PLISKA
STUDIA MATHEMATICA
BULGARICA

ПЛИСКА
БЪЛГАРСКИ
МАТЕМАТИЧЕСКИ
СТУДИИ

Pliska Stud. Math. Bulgar. **12** (1998), 201-212

PLISKA
STUDIA MATHEMATICA
BULGARICA

CONTROL SYSTEMS WITH CONSTRAINTS AND UNCERTAIN INITIAL CONDITIONS

M. Quincampoix, V. Veliov

We study the problem of finding a control such that all solutions of a control systems, starting from a given set of initial conditions, satisfy a given constraint. This problem is an extension of the well-known Viability Problem when the initial condition is a set. The present paper is mainly a survey of results recently obtained by the authors, but some new results with proofs are also included.

Keywords: viability, uncertainty, set equations.

AMS subject classification: 49N55, 93B52, 93C15, 93C10, 26E25.

1 Introduction

Let us consider the control system

$$x'(t) = f(x(t), u(t)) \text{ for almost every } t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^r$, $f : \mathbb{R}^n \times U \mapsto \mathbb{R}^n$. Let a state constraint $x \in K \subset \mathbb{R}^n$ be given.

We denote by $t \mapsto x[x_0, u(\cdot)](t)$ the solution to (1) with initial condition $x[x_0, u(\cdot)](0) = x_0$ and by $\mathcal{U}$ – the set of all measurable functions $[0, +\infty) \mapsto U$.

The problem that we investigate can be presented in two different ways:

- The first way comes from Viability Theory (cf. [2]) where the basic problem is: given $x_0 \in K$, is there a measurable control $u(\cdot) \in \mathcal{U}$ such that the solution $x[x_0, u(\cdot)]$ is *viable* in K, namely $x[x_0, u(\cdot)](t) \in K$ for every $t \geq 0$? If this problem is solvable for any initial condition x_0 in K, then the set K is called viable. Let us consider, for a moment, such a set K and $x_0 \in K$. Since K is viable, there exists some $u(\cdot) \in \mathcal{U}$ such that $x[x_0, u(\cdot)]$ is a viable trajectory. But one can expect that *for the same control* $u(\cdot)$ there are some points x_1 near enough to x_0 such that $x[x_1, u(\cdot)]$ is also viable in K. The problem we investigate consists of finding and characterizing sets E_0 of such points x_1. We also look for the maximal set containing x_0 that has this property.

- The second way, which reveals the main motivation for this paper, is related to the lack of information about the initial condition: we consider a control system with an initial condition x_0 that is not exactly known, rather it is only known that x_0 belongs to some set $E_0 \subset K$.

The problem is then the following: given $E_0 \subset K$, is there a measurable control $u(\cdot) : [0, +\infty) \mapsto U$ such that all trajectories of (1) starting from E_0 are viable in K. Using the notation $x[E_0, u(\cdot)](t) := \{x[x_0, u(\cdot)](t) : \; x_0 \in E_0\}$, the problem is to find $u(\cdot)$ such that

$$x[E_0, u(\cdot)](t) \subset K \quad \forall t \geq 0. \tag{2}$$

We shall characterize the family $\hat{\mathcal{E}}$ of those initial sets E_0 for which there exist a viable control $u(\cdot) \in \mathcal{U}$, that is, a function $u(\cdot)$ for which (2) is fulfilled.

If $E_0 \in \hat{\mathcal{E}}$ and if $u(\cdot)$ is a control for which (2) is fulfilled, then the set

$$E(t) := x[E_0, u(\cdot)](t) \in \hat{\mathcal{E}} \quad \forall t \geq 0. \tag{3}$$

is the exact guaranteed estimation of the state of the system, given the initial set E_0 and the control $u(\cdot)$ up to the moment t. We shall see that the determination of $\hat{\mathcal{E}}$ is related to the evolution of the set $E(t)$.

