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VIABILITY KERNELS OF HIGHER ORDER*

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The aim of this article is to establish a definition of the viability kernel associated with a differential inclusion of high order, which generalizes this notion for first order differential inclusion. We present a sufficient condition ensuring the existence of the viability kernel of high order. Some examples in the second order case are analysed.

Keywords: Multivalued differential inclusions, viability, differential inclusions with constraints

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1 Introduction

A viability problem for a first order differential inclusion consists in looking for absolutely continuous functions $x(\cdot)$ such that $x'(t) \in F(x(t))$ for almost all $t \geq 0$, starting from x_0 (i.e. $x(0) = x_0$) and satisfying the viability condition $x(t) \in K$. Where $F : D(F) \subset X \to 2^X$ is a set-valued map, K a closed subset of the finite dimensional space X and $x_0 \in K$ is the initial state. Viability theorems for first order differential inclusions have been studied in recent years (see [3], [5] or [9] for more details).

It is known that a subset $K \subset D(F)$ is a viability domain of F if and only if K is locally viable under F provided that F is upper semicontinuous (usc) with convex compact values and K is locally compact.

If a closed subset K is not a viability domain, the question arises as to whether there are closed viability subsets of K viable under F, whether there exists a largest closed subset of K viable under F. The answer is positive assuming restrictions on the set-valued map F and the largest closed subset viable under F contained in K is called **Viability Kernel** of a closed subset K with respect to the set-valued map F ([5]).

The notion of viability kernel appeared in the framework of differential inclusions in [1], and the relationship between viability kernels and zero dynamics in [2]. Properties of

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the viability kernels can be found in [7], [13], [14] and algorithms for approaching them in [10] and [15].

In this paper we continue with the search on viability problems for differential inclusions of high order started in [11, 12] by introducing and analysing the viability kernel of higher order. Section 2 presents some topics in the framework and sets up notation and terminology. Section 3 is devoted to introduce the notion of tangent set of n-th order very useful in this framework. In section four we obtain sufficient and necessary conditions ensuring the existence of viable global solutions. A counterexample showing that we can not replace local existence by global existence even if the set valued map is Marchaud is presented. The next section deals with viability kernels of high order and we give a definition and a sufficient condition for its existence. We conclude with three examples of viability kernels of second order.

2 Preliminaries

Let us first recall some notions and notation. Let X, Y be metric spaces. Given a set-valued map $F : X \to 2^Y$, we will denote by *domain of* F the set $D(F) = \{x \in X : F(x) \neq \emptyset\}$ and we say that F is nontrivial if $D(F) \neq \emptyset$. The *graph of* F is said to be the subset of the space $X \times Y$ defined by $\mathcal{G}(F) = \{(x, y) \in X \times Y : y \in F(x)\}$. It is said that F is *upper semicontinuous* (u.s.c.) if $F^{-1}(C) = \{x \in X : F(x) \cap C \neq \emptyset\}$ is closed in $D(F)$ for every closed set $C \subseteq Y$. We say that

$$\limsup_{x' \to x} F(x') = \{y \in Y : \liminf_{x' \to_F x} d(y, F(x')) = 0\}$$

is the upper limit of $F(x')$ when $x' \to x$, where the notation $x' \to_F x$ means that $x' \in D(F)$ and converges to x.

In the sequel we consider X a finite dimensional space, $K \subseteq X$ a non-empty set and $F : X^n \to 2^X$ an usc non trivial set-valued map with convex compact values.

Let $(x_0, v_1, \cdots, v_{n-1}) \in K \times X^{(n-1)}$, we deal with the *viability problem of order* n which consists in looking for absolutely continuous functions $x(\cdot) : [0, T[\to X$ possesing absolutely continuous derivatives up to order $n-1$ (i.e. $x \in W^{n,1}_{loc}(0, T, X)$, such that

$$x^{(n)}(t) \in F(x(t), x'(t), \cdots, x^{(n-1)}(t)); \qquad \text{(a.e) } t \in [0, T[\tag{2.1}$$

$$x(0) = x_0; \, x'(0) = v_1, \ldots, x^{(n-1)}(0) = v_{n-1} \tag{2.2}$$

$$x(t) \in K, \qquad \forall\, t \in [0, T[\tag{2.3}$$

We denote by $S(x_0, v_1, \ldots, v_{n-1}; T)$ the set of solutions on $[0, T[$ of (2.1) satisfying initial condition (2.2). We denote by $\mathcal{K}(T)$ the set of functions $\varphi : [0, T[\to X$ such that $\varphi(t) \in K$ for all $t \in [0, T[$ (where $0 < T \leq \infty$). We say that the set valued map F has **linear growth** if there exists $c > 0$ such that

$$\|F(x_1, \ldots, x_n)\| \leq c(1 + |x_1| + \ldots + |x_n|)$$

for all $(x_1, \dots, x_n) \in D(F)$ where $\|F(x_1, \dots, x_n)\| = \sup_{y \in F(x_1,\dots,x_n)} |y|$. We will say that an u.s.c. map with convex compact values and linear growth is a Marchaud map.

As usual we regard the n-th order differential inclusion (2.1) as the system of first order differential inclusion $y'(t) \in \widetilde{F}(y(t))$ where

$$y = (x, x', \dots, x^{(n-1)}) \in X^n, \quad \text{and} \quad \widetilde{F}(y) = (y_2, \dots, y_n, F(y))$$

Thus initial condition (2.2) becomes $y(0) = (x_0, v_1, \dots, v_{n-1})$. In order to establish the viable condition in terms of variable y we generalize the notion of Bouligand's cone to high order, by introducing the so called *Tangent sets of High Order.*

3 Higher order tangent sets

Definition 3.1 *Let $K \subseteq X$ be a non-empty set, and $x_1, x_2, \dots, x_n \in X$. We denote by $A_K^{(n)}(x_1, \dots, x_n)$ the n*-th. order tangent set *of K at $(x_1, \dots, x_n)$ defined by:*

$$A_K^{(n)}(x_1, \dots, x_n) := \limsup_{h \to 0^+} \frac{n!}{h^n} \left(K - x_1 - hx_2 - \cdots - \frac{h^{n-1}}{(n-1)!} x_n \right).$$

We have the following useful characterization.

Proposition 3.1 *A vector $y \in X$, belongs to $A_K^{(n)}(x_1, \dots, x_n)$ if and only if it satisfies one of these equivalent statements*

1. *There exist $h_m \to 0^+$ and $y_m \to y$ such that*

$$x_1 + h_m x_2 + \ldots + \frac{h_m^{n-1}}{(n-1)!} x_n + \frac{h_m^n}{n!} y_m \in K; \qquad (m \in \mathbb{N})$$

2. $$\liminf_{h \to 0^+} \frac{n!}{h^n} d\left(x_1 + hx_2 + \cdots + \frac{h^{n-1}}{(n-1)!} x_n + \frac{h^n}{n!} y,\, K \right) = 0$$

We now list several basic facts about these tangent sets of higher order.

Proposition 3.2 *$A_K^{(1)}(x_1)$ is a closed cone, (it is equal to Bouligand's cone).*

$A_K^{(n)}(x_1, \dots, x_n)$ is closed and

$$A_K^{(n)}(x_1, \lambda x_2, \dots, \lambda^{n-1} x_n) = \lambda^n A_K^{(n)}(x_1, \dots, x_n)$$

for all $\lambda > 0$ and $n \geq 2$.

If $A_K^{(n)}(x_1, \dots, x_n) \neq \emptyset$ then $x_1 \in \overline{K}$ and[1] *$x_r \in A_K^{(r-1)}(x_1, \dots, x_{r-1})$ for each $r = 2, \cdots, n$.*

[1] If $x_1 \in \mathrm{int} K$, then $A_K^{(n)}(x_1, \dots, x_n) = X$.

From Proposition 3.2 it follows the following relationship between the domain of $A_K^{(n)}$ and the graph of $A_K^{(n-1)}$:

$$D(A_K^{(n)}) \subseteq \mathcal{G}(A_K^{(n-1)}).$$

Notice that $A_K^{(2)}$ coincides with Ben-Tal's second order tangent set (see 1.7 in [6])[2].

Proposition 3.3 *Let $\varphi : [0,T[\to X$ be a solution of (2.1). If $\varphi(t) \in K$ for all $t \in [0,T[$ then*

$$(\varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t)) \in \mathcal{G}(A_K^{(n-1)}), \quad \forall\, t \in [0,T[\tag{3.1}$$

Conversely, if K is closed and (3.1) *holds then $\varphi(t) \in K$ for all $t \in [0,T[$*

According to the above result the viable condition on $x(t)$ (2.3) involves an underlying viability condition on $x(t)$ and its derivatives up to order $n-1$, that is, $y(t) = (x(t), x'(t), \ldots, x^{n-1}(t)) \in \mathcal{G}(A_K^{(n-1)})$.

4 Local viability theorem of high order

We can obtain the following local viability theorem for higher order. (For the proof we refer the reader to [11], [12])

Theorem 4.1 *Let $K \subseteq X$ such that $\mathcal{G}(A_K^{(n-1)})$ is locally compact and $D(F) \subset \mathcal{G}(A_K^{(n-1)})$. Are equivalent the following statements.*
(i) For each $(x, u_1, \ldots, u_{n-1}) \in X^n$ there exists $T > 0$ such that

$$S(x, u_1, \ldots, u_{n-1}; T) \cap \mathcal{K}(T) \neq \emptyset$$

(ii) $F(x, u_1, \ldots, u_{n-1}) \cap DA_K^{(n-1)}(x, u_1, \ldots, u_{n-1})[u_1, \ldots, u_{n-1}] \neq \emptyset$ for all $(x, u_1, \cdots, u_{n-1}) \in \mathcal{G}(A_K^{(n-1)})$.

Moreover given $(x_0, v_1, \ldots, v_{n-1}) \in \mathcal{G}(A_K^{(n-1)})$, there exist $\eta > 0$ and $T_0 > 0$, such that $S(x, u_1, \ldots, u_{n-1}, T_0) \cap \mathcal{K}(T_0) \neq \emptyset$, for each initial condition $(x, u_1, \ldots, u_{n-1}) \in \mathcal{G}(A_K^{(n-1)}) \cap ((x_0, v_1, \ldots, v_{n-1}) + \eta\mathcal{U})$, where $\mathcal{U}$) is the unit ball.

Theorem 4.2 *Let K be a closed subset such that $\mathcal{G}(A_K^{(n-1)})$ is locally compact and contained in $D(F)$. Under the assumption (ii) of Theorem* 4.1 *for every $(x_0, x_1, \ldots, x_{n-1}) \in$*

[2] *The contingent set of high order*, was introduced by Aubin & Frankowska in [4] by

$$T_K^{(n)}(x_1, \ldots, x_n) = \limsup_{h\to 0^+} h^{-n}(K - x_1 - hx_2 - \cdots - h^{n-1}\, x_n)$$

There exists the following relationship between $T_K^{(n)}$ and $A_K^{(n)}$

$$A_K^{(n)}(x_1, \ldots, x_n) = n!\, T_K^{(n)}\left(x_1, x_2, \cdots, \frac{1}{(n-1)!}x_n\right)$$

$\mathcal{G}(A_K^{(n-1)})$ there exists a maximal solution $\varphi \in S(x_0, x_1, \ldots, x_{n-1}, T)$ viable in K, such that either $T = \infty$ or $T < \infty$ and in this case either

$$\limsup_{t\to T-} \|(\varphi(t), \ldots, \varphi^{(n-1)}(t))\| = \infty$$

or there exists $\lim_{t\to T} \varphi^{(j)}(t) = \varphi^{(j)}(T)$ for $j = 0, \ldots, n-1$ and

$$(\varphi(T), \ldots, \varphi^{(n-1)}(T)) \in \overline{\mathcal{G}(A_K^{(n-1)})} \setminus \mathcal{G}(A_K^{(n-1)})$$

Remark 4.1 *Let us mention an important consequence about the existence of global viable solutions in the preceding theorem:*

Replacing the assumptions "F u.s.c. with convex compact values*" by "F* is a Marchaud map*" we obtain $\limsup_{t\to T-} \|(\varphi(t), \ldots, \varphi^{(n-1)}(t)\| < \infty$. Thus, assuming the hypotheses $\mathcal{G}(A_K^{(n-1)})$ is closed and F is a Marchaud map we obtain $T = \infty$ in the preceding theorem.*

A sufficient condition to prove local existence of viable solution for second order differential inclusions without the assumption of $\mathcal{G}(T_K)$ is closed is given by:

$$(x_0, v_0) \in \mathcal{G}(AI_K)$$

where AI_K is the Dubovickii-Miljutin cone (see [8]).

The following example shows that this result is not longer true when "local existence" is replaced by "global existence", even if F is a Marchaud map.

Example 4.1 *Let $K = [a, b]$ with $0 < a < b$. We show that* **sufficient conditions of the local existence** *theorem are satisfied by the problem,*

$$x''(t) = x(t), \quad x(0) = x_0 \in]a, b[, \quad x'(0) = 0 \tag{4.1}$$

Howewer **there is not any viable solution** *in K on $[0, \infty[$.*

- *Firstly we show that $(x_0, 0) \in \mathcal{G}(AI_K)$. In fact since K is convex $AI_K(x) = int\, T_K(x)$ so if $x_0 \in]a, b[$ then $0 \in AI_K(x_0)$.*
- *Next we prove that tangential condition*

  $$F(x, v) \cap DT_K(x, v)[v] \neq \emptyset; \quad \forall\, (x, v) \in \mathcal{G}(T_K)$$

  holds.

It is easily seen that $T_K(x) = \begin{cases} [0, +\infty[& \text{for} \quad x = a \\ \mathbb{R} & \text{for} \quad x \in]a, b[\\ ]-\infty, 0[& \text{for} \quad x = b \end{cases}$ hence the graph of T_K is not a closed set (see Fig. 1).

In a similiar way we check that

$$T_{\mathcal{G}(T_K)}(x, v) = \begin{cases} [0, \infty[\times\mathbb{R} & \text{for} \quad x = a,\ v \geq 0 \\ \mathbb{R}^2 & \text{for} \quad x \in]a, b[,\ v \in \mathbb{R} \\ ]-\infty, 0] \times \mathbb{R} & \text{for} \quad x = b,\ v \leq 0 \end{cases}$$

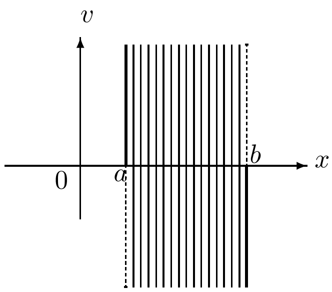

Figure 1: $\mathcal{G}(T_K)$

for all $(x,v) \in \mathcal{G}(T_K)$*. Hence we have*

$$(v,a) \in T_{\mathcal{G}(T_K)}(a,v); \quad v \geq 0$$

$$(v,x) \in T_{\mathcal{G}(T_K)}(x,v), \quad x \in]a,b[, \; v \in \mathbb{R}$$

$$(v,b) \in T_{\mathcal{G}(T_K)}(b,v); \quad v \leq 0$$

which implies,

$$x = F(x,v) \in DT_K(x,v)[v] \neq \emptyset$$

for all $(x,v) \in \mathcal{G}(T_K)$*, that is the tangential condition.*

Finally, we prove that there is no viable solution in K*. The only solution of the second order problem (4.1) is* $x(t) = \frac{x_0}{2}(e^t + e^{-t})$ *that is not viable on* $[0,\infty[$ *in* $[a,b]$*. Notice that* $x(t) \to +\infty$ *as* $t \to +\infty$*.*

5 Existence of viability kernel of high order

We introduce the notion of **Viability Kernel of n-th order** First of all we give the following

Definition 5.1 *A subset* L *contained in* $\mathcal{G}(A_K^{(n-1)})$ *is said to be a* **viable set under** F *if and only if*

$$S(x_0,\ldots,x_{n-1};\infty) \cap \mathcal{K} \neq \emptyset$$

for all $(x_0,x_1,\ldots,x_{n-1}) \in L$*, here* $\mathcal{K} = \mathcal{K}(\infty)$*.*

Let K be a closed subset, if $\mathcal{G}(A_K^{(n-1)})$ is not a viable set under F the question arises as to whether there are closed subsets of $\mathcal{G}(A_K^{(n-1)})$ viable under F, whether there exists a largest closed subset of $\mathcal{G}(A_K^{(n-1)})$ viable under F.

Definition 5.2 *We call* **the viability kernel** *of K of n-th order associated with* (2.1), *denoted by $Viab_F^{(n)}(K)$, the largest closed subset in $\mathcal{G}(A_K^{(n-1)})$ viable under F.*

Let us note that $Viab_F^{(n)}(K)$ is equal to $Viab_{\widetilde{F}}^{(1)}(\mathcal{G}(A_K^{(n-1)}))$. We prove the following technical lemma

Lemma 5.1 *Let F be a Marchaud map and K a closed non-empty set such that $\mathcal{G}(A_K^{(n-1)}) \subset D(F)$. Let $\varphi \in S(x_0, \ldots, x_{n-1}; T) \cap \mathcal{K}$ then:*

$$|\varphi^{(j)}(t)| \leq (1 + \|y_0\|) \exp{(at)}; \qquad t \in [0, T[, \quad (0 \leq j \leq n-1)$$

$$|\varphi^{(n)}(t)| \leq c(1 + \|y_0\|) \exp(at); \qquad t \in [0, T[\text{ (a.e.)}$$

where $\|y_0\| = (|x_0|^2 + \ldots + |x_{n-1}|^2)^{1/2}$ and $a = 1 + c\sqrt{n}$.

Proof. We set $\psi = (\varphi, \ldots, \varphi^{(n-1)})$ and

$$\|\psi(t)\| = \left(|\varphi(t)|^2 + \ldots + |\varphi^{(n-1)}(t)|^2\right)^{1/2}.$$

It is sufficient to prove that

$$\|\psi(t)\| \leq (1 + \|y_0\|) \exp{(at)} - 1.$$

In fact

$$\|\psi'(t)\| = (\sum_{j=1}^{n} |\varphi^{(j)}(t)|^2)^{1/2}$$

$$\leq (\sum_{j=0}^{n-1} |\varphi^{(j)}(t)|^2)^{1/2} + c(1 + \sum_{j=0}^{n-1} |\varphi^{(j)}(t)|)$$

$$\leq \|\psi(t)\| + c(1 + \sqrt{n}\|\psi(t)\|) = (1 + c\sqrt{n})\|\psi(t)\| + c \leq a(\|\psi(t)\| + 1)$$

and the result follows by Gronwall's Inequality. $\square$

Note 5.1 *We observe that if $\mathcal{G}(A_K^{(n-1)})$ is not closed (or locally compact) we can not obtain (local) existence of viable solutions of the n-th order inclusion by using the first order theory because one of the main assumptions on the local viability theory for first order differential inclusion is that the viability set must be closed. Howewer we show that with respect to the existence of viability kernel of high order we can obtain a proof without using the results of first order.*

The existence and a characterization of viability kernels of high order are established by our next theorem.

Theorem 5.1 *Let F be a Marchaud map and K a closed non-empty set such that $\mathcal{G}(A_K^{(n-1)}) \subset D(F)$. The Viability Kernel of K of n-th order associated to* (2.1)*, exists (possibly empty) and is equal to the subset of the n-th initial states such that at least one solution starting from them is viable in K.*

Proof. We set $A = \{y \in \mathcal{G}(A_K^{(n-1)}) \colon S(y,\infty) \cap \mathcal{K} \neq \emptyset\}$ The proof will be divided into three steps.

Step 1. First of all notice that $Viab_F^{(n)}(K) \subseteq A$.

Step 2. We check that A is closed.

Let a sequence $y_m = (x_{0m}, \dots, x_{n-1,m}) \in A$ converging to the vector

$y = (x_0, \dots, x_{n-1})$. By definition of A there exists $\varphi_m \in S(y_m,\infty) \cap \mathcal{K}$ for each $m \in \mathbb{N}$.

Since $\{y_m\}$ is convergent, from the preceding lemma it follows

$$|\varphi_m^{(j)}(t)| \leq M \exp aT; \quad t \in [0,T]$$

for each $j = 0, 1, \dots, n-1$, and

$$|\varphi_m^{(n)}(t)| \leq cM \exp aT; \quad t \in [0,T] \quad \text{a.e}$$

Thus by Ascoli's theorem a subsequence (again denoted by) $\varphi_m^{(j)}(.)$ converges uniformly in $W^{1,1}(0,T,X)$ to some function $v_j(.) \in W_{loc}^{1,1}(0,\infty,X)$ for each $j = 0,1,\dots,n-1$ and by the other hand Alaoglu's theorem implies that a subsequence (again denoted) $\varphi_m^{(n)}(.)$ converges weakly to some function $w \in L_{loc}^1(0,\infty)$.

Writing $\varphi = u_0$ we prove that $v_j = \varphi^{(j)}$ for all $j = 1,\dots,n-1$ and w is actually the n^{th}-derivative, $\varphi^{(n)}$, of φ. In fact, letting $m \to \infty$ in

$$\varphi_m^{(j)}(t) - \varphi_m^{(j)}(s) = \int_s^t \varphi_m^{(j+1)}(\tau)\, d\tau$$

for $j = 0,1,\dots,n-1$, the desired result is obtained. We proceed to show that $\varphi \in S(y,\infty) \cap K$.

Obviously, $\varphi(t) \in K$ for all $t \in [0,\infty[$ being K closed.

By the other hand, since $(\varphi_m(t),\dots,\varphi_m^{(n)}(t)) \in \mathcal{G}(F)$ almost everywhere and F is *u.s.c.*, a convergence theorem (see p. 67 in [5] for instance) shows: $(\varphi(t),\dots,\varphi^{(n)}(t)) \in \mathcal{G}(F)$. Finally, by Proposition 3.3 we have

$$(\varphi(t),\dots,\varphi^{(n-1)}(t)) \in \mathcal{G}(A_K^{(n-1)})$$

hence $y \in A$, which completes the proof of step 2.

Step 3. A is viable under F and the largest one.

Let $y = (x_0,\dots,x_{n-1}) \in A$, then there exists a viable solution $x(\cdot)$ such that $(x(0),\dots,x^{n-1)}(0)) = (x_0,\dots,x_{n-1})$. For all $t > 0$, the function

$$\phi(s) = x(t+s); \quad s \in [0,+\infty[$$

is also a viable solution in K such that

$$(\phi(0),\dots,\phi^{(n-1)}(0)) = (x(t),\dots,x^{n-1)}(t))$$

and by Proposition 3.3 $(\phi(s),\dots,\phi^{n-1)}(s)) \in \mathcal{G}(A_K^{(n-1)})$ hence we obtain $(x(t),\dots,x^{n-1)}(t)) \in A$. $\square$

Next example shows a viability kernel of second order $Viab_F^{(2)}(K)$ empty. In this case we said that K is a **repeller** for F because any solution leaves K in finite time.

Example 5.1 *Let $K = [a, b]$ with $0 < a < b$ and $F(x, v) = x$. Given $(x_0, v_0) \in \mathcal{G}(T_K)$. The problem:*

$$x'' = x, \quad x(0) = x_0, \quad x'(0) = v_0$$

has the only solution

$$x(t) = \frac{x_0 + v_0}{2} e^t + \frac{x_0 - v_0}{2} e^{-t} \tag{5.1}$$

If $\frac{x_0 + v_0}{2} > 0$ we have $x(t) \to \infty$ as $t \to \infty$. If $\frac{x_0 + v_0}{2} < 0$ we have $x(t) \to -\infty$ as $t \to \infty$. If $\frac{x_0 + v_0}{2} = 0$ we have $x(t) = x_0 e^{-t} \to 0$ as $t \to \infty$. Thus, $x(\cdot)$ is not viable in K on $[0, +\infty[$, because of $a > 0$. Hence, there is no global viable solution for this problem. (i.e. every solution $x(\cdot)$ is not viable in K on $[0, +\infty[$).

However, if we take $a \leq 0$ in the preceding example the viability kernel is not empty.

Example 5.2 *Let $K = [a, b]$ with $a \leq 0 < b$ and $F(x, v) = x$. We show that*

$$Viab_F^{(2)}(K) = \{(x, -x);\ x \in [a, b]\}$$

Let $(x_0, v_0) \in \mathcal{G}(T_K)$ such that $x_0 + v_0 = 0$ we prove that there exists a viable solution in $[a, b]$ on $[0, \infty[$.

In fact, let $x_0 \in [a, b]$ then $v_0 = -x_0 \in T_K(x_0)$. The solution of

$$x''(t) = x(t); \quad x(0) = x_0, \quad x'(0) = v_0$$

is $x(t) = x_0 e^{-t}$. So if $b \geq x_0 > 0$ then $a \leq 0 \leq x_0 e^{-t} \leq x_0 \leq b$ for all $t \in [0, \infty[$. If $a \leq x_0 \leq 0$ then $a \leq x_0 \leq x_0 e^{-t} < 0 < b$ for all $t \in [0, \infty[$.

Conversely, if $(x_0, v_0) \in Viab_F^{(2)}(K)$ and $x_0 + v_0 \neq 0$ analysis carried out in Example 5.1 shows that $x(\cdot)$ given by (5.1) is not viable in $[a, b]$ on $[0, \infty[$, which is impossible.

Notice that $x(t) + x'(t) = 0$ for all $t \geq 0$. That is, $(x(t), x'(t)) \in Viab_F^{(2)}(K)$ i.e. the viability kernel is also viable under F.

In Example 5.2 a non-empty viability kernel is presented, but it has empty interior. Next example provides a viability kernel with non-empty interior.

Example 5.3 *Let $K = [a, b]$ with $0 < a < b$ and $F(x, v) = -v$. We obtain*

$$Viab_F^{(2)}(K) = \{(x, v) \in \mathcal{G}(T_K); \quad a - x \leq v \leq b - x\}$$

Let $(x_0, v_0) \in \mathcal{G}(T_K)$. The second order problem:

$$x''(t) = -x'(t); \quad x(0) = x_0, \quad x'(0) = v_0$$

has the only solution $x(t) = x_0 + v_0(1 - e^{-t})$. If $(x_0, v_0) \in Viab_F^{(2)}(K)$ then $a \leq x_0 + v_0(1 - e^{-t}) \leq b$ for all $t \in [0, \infty[$ so $a - x_0 \leq v_0 \leq b - x_0$. Conversely, if this condition holds we have:

If $v_0 > 0$ $(v_0 < 0)$ *then* $v_0(1-e^{-t})$ *is increasing (uncreasing). Therefore* $a - x_0 \leq 0 < v_0(1-e^{-t}) < v_0 \leq b - x_0$, $(a - x_0 \leq v_0 < v_0(1-e^{-t}) \leq 0 \leq b - x_0)$. *If* $v_0 = 0$ *then* $x(t) = x_0$ *and* $a - x_0 \leq 0 \leq b - x_0$. *Hence,* $x(\cdot)$ *is a global viable solution, which completes our claim.*

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