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METHOD OF AVERAGING FOR IMPULSIVE DIFFERENTIAL INCLUSIONS*

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The paper deals with impulsive differential inclusions in the euclidean space. The main purpose is to justify the method of averaging in the case of bounded and asymptotically small impulses. The obtained results, which are based on the condition of an integral continuity, generalize the first Bogoljubov's theorem for the method of averaging.

Keywords: method of averaging, differential inclusion, impulsive differential inclusion, small parameter.

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1 Introduction

The purpose of the present paper is to contribute to the method of averaging theory for the impulsive differential inclusions, when the trajectories jump at nonfixed moments. We consider the both bounded (finite) and asymptotically small impulses. We refer the reader to [2, 3, 5, 6, 7, 9, 10, 11], where the justification of averaging method is considered for differential equations with asymptotically small impulses in fixed moments. The paper [11] also deals with differential inclusions. There are various algorithms of an averaging method which permit generalizations (see f.e. [8, 9]) and the classical statement of the problem can be obtained by suitable substitutions. This is the reason why we prefer to set the problem in a nonclassical way.

Let $\mathrm{comp}\,(E)$ $[\mathrm{conv}\,(E)]$ be the metric space of nonempty compact [and convex] subsets of $E \subset \mathbb{R}^n$. The metric in these spaces is the common Hausdorff distance:

$$h(A,B) = \inf\{d \geq 0 \,|\quad S_d(A) \supset B \quad S_d(B) \supset A\},$$

where $A, B \in \mathrm{comp}\,(\mathbb{R}^n)$, $S_d(A) = \{x \in \mathbb{R}^n \mid \min_{y\in A} |x-y| \leq d\}$ is the closed d-neighborhood of the set A.

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Consider, in the domain $Q = D \times [0, L] \subset \mathbb{R}^n \times \mathbb{R}$, the following differential inclusions with impulsive effects:

$$\dot{x} \in F^1(t, x, \varepsilon), \quad x(0) = x_0, \quad t \neq \varepsilon\tau_i^1(x), \quad t \neq \sigma_i^1(x), \tag{1}$$

$$\Delta x|_{t=\varepsilon\tau_i^1(x)} \in \varepsilon I_i^1(x), \tag{2}$$

$$\Delta x|_{t=\sigma_i^1(x)} \in K_i^1(x). \tag{3}$$

To the inclusion (1)–(3) we assign the following differential inclusion:

$$\dot{y} \in F^2(t, y, \varepsilon), \quad y(0) = x_0, \quad t \neq \varepsilon\tau_i^2(y), \quad t \neq \sigma_i^2(y), \tag{4}$$

$$\Delta y|_{t=\varepsilon\tau_i^2(y)} \in \varepsilon I_i^2(y), \tag{5}$$

$$\Delta y|_{t=\sigma_i^2(y)} \in K_i^2(y), \tag{6}$$

where $F^j(t, x, \varepsilon)$ are multi-functions, defined on $(n+1)$-dimensional euclidean space with nonempty compact and convex values, i.e. $F^j : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \operatorname{conv}(\mathbb{R}^n)$, $I_i^j : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^n)$, $K_i^j : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^n)$, $\tau_i^j : \mathbb{R}^n \to \mathbb{R}$, $\sigma_i^j : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, k$, $j = 1, 2$ $t \in I = [0, L]$, $x \in D \subset \mathbb{R}^n$, Δx, respectively Δy, is the jump of the solution $x(t)$, respectively $y(t)$, of the differential inclusion and $\varepsilon \in (0, \varepsilon_1)$ is a small parameter, ε_1 is fixed.

We suppose, that

$$\lim_{\varepsilon \to 0} h\Bigg(\frac{1}{\Delta t}\Big(\int_t^{t+\Delta t} F^1(t, x, \varepsilon)\, dt + \varepsilon \sum_{t < \varepsilon\tau_i^1(x) < t+\Delta t} I_i^1(x)\Big), \frac{1}{\Delta t}\Big(\int_t^{t+\Delta t} F^2(t, x, \varepsilon)\, dt + \varepsilon \sum_{t < \varepsilon\tau_i^2(x) < t+\Delta t} I_i^2(x)\Big)\Bigg) = 0. \tag{7}$$

The equality (7) is said to be the condition of an integral continuity. It should be considered as a generalization of the Bogoljubov's averaging method (see **Remark 2.1**).

Following the Bogoljubov's theorem, the classical form of the above differential inclusions have to be:

$$\dot{x} \in \varepsilon X(t, x), \quad x(0) = x_0, \quad t \neq \tau_i^1, \quad t \in [0, L\varepsilon^{-1}],$$

$$\Delta x|_{t=\tau_i^1(x)} \in \varepsilon I_i^1(x)$$

with its corresponding averaged impulsive differential inclusion:

$$\dot{y} \in \varepsilon Y(t, y), \quad y(0) = x_0, \quad t \neq \tau_i^2, \quad t \in [0, L\varepsilon^{-1}], \tag{8}$$

$$\Delta y|_{t=\tau_i^2(y)} \in \varepsilon I_i^2(y), \tag{9}$$

The condition (7) of the integral continuity, which is a modification of the respective main value Bogoljubov's condition (see f.e. Plotnikov [9]), has the following form:

$$\lim_{T \to \infty} h\Bigg(\frac{1}{T}\Big(\int_0^T X(t, x)\, dt + \sum_{0 \le \tau_i^1 < T} I_i^1(x)\Big), \frac{1}{T}\Big(\int_0^T Y(t, x)\, dt + \sum_{0 \le \tau_i^2 < T} I_i^2(x)\Big)\Bigg) = 0. \tag{10}$$

By the simple substitution $s = \varepsilon t$ one comes to (1)–(2) and (4)–(5) with the condition (7), where $F^1(s,x,\varepsilon) = X(\frac{s}{\varepsilon}, x)$ and $F^2(s,x,\varepsilon) = Y(\frac{s}{\varepsilon}, x)$.

Remark 1.1 *If there exist the limits:*

$$\overline{X}(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T X(s,x)\,ds,$$

$$\overline{I}(x) = \lim_{T\to\infty} \frac{1}{T}\sum_{0\le\tau_i^1<T} I_i^1(x),$$

then one changes the system (8)–(9) *to the following differential inclusion without impulses:*

$$\dot{y} \in \varepsilon[\overline{X}(y) + \overline{I}(y)], \quad y(0) = x_0.$$

Remark 1.2 *(The periodical case.) If the multi-function $X(t,x)$ has a time-period $t = 2\pi$, i.e. $X(t,x) = X(t+2\pi, x)$, $x \in D$, and there exists a number p for which $\tau_{i+p}^1 = \tau_i^1 + 2\pi$, $I_{i+p}^1 = I_i^1$, $x \in D$, then we can set*

$$\overline{X}(x) = \frac{1}{2\pi}\int_0^{2\pi} X(t,x)\,dt,$$

$$\overline{I}(x) = \frac{1}{2\pi}\sum_{0\le\tau_i^1<2\pi} I_i^1(x).$$

In this case, the averaging system (8)–(9) *can be written, for example, in the following forms:*

$$\dot{y} \in \varepsilon\overline{X}(y), \quad t \neq 2\pi i, \quad \Delta y|_{t=2\pi i} \in \varepsilon\overline{I}(y),$$

or

$$\dot{y} = 0, \quad t \neq 2\pi i, \quad \Delta y|_{t=2\pi i} \in [\overline{X}(y) + \overline{I}(y)].$$

The last one is a discrete system.

2 Main Results

We need some technical results. One proves the following technical lemma:

Lemma 2.1 *Let $\{\delta_i^-\}_{i=1}^\infty$ and $\{\delta_i^+\}_{i=0}^\infty$ be two scalar sequences for which*

$$\delta_{i+1}^+ \le a_1\delta_{i+1}^- + a_2,$$

$$\delta_{i+1}^- \le a_3\delta_i^+ + a_4,$$

where $a_j \ge 0, j = 1,\ldots,4$ and $i = 0,1,2,\ldots$. Then

$$\delta_{i+1}^- \le (a_2a_3 + a_4)\frac{(a_1a_3)^i - 1}{a_1a_3 - 1} + (a_1a_3)^i\delta_0^+ \quad \text{if} \quad a_1a_3 \neq 1,$$

$$\delta_{i+1}^- \le (a_2a_3 + a_4)i + \delta_0^+, \quad \text{if} \quad a_1a_3 = 1.$$

First we consider the differential inclusions (1) and (4) with the only asymptotically small impulsive effects (2) and respective effects (5). Denote $|F| = h(0, F)$ ($F \in \mathrm{comp}\,(\mathbb{R}^n)$) and $T_D(x)$ – the Bouligand contingent cone for $x \in D$ (see f.e. Aubin and Frankowska [1]), i.e.

$$T_D(x) = \{y \in \mathbb{R}^n \mid \lim_{s \to +0} s^{-1} \inf_{z \in D} |x + sy - z| = 0\}.$$

and $J_j(t, t+\Delta)$, $j = 1, 2$, the number of impulses on the interval $[t, t+\Delta]$ ($0 \le t, t+\Delta \le L$) of the solutions of (1)–(2) and (4)–(5) respectively. Further, we suppose that $\frac{1}{\Delta} J_j(t, t+\Delta) \le \frac{A}{\varepsilon} < \infty$, $j = 1, 2,$. The trivial case of such kind of restrictions is the time-fixed moments of impulses.

Theorem 2.1 *Let in the domain Q the following three conditions be fulfilled:*

1) *The multi-valued maps $F^j(t, x, \varepsilon)$ and the functions $\tau_i^j(x)$ are bounded ($|F^j(t, x, \varepsilon)|$, $|I_i^j(x)|$, $|\tau_i^j(x)| \le M < \infty$), Lipschitz continuous in x with a constant λ and, moreover, $F^j(t, x, \varepsilon) \subset T_D(x)$ are continuous in t, $t \in [0, L]$, $\varepsilon \in (0, \varepsilon_1)$, $x + I_i^j(x) \subset D$, $x \in D$, $j = 1, 2$.*

2) *The limit in the condition* (7) *of an integral continuity is uniform with respect to $(t, x) \in Q$.*

3) *The numbers $J_j(t, t+\Delta)$, $j = 1, 2$, of the asymptotically small impulses on the interval $[t, t+\Delta] \subset [0, L]$ satisfy the following inequalities: $\frac{1}{\Delta} J_j(t, t+\Delta) \le \frac{A}{\varepsilon} < \infty$ ($t, t+\Delta \in [0, L]$, A is a constant), the surfaces $t = \varepsilon \tau_i^j(x)$ do not intersect each other and for every $x \in D$, $z \in I_i^j$, $j = 1, 2$, the following inequalities hold:*

$\tau_i^j(x) \ge \tau_i^j(x + z)$.

Then for every $\xi > 0$ there exists $\varepsilon(\xi) \in (0, \varepsilon_1)$ for which:

1) *If $y(t) = y(t; \varepsilon)$ is a solution of* (4)–(5) *with $\varepsilon \in (0, \varepsilon(\xi))$, then there exists a solution $x(t) = x(t; \varepsilon)$ of (1)–(2) such that*

$$|x(t) - y(t)| \le \xi, \quad x(0) = y(0) \tag{11}$$

2) *If $x(t) = x(t; \varepsilon)$ ($\varepsilon \in (0, \varepsilon(\xi))$) is a solution of* (1)–(2), *then there exists a solution $y(t) = y(t; \varepsilon)$ of* (4)–(5) *such that the inequality* (11) *holds.*

Proof. Let $y(t)$ be any solution of (4)–(5). Note that under the condition 1) of the theorem there exist solutions of (1)–(2) and (4)–(5) which are extendable on $[0, L]$ and all solutions belong to the domain D.

We claim that the "beating phenomena" is avoided by choosing $\varepsilon_1 > 0$ sufficiently small. Suppose that the claim is not valid for the solution $y^j(t)$ of (1)–(2) or respectively of (4)–(5). Let $t_0 = \varepsilon \tau_i^j(y^j(t_0))$ and let the solution $y^j(t)$ with initial condition $y^j(t_0) + z \in y^j(t_0) + \varepsilon I_i^j(y^j(t_0))$ intersect the same surface at the moment t^*, i.e. $t^* = \varepsilon \tau_i^j(y^j(t^*))$, where $y^j(t)$ is continuous on the interval (t_0, t^*). Integrating we obtain

$$y^j(t^*) = y^j(t_0) + \varepsilon I_i^j(y^j(t_0)) + \int_{t_0}^{t^*} u(\tau)\, d\tau, \quad u(t) \in F^j(t, y^j(t), \varepsilon),$$

where, without any loss of the generality, we replace z with $\varepsilon I_i^j(y^j(t_0))$. Applying the Lipschitz condition we have

$$|\varepsilon\tau_i^j(y^j(t^*)) - \varepsilon\tau_i^j\Big(y^j(t_0) + \varepsilon I_i^j(y^j(t_0))\Big)| \leq \varepsilon\lambda \int_{t_0}^{t^*} u(\tau)\, d\tau \leq \varepsilon\lambda M(t^* - t_0).$$

Choosing $0 < \varepsilon < \varepsilon_1 < \dfrac{1}{\lambda M}$ one becomes to a contradiction with the condition 3) of the theorem:

$$t^* - t_0 = \varepsilon\tau_i^j(y^j(t^*)) - \varepsilon\tau_i^j(y^j(t_0)) =$$

$$\varepsilon\tau_i^j(y^j(t^*)) - \varepsilon\tau_i^j(y^j(t_0) + \varepsilon I_i^j(y^j(t_0))) + \varepsilon\tau_i^j(y^j(t_0) + \varepsilon I_i^j(y^j(t_0))) - \varepsilon\tau_i^j(y^j(t_0)) \leq$$

$$\leq \varepsilon\lambda M(t^* - t_0) + \varepsilon\tau_i^j(y^j(t_0) + \varepsilon I_i^j(y^j(t_0))) - \varepsilon\tau_i^j(y^j(t_0)).$$

Thus,

$$(1 - \varepsilon\lambda M)(t^* - t_0) \leq \varepsilon\tau_i^j(y^j(t_0) + \varepsilon I_i^j(y^j(t_0))) - \varepsilon\tau_i^j(y^j(t_0))$$

which contradicts the condition 3) of the theorem.

Moreover, if we suppose that the solution $y^j(t)$ intersects another surface $t = \varepsilon\tau_k^j(x)$ at the moment $t_* \in (t_0, t^*)$ then we have:

$$t_0 = \varepsilon\tau_i^j(y^j(t_0)) > \varepsilon\tau_i^j(y^j(t)),\ t_0 < t \leq t_*;$$

$$t_* = \varepsilon\tau_k^j(y^j(t_*)) > \varepsilon\tau_k^j(y^j(t)),\ t_* < t \leq t^*;$$

$$t^* = \varepsilon\tau_i^j(y^j(t^*)).$$

For every arbitrarily chosen continuous function $z(t)$ such that $z(t_*) = y^j(t_*)$ and $z(t^*) = y^j(t^*)$ there exists $\overline{t} \in (t_*, t^*]$ for which $\tau_i^j(z(\overline{t})) = \tau_k^j(z(\overline{t}))$. The last one equality contradicts the condition that surfaces $t = \varepsilon\tau_i^j(x)$ and $t = \varepsilon\tau_k^j(x)$ does not intersect each other. One concludes that every solution intersects every surface no more than one time.

We intend to prove the theorem by the following way:

Discretizing the interval $[0, L]$ we find a function $y^1(t)$ which is sufficiently close to $y(t)$ and for which we can directly apply the condition (7) of an integral continuity. According to (7), we find a function $x^1(t)$ which is close to $y^1(t)$ and to the graph of the solutions set of (1)–(2). Applying the Filippov's theorem (see f.e. Aubin and Frankowska [1]) and choosing suitable impulses, we find the needed solution $x(t)$ of (1)–(2).

The proof of the second conclusion of the theorem will be miss because of changing $y(t)$ to $x(t)$ one can do it.

Let $[t_k, t_{k+1}]$, where $t_k = k\dfrac{L}{m}$, $k = 0, 1, 2, \ldots, m$, $t_0 = 0$, $t_m = L$ be the partition of the interval $[0, L]$. As long as $y(t)$ is a solution of (4)–(5) there is a measurable selection $v(t) \in F^2(t, y(t), \varepsilon)$ (see f.e. Blagodatskikh and Filippov [4]) and impulsive vectors $p_i \in I_i^2(y(\varepsilon\tau_i^2))$ for which

$$y(t) = y(t_k) + \int_{t_k}^{t} v(\tau)\, d\tau + \varepsilon \sum_{t_k \leq \varepsilon\tau_i^2(y) < t} p_i, \quad y(0) = x_0, \quad t \in [t_k, t_{k+1}].$$

For every $k = 1, 2, \dots, m$ we define

$$y^1(t) = y^1(t_k) + \int_{t_k}^{t} \overline{v}(\tau)\, d\tau + \varepsilon \sum_{t_k \le \varepsilon\tau_i^2(y) < t} \overline{p}_i, \quad y^1(0) = x_0, \quad t \in [t_k, t_{k+1}],$$

where $\overline{v}(\tau) \in F^2(\tau, y^1(t_k), \varepsilon)$, $\overline{p}_i \in I_i^2(y^1(t_k))$ for $\varepsilon\tau_i^2(y) \in [t_k, t_{k+1})$ and additionally they satisfy

$$\left| \int_{t_k}^{t_{k+1}} \overline{v}(\tau)\, d\tau - \int_{t_k}^{t_{k+1}} v(\tau)\, d\tau \right| = \min_{z(\tau) \in F^2(\tau, y^1(t_k), \varepsilon)} \left| \int_{t_k}^{t_{k+1}} z(\tau)\, d\tau - \int_{t_k}^{t_{k+1}} v(\tau)\, d\tau \right|,$$

$$|\overline{p}_i - p_i| = \min_{r \in I_i^2(y^1(t_k))} |r - p_i|.$$

Let us denote $\delta_k = |y(t_k) - y^1(t_k)|$ and remember that $\dfrac{1}{\Delta} J_2(t, t+\Delta) \le \dfrac{A}{\varepsilon} < \infty$ one can write

$$|y(t) - y^1(t_k)| \le |y(t) - y(t_k)| + |y(t_k) - y^1(t_k)| \le$$

$$\delta_k + (t - t_k)M + \varepsilon M \frac{A}{\varepsilon}(t - t_k) \le \delta_k + M(1 + A)(t - t_k). \tag{12}$$

For $|y(t_{k+1}) - y^1(t_{k+1})|$ we have

$$\delta_{k+1} = |y(t_{k+1}) - y^1(t_{k+1})| \le \delta_k + \left| \int_{t_k}^{t_{k+1}} \overline{v}(\tau)\, d\tau - \int_{t_k}^{t_{k+1}} v(\tau)\, d\tau \right| +$$

$$\varepsilon \Big| \sum_{t_k \le \varepsilon\tau_i^2(y) < t_{k+1}} (\overline{p}_i - p_i) \Big| \le$$

$$\delta_k + \int_{t_k}^{t_{k+1}} h\Big(F^2(\tau, y(\tau), \varepsilon), F^2(\tau, y^1(t_k), \varepsilon)\Big)\, d\tau +$$

$$\varepsilon \sum_{t_k \le \varepsilon\tau_i^2(y) < t_{k+1}} h\Big(I_i^2(y^1(t_k)), I_i^2(y(\varepsilon\tau_i^2))\Big) \le$$

$$\delta_k + \int_{t_k}^{t_{k+1}} \lambda |y(t) - y^1(t_k)| + \varepsilon \sum_{t_k \le \varepsilon\tau_i^2(y) < t_{k+1}} \lambda |y(\varepsilon\tau_i^2) - y^1(t_k)| \le$$

$$\le \delta_k + \lambda[\delta_k(t_{k+1} - t_k) + M(1 + A)\frac{(t_{k+1} - t_k)^2}{2}] + A\lambda[\delta_k + M(1 + A)(t_{k+1} - t_k)](t_{k+1} - t_k).$$

Setting $a = \lambda(1 + A)$, $\quad b = \lambda M(\dfrac{1 + A}{2}) + A\lambda M(1 + A)$, $\quad \Delta t = \dfrac{L}{m}$ we have

$$\delta_{k+1} \le (1 + a\Delta t)\delta_k + b\Delta t^2.$$

Thus, we have the following sequence of estimates:

$$\delta_0 = |y^1(0) - y(0)| = 0$$

$$\delta_1 \le b\Delta t^2$$
$$\delta_2 \le (1 + a\Delta t)b\Delta t^2 + b\Delta t^2$$
$$\delta_3 \le (1 + a\Delta t)^2 b\Delta t^2 + (1 + a\Delta t)b\Delta t^2 + b\Delta t^2$$
$$\delta_{k+1} \le (1 + a\Delta t)^k b\Delta t^2 + (1 + a\Delta t)^{k-1} b\Delta t^2 + \cdots + (1 + a\Delta t)b\Delta t^2 + b\Delta t^2.$$

One obtains:

$$\delta_{k+1} \le \frac{(1 + a\Delta t)^{k+1} - 1}{a\Delta t} b\Delta t^2 \le \frac{b}{a}\frac{L}{m}[(1 + a\frac{L}{m})^m - 1] \le \frac{b}{a}\frac{L}{m}(e^{aL} - 1).$$

Therefore

$$\delta_{k+1} \le \frac{ML(0,5 + A)}{m}(e^{\lambda(1+A)L} - 1) \tag{13}$$

Then from (12), (13) one obtains:

$$\begin{aligned} |y(t) - y^1(t)| &\le |y(t) - y^1(t_k)| + |y^1(t_k) - y^1(t)| \\ &\le \tfrac{ML(1+A)}{m}(e^{\lambda(1+A)L} + 1), \quad t \in [0, L]. \end{aligned} \tag{14}$$

On every interval $[t_k, t_{k+1}]$ we are going to define a function

$$x^1(t) = x^1(t_k) + \int_{t_k}^t \overline{u}(\tau)\, d\tau + \varepsilon \sum_{t_k \le \varepsilon\tau_i^1(y) < t} \overline{q}_i, \quad x^1(0) = x_0$$

which is sufficiently close to $y^1(t)$. Under the condition (7) of an integral continuity, for every $\eta > 0$ there are $\varepsilon(\eta) > 0$, $\overline{u}(\tau) \in F^1(t, y^1(t_k), \varepsilon)$ and $\overline{q}_i \in I_i^1(y^1(t_k)$ such that

$$\left| \Big[\int_{t_k}^t \overline{u}(\tau)\, d\tau + \varepsilon \sum_{t_k \le \varepsilon\tau_i^1(y) < t} \overline{q}_i\Big] - \Big[\int_{t_k}^t \overline{v}(\tau)\, d\tau + \varepsilon \sum_{t_i \le \varepsilon\tau_i^2(y) < t} \overline{p}_i\Big] \right| < \eta\Delta t, \quad 0 < \varepsilon < \varepsilon(\eta).$$

Let us fix m and choose $\eta = \Delta t = \dfrac{L}{m}$. For $t \in [t_k, t_{k+1}]$, $k = 1, 2, \ldots, (m - 1)$, one obtains

$$|x^1(t) - y^1(t)| \le |x^1(t_k) - y^1(t_k)| + \eta\Delta t = \delta_k + \eta\Delta t = \delta_k + \Delta t^2.$$

Remember $\delta_0 = |x^1(0) - y^1(0)| = 0$, we have

$$|x^1(t) - y^1(t)| \le \frac{L^2}{m}, \quad t \in [0, L]. \tag{15}$$

We claim that there exists a solution

$$x(t) = x(t_k) + \int_{t_k}^t u(\tau)\, d\tau + \varepsilon \sum_{t_k \le \varepsilon\tau_i^1(x) < t} q_i, \quad x(0) = x_0, \quad t \in [t_k, t_{k+1}),$$

of (1)–(2) which is sufficiently close to $x^1(t)$.

Let us denote $\rho(x, F) = \min\limits_{y \in F} |x - y|$ the distance between the point x and the set F. Remember that $\dot{x}^1(t) \in F^1(t, y^1(t_k), \varepsilon)$, by the Lipschitz condition and (15), we can write:

$$\rho\Big(\dot{x}^1(t), F^1(t, x^1(t), \varepsilon))\Big) \le$$

$$h(F^1(t,y^1(t_k),\varepsilon),F^1(t,x^1(t),\varepsilon))\le\lambda|y^1(t_k)-x^1(t)|\le$$

$$\lambda(\frac{L^2}{m}+M(1+A)\Delta t)=\frac{\lambda L(L+(1+A)M)}{m}=\gamma.$$

On every interval without impulses we can apply the Filippov's theorem (see Aubin and Frankowska [1]) to obtain the solution $x(t)$ on these intervals. Let $[s_1,s_2)$ be any of these intervals. If $\delta=|x(s_1)-x^1(s_1)|$ and $\eta(t)=e^{\lambda(t-t_k)}(\delta+\gamma(t-t_k))$ then, under the Filippov's theorem, the following inequalities

$$|x(t)-x^1(t)|\le\eta(t),\quad |\dot x(t)-\dot x^1(t)|\le\lambda\eta(t)+\gamma,\quad t\in[s_1,s_2),$$

are valid.

We are going to estimate the differences between $x^1(t)$ and $x(t)$ at the impulsive time-points. Let s_i^- and s_i^+ $(s_i^-\le s_i^+)$ be the moments when $x^1(t)$ and the solution $x(t)$ reaches the surfaces $t=\varepsilon\tau_i^1(x)$.

For any solution $x(t)$ of (1)–(2) we denote $\delta_i^-=|x^1(s_i^-)-x(s_i^-)|$ and $\delta_i^+=|x^1(s_i^+)-x(s_i^++0)|$ $(\delta_0^+=0)$. Without any loss of the generality we suppose that $\tau_i^1(x_0)\ne0$, s_i^- is the moment when $x^1(t)$ reaches the surface $t=\varepsilon\tau_i^1(x)$ and s_i^+ is the moment when $x(t)$ reaches the same surface.

One can write the following estimations:

$$|x^1(s_i^-)-x(s_i^+)|\le|x^1(s_i^-)-x(s_i^-)|+|x(s_i^-)-x(s_i^+)|\le\delta_i^-+M(s_i^+-s_i^-),$$

$$s_i^+-s_i^-=|\varepsilon\tau_i^1(x^1(s_i^-))-\varepsilon\tau_i^1(x(s_i^+))|\le$$

$$\varepsilon\lambda(\delta_i^-+M(s_i^+-s_i^-)),$$

i.e.

$$(s_i^+-s_i^-)\le\frac{\varepsilon\lambda\delta_i^-}{1-\varepsilon\lambda M}.$$

We take $x(s_1^++0)$ such that:

$$\delta_i^+=|x^1(s_i^+)-x(s_i^++0)|=\min_{x\in x(s_i^+)+I_i^1(x(s_i^+))}|x^1(s_i^+)-x|\le$$

$$h\Big(x^1(s_1^-)+\varepsilon I_1^i(x^1(s_i^-))+\int_{s_1^-}^{s_1^+}\overline{u}(s)\,ds,x(s_i^-)+\varepsilon I_i^1(x(s_i^+))+\int_{s_i^-}^{s_i^+}u(s)\,ds\Big)\le$$

$$|x^1(s_i^-)-x(s_i^-)|+\Big|\int_{s_1^-}^{s_1^+}(\overline{u}(s)-u(s))\,ds\Big|+\varepsilon h\Big(I_i^1(x^1(s_i^-)),I_i^1(x(s_i^+))\Big)\le$$

$$\delta_i^-+2M(s_i^+-s_i^-)+\varepsilon\lambda(\delta_i^-+M(s_i^+-s_i^-))=$$

$$\frac{1+\varepsilon\lambda+\varepsilon\lambda M}{1-\varepsilon\lambda M}\delta_i^-,$$

i.e. for all sufficiently small $\varepsilon>0$ we can write

$$\delta_i^+\le\frac{1+\varepsilon\lambda+\varepsilon\lambda M}{1-\varepsilon\lambda M}\delta_i^-\le a_1\delta_i^-.$$

By analogy, one obtains:

$$\delta_i^- \le a_3\delta_{i-1}^+ + a_4\varepsilon,\ i = 1, 2, \ldots.$$

According to the Lemma 1 one obtains:

$$\delta_{i+1}^- = K_1\varepsilon + K_2\delta_0^+, \tag{17}$$

where, for all sufficiently small $\varepsilon > 0$, K_1, K_2 are constant, which depend on λ, M. Taking suitable ε ($\delta_0^+ = 0$) from (14), (15), (16) and (17) one obtains validity (11). □

Remark 2.1 *To obtain the classical Bogoljubov's integral continuity condition we leave ε to zero and T to infinity in the condition* (10). *In this sense Theorem* 1 *partially generalizes the first Bogoljubov's theorem for the method of averaging.*

Remark 2.2 *The inclusion $F^j(t, x, \varepsilon) \subset T_D(x)$, $t \in [0, L]$, $\varepsilon \in (0, \varepsilon_1)$, can be replaced by the following condition: For every $x_0 \in D' \subset D$ there exists a constant $\rho > 0$ for which all solutions of* (1)–(2) *and* (4)–(5) *belong to the domain D on the interval $[0, L]$ with a ρ-neighborhood. In this case the condition $x + I_i^j(x) \subset D$, $x \in D$ is superfluous. Note that, in generally, the above conditions are not equivalent.*

Corollary 2.1 *Let the conditions of Theorem* 1 *be fulfilled. Consider the inclusion* (4)–(5) *with $y(0) = y_0$ and denote $\delta = |x_0 - y_0|$. Then for any $\xi > 0$ and for every solution $y(t)$ of* (4)–(5) *there exist $\varepsilon(\xi) > 0$, $\delta(\xi) > 0$, constant C and a solution $x(t)$ of* (1)–(2) *such that*

$$|x(t) - y(t)| < C\delta + \xi, \quad t \in [0, L], \quad 0 < \varepsilon < \varepsilon(\xi), \quad 0 < \delta < \delta(\xi).$$

In generally, the proof of the Corollary 1 repeats the proof of the Theorem 1 and we miss it.

We are going to prove the theorem about the method of averaging for the impulsive differential inclusion (1)–(3) and for the disturbed system (4)–(6). Denote $[s_i^-, s_i^+]$ the time-intervals between the respective intersections of $x(t)$ and $y(t)$ with the surfaces $t = \sigma_i^j$ ($j = 1, 2$).

Theorem 2.2 *Let in the domain Q all conditions of Theorem* 1 *be fulfilled and additionally:*

4) *The maps $K_i^j(x)$ and the functions $\sigma_j^i(x)$ $i = 1, 2, \ldots$, ($j = 1, 2$) satisfy the Lipschitz condition with a constant μ, $x + K_i^j(x) \subset D$, $x \in D$.*

5) *The surfaces $t = \sigma_j^i(x)$ do not intersect each other and $\sigma_j^i(x) \ge \sigma_j^i(x+z)$ for every $x \in D$ and $z \in K_i^j$, $j = 1, 2$.*

6) *The following inequalities hold:*

$$h(K_i^1(x), K_i^2(x)) \le \eta, \quad |\sigma_i^1(x) - \sigma_i^2(x)| \le \eta, \quad |x_0 - y_0| \le \delta.$$

If $\mu M < 1$ then for every $\xi > 0$ there exists $\eta > 0$ and $\delta_0^+ > 0$ such that for every solution $x(t)$ of (1)–(3) *there exists solution $y(t)$ of* (4)–(6) *for which the following estimate*

$$|x(t) - y(t)| \le \xi \tag{16}$$

is valid for all $t \in [0, L] \setminus \bigcup_i [s_i^-, s_i^+]$, where $\sum_i [s_i^+ - s_i^-] \le \xi$.

The proof of the Theorem 2, in generally, repeats the proof of the Theorem 1 if the citing of the Filippov's theorem is replaced by the citing of the Theorem 1.

It is available to suppose that the surfaces $t = \sigma_i^2(y)$ should be intersect each other. The proofs of the theorems in this case have to pass through the technical problems with the numeration of the surfaces.

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