In fact we investigate an essentially more general problem. Instead of looking for the maximal set $\hat{\mathcal{E}}$ such that (3) is fulfilled for each $E_0 \in \hat{\mathcal{E}}$, we suppose that a collection of sets $\mathcal{E}^*$ is a priori given (presumably consisting of subsets of K). Then the problem is to find the maximal subcollection (denoted further by $\mathrm{Viab}(\mathcal{E}^*)$) such that for each $E_0 \in \mathrm{Viab}(\mathcal{E}^*)$ there exists a measurable $u(\cdot) : [0, +\infty) \mapsto U$ for which

$$x[E_0, u(\cdot)](t) \subset E(t) \in \mathrm{Viab}(\mathcal{E}^*) \quad \forall t \geq 0. \tag{4}$$

The reason for this generality is rather practical. It allows to restrict all considerations to sets from $\mathcal{E}^*$ only – both the current estimations $E(t)$, and the collection $\mathrm{Viab}(\mathcal{E}^*)$ sought for, should consist of sets from $\mathcal{E}^*$. Choosing a relatively simple collection $\mathcal{E}^*$ (balls, ellipsoids, boxes, etc., in K) one may come up with constructive approximation schemes for finding (approximating) $\mathrm{Viab}(\mathcal{E}^*)$.

The organization of the paper is the following. The first section is devoted to some preliminaries and to extending the notion of contingent cone (used in Viability Theory) to the present context. In Section 2, we appropriately generalize the already "classical" concepts of Viability Domains and Kernels. The third section deals with stating our main results of characterization of Viability with an uncertain initial condition. In Section 4, we describe an approximation procedure for Viability Kernels.

2 Contingent fields

Before extending the notion of contingent cone in a way that is relevant to the viability control problems under uncertainties, we introduce some notions and notations. Let comp($\mathbb{R}^n$) be the set of all compact subsets of $\mathbb{R}^n$ and let $\mathbf{B}$ be the unit ball in $\mathbb{R}^n$ with respect to Euclidean norm $|\cdot|$. Let dist(x, A) be the distance from x to the set A. We denote by

$$H^+(A,B) := \max_{a\in A} \text{dist}(a,B) = \inf\{\varepsilon \geq 0 : \ A \subset B + \varepsilon\mathbf{B}\}$$

the Hausdorff semidistance from A to B and by $H(A,B)$ the Hausdorff distance between A and B. With this distance the set comp($\mathbb{R}^n$) is a complete metric space.

Further on, speaking about a collection (of sets) we shall always mean a nonempty subset $\mathcal{E} \subset$ comp($\mathbb{R}^n$) that satisfies the following condition.

Condition A. For every nonnegative M there is N such that whenever a set $E \in \mathcal{E}$ intersects $M\mathbf{B}$, this set is contained in $N\mathbf{B}$.

This condition is obviously fulfilled if $\mathcal{E}$ consists of compact subsets of a given bounded set in $\mathbb{R}^n$ but in general, boundedness of $\cup\{E : \ E \in \mathcal{E}\}$ is not necessary.

Viability Theory uses in a crucial way the notion of *contingent cone* introduced by Bouligand. The contingent cone $T_K(x)$ to $K \subset \mathbb{R}^n$ at the point $x \in K$ is the set of all $l \in \mathbb{R}^n$ such that

$$\liminf_{h\to 0+} \frac{1}{h}\text{dist}(x + hl, K) = 0.$$

Below we introduce the notion of *contingent field* to a collection of sets, which differs from the already existing extensions (cf. [4, 5, 1]) extensions in that it is based on the semimetric H^+ in comp($\mathbb{R}^n$) rather than on the Hausdorff metric.

Definition 2.1 *Let $\mathcal{E} \subset$ comp($\mathbb{R}^n$) be a collection of sets and let $Z \subset \mathbb{R}^n$. A continuous mapping $l(\cdot) : Z \mapsto \mathbb{R}^n$ is called contingent field to $\mathcal{E}$ at Z if and only if*

$$\liminf_{h\to 0+} \inf_{\tilde{E}\in\mathcal{E}} \sup_{x\in Z} \text{dist}(l(x), \frac{\tilde{E} - x}{h}) = 0.$$

$\mathcal{T}_{\mathcal{E}}(Z)$ will denote the set of all contingent fields to $\mathcal{E}$ at Z.

The motivation for this definition will be given in the third section. Let us provide an equivalent formulation which is easy to deduce from the definition.

Proposition 2.2 *The continuous mapping $l(\cdot) : Z \mapsto \mathbb{R}^n$ belongs to $\mathcal{T}_{\mathcal{E}}(Z)$ if and only if there are sequences $h_k \to 0+$ and $\gamma_k \to 0$ and corresponding $E_k \in \mathcal{E}$ such that*

$$x + h_k l(x) \in E_k + h_k\gamma_k\mathbf{B} \qquad \forall x \in Z. \tag{5}$$

When the collection consists only of singletons

$$\mathcal{E}^s_K = \{\{x\} : \ x \in K\}$$

then the contingent field $\mathcal{T}_{\mathcal{E}}(\{x\})$ consists of all constant functions whose values belong to $T_K(x)$.

In the general case $\mathcal{T}_{\mathcal{E}}(Z)$ shares some of the properties of $T_K(x)$.

Proposition 2.3 *Let $\mathcal{E}$ and $\mathcal{E}'$ be collections, Z and Z' be compact sets.*

1) *$\mathcal{T}_{\mathcal{E}}(Z) \neq \emptyset$ if and only if $Z \subset E$ for some $E \in$ cl$(\mathcal{E})$.*

2) *If nonempty,* $\mathcal{T}_{\mathcal{E}}(Z)$ *is a cone in the space of continuous functions* $C(Z, \mathbb{R}^n)$ *(from* Z *to* $\mathbb{R}^n$*) with vertex* $l(\cdot) = 0$*.*
3) *If* $\mathcal{E} \subset \mathcal{E}'$ *then* $\mathcal{T}_{\mathcal{E}}(Z) \subset \mathcal{T}_{\mathcal{E}'}(Z)$*.*
4) *If* $Z \subset Z'$ *then* $\mathcal{T}_{\mathcal{E}}(Z') \subset \mathcal{T}_{\mathcal{E}}(Z)$*.*
5) $\mathcal{T}_{\mathcal{E} \cup \mathcal{E}'}(Z) = \mathcal{T}_{\mathcal{E}}(Z) \cup \mathcal{T}_{\mathcal{E}'}(Z)$*.*
6) $\mathcal{T}_{\mathcal{E} \cap \mathcal{E}'}(Z) \subset \mathcal{T}_{\mathcal{E}}(Z) \cap \mathcal{T}_{\mathcal{E}'}(Z)$*.*

3 Viability domains and kernels

Here we give the definition of a Viability domain; its relevance to the problem described in the introduction will become clear in the next section.

Definition 3.1 *Let* $\mathcal{L}$ *be a convex subset of the space* $C(\mathbb{R}^n, \mathbb{R}^n)$ *of continuous mappings* $\mathbb{R}^n \mapsto \mathbb{R}^n$ *such that the restrictions of the functions from* $\mathcal{L}$ *to any compact subset* X *of* $\mathbb{R}^n$ *are equi-Lipschitz and uniformly bounded. The set of these restrictions will be denoted by* $\mathcal{L}_{|X}$*.*

The collection $\mathcal{E}$ *is called a* viability domain *for* $\mathcal{L}$ *if*

$$\mathcal{T}_{\mathcal{E}}(E) \cap \mathcal{L}_{|E} \neq \emptyset \qquad \forall E \in \mathrm{cl}\,(\mathcal{E}).$$

Obviously $\mathcal{E}$ is a viability domain if and only if $\mathrm{cl}\,(\mathcal{E})$ is such. If a collection $\mathcal{E}$ is not a viability domain for $\mathcal{L}$, then it may happen that it contains a collection $\mathcal{E}' \subset \mathcal{E}$ which is a viability domain. In this case we encounter the question if a largest viability domain $\hat{\mathcal{E}} \subset \mathcal{E}$ exists. Obviously if it exists, it must coincide with the closure of the union of all viability domains contained in $\mathcal{E}$. Property 5 in Proposition 2.3 implies that the union of finite number of viability domains is also a viability domain. However, this is not obvious for the union of infinitely many viability domains. Therefore we need some properties shared by all viability domains included in a given collection:

