{q,r} - \mathrm{V}_q^t(\mathrm{V}[\{q, r\}^c])^{-1} \mathrm{V}_r \\ v_{q,r} - \mathrm{V}_q^t(\mathrm{V}[\{q, r\}^c])^{-1} \mathrm{V}_r & v_{r,r} - \mathrm{V}_r^t(\mathrm{V}[\{q, r\}^c])^{-1} \mathrm{V}_r \end{pmatrix}. \tag{11}$$

The Schur complements of $\mathrm{V}[\{q, r\}^c]$ in $\mathrm{V}[\{q\}^c]$ and $\mathrm{V}[\{r\}^c]$ are numbers,

$$\mathrm{V}[\{q\}^c] / \mathrm{V}[\{q, r\}^c] = v_{r,r} - \mathrm{V}_r^t(\mathrm{V}[\{q, r\}^c])^{-1} \mathrm{V}_r = \frac{\det \mathrm{V}[\{q\}^c]}{\det \mathrm{V}[\{q, r\}^c]},$$

$$\mathrm{V}[\{r\}^c] / \mathrm{V}[\{q, r\}^c] = v_{q,q} - \mathrm{V}_q^t(\mathrm{V}[\{q, r\}^c])^{-1} \mathrm{V}_q = \frac{\det \mathrm{V}[\{r\}^c]}{\det \mathrm{V}[\{q, r\}^c]}.$$

The Schur complement of the matrix $\mathrm{V}[\{q, r\}^c]$ in the matrix

$$\mathrm{V}[\{q, r\}^c]_{q,r} = \begin{pmatrix} \mathrm{V}[\{q, r\}^c] & \mathrm{V}_r \\ \mathrm{V}_q^t & v_{q,r} \end{pmatrix}$$

is again a number,

$$\mathrm{V}[\{q, r\}^c]_{q,r} / \mathrm{V}[\{q, r\}^c] = v_{q,r} - \mathrm{V}_q^t(\mathrm{V}[\{q, r\}^c])^{-1} \mathrm{V}_r = \frac{\det \mathrm{V}[\{q, r\}^c]_{q,r}}{\det \mathrm{V}[\{q, r\}^c]}.$$

Replacing in (11), we obtain the representation

$$\mathrm{V} / \mathrm{V}[\{q, r\}^c] = \frac{1}{\det \mathrm{V}[\{q, r\}^c]} \begin{pmatrix} \det \mathrm{V}[\{r\}^c] & \det \mathrm{V}[\{q, r\}^c]_{q,r} \\ \det \mathrm{V}[\{q, r\}^c]_{q,r} & \det \mathrm{V}[\{q\}^c] \end{pmatrix}.$$

Let α be a nonempty set of indexes. A square matrix V is positively definite if and only if the matrices $\mathrm{V}[\alpha]$ and $\mathrm{V} / \mathrm{V}[\alpha]$ are positively definite (see [6]). Using this property of the Schur complement, the matrix V is positively definite if and only if the matrices $\mathrm{V}[\{q, r\}^c]$ and $\mathrm{V} / \mathrm{V}[\{q, r\}^c]$ are both positively definite. Consequently, the positively definiteness of the matrix V is equivalent to the conditions:

1.1. The matrix $\mathrm{V}[\{q, r\}^c]$ is positively definite;

1.2. $\det \mathrm{V}[\{r\}^c] > 0$;

1.3. $\det \mathrm{V}[\{q\}^c] > 0$;

1.4. $-\sqrt{\det \mathrm{V}[\{q\}^c] \det \mathrm{V}[\{r\}^c]} < \det \mathrm{V}[\{q, r\}^c]_{q,r} < \sqrt{\det \mathrm{V}[\{q\}^c] \det \mathrm{V}[\{r\}^c]}$.

Let us consider the matrix $\mathrm{V}[\{q,r\}^c]_{q,q} = \begin{pmatrix} \mathrm{V}[\{q,r\}^c] & \mathrm{V}_q \\ \mathrm{V}_q^t & v_{q,q} \end{pmatrix}$, which can be obtained from the matrix $\mathrm{V}[\{r\}^c]$, placing its q-th row and column after the last row and column, respectively. With this transformation the determinant remains unchanged, $\det \mathrm{V}[\{q,r\}^c]_{q,q} = \det \mathrm{V}[\{r\}^c]$. A symmetric matrix is positively definite if and only if all principal minors of the matrix are positive (see [2]). Hence, the conditions 1.1 and 1.2 are equivalent to

2.1. The matrix $\mathrm{V}[\{q,r\}^c]_{q,q}$ is positively definite.

Another well-known property of the positively definite matrices is that their eigenvalues are all positive. Since, obviously, the matrices $\mathrm{V}[\{q,r\}^c]_{q,q}$ and $\mathrm{V}[\{r\}^c]$ have the same eigenvalues, the condition 2.1 is equivalent to

3.1. The matrix $\mathrm{V}[\{r\}^c]$ is positively definite.

Analogically, the conditions 1.1 and 1.3 are equivalent to

3.2. The matrix $\mathrm{V}[\{q\}^c]$ is positively definite.

From the expansion of $\det \mathrm{V}[\{q,r\}^c]_{q,r}$by the elements of its last row, we have

$$\det \mathrm{V}[\{q,r\}^c]_{q,r} = v_{q,r} \det \mathrm{V}[\{q,r\}^c] + \det \begin{pmatrix} \mathrm{V}[\{q,r\}^c] & \mathrm{V}_r \\ \mathrm{V}_q^t & 0 \end{pmatrix}. \tag{12}$$

The last matrix in (12) can be obtained from the matrix $\mathrm{V}_0[\{r\}^c,\{q\}^c]$, placing its q-th row below the last row and its $(r-1)$-th column after its last column. Consequently,

$$\det \begin{pmatrix} \mathrm{V}[\{q,r\}^c] & \mathrm{V}_r \\ \mathrm{V}_q^t & 0 \end{pmatrix} = (-1)^{r-q-1} \det \mathrm{V}_0[\{r\}^c,\{q\}^c]. \tag{13}$$

Since the transposition preserves the value of a determinant,

$$\det \mathrm{V}_0[\{r\}^c,\{q\}^c] = \det \mathrm{V}_0[\{q\}^c,\{r\}^c]. \tag{14}$$

Now, using (12)–(14) and 1.1, we obtain that the condition 1.4 is equivalent to

3.3. The element $v_{q,r}$ satisfies the inequalities (6).

Appendix B.

Proof of Lemma 2. From equality (8) we have that

$$t = \frac{v_{q,r} \det \mathrm{V}[\{q,r\}^c] + (-1)^{r-q-1} \det \mathrm{V}_0[\{q\}^c,\{r\}^c]}{\sqrt{\det \mathrm{V}[\{q\}^c] \det \mathrm{V}[\{r\}^c]}}.$$

From the expansion of $\det \mathrm{V}[\{q\}^c,\{r\}^c]$ by the elements of its r-th row it can be seen that $\det \mathrm{V}[\{q\}^c,\{r\}^c] = v_{r,q}(-1)^{r-q-1} \det \mathrm{V}[\{q,r\}^c] + \det \mathrm{V}_0[\{q\}^c,\{r\}^c]$. Consequently,

$$t = \frac{(-1)^{r-q-1} \det \mathrm{V}[\{q\}^c,\{r\}^c]}{\sqrt{\det \mathrm{V}[\{q\}^c] \det \mathrm{V}[\{r\}^c]}}.$$

Hence,

$$1 - t^2 = \frac{\det \mathrm{V}[\{q\}^c] \det \mathrm{V}[\{r\}^c] - (\det \mathrm{V}[\{q\}^c,\{r\}^c])^2}{\det \mathrm{V}[\{q\}^c] \det \mathrm{V}[\{r\}^c]}.$$

Now, using the equality

$$\det \mathrm{V} \det \mathrm{V}[\{q,r\}^c = \det \mathrm{V}[\{q\}^c] \det \mathrm{V}[\{r\}^c] - (\det \mathrm{V}[\{q\}^c,\{r\}^c])^2,$$

which is a special case of the Sylvester's determinant identity (see [5]), we obtain that

$$1 - t^2 = \frac{\det \mathrm{V} \det(\mathrm{V}[\{q,r\}^c])}{\det \mathrm{V}[\{q\}^c] \det \mathrm{V}[\{r\}^c]}.$$

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Evelina Veleva
Department of Numerical Methods and Statistics
Rouse University
8, Studentska Str.
7004 Rouse, Bulgaria
e-mail: eveleva@uni-ruse.bg

НЯКОИ МАРГИНАЛНИ ПЛЪТНОСТИ НА РАЗПРЕДЕЛЕНИЕТО НА УИШАРТ

Евелина Илиева Велева

Разпределението на Уишарт се среща в практиката като разпределението на извадъчната ковариационна матрица за наблюдения над многомерно нормално разпределение. Изведени са някои маргинални плътности, получени чрез интегриране на плътността на Уишарт разпределението. Доказани са необходими и достатъчни условия за положителна определеност на една матрица, които дават нужните граници за интегрирането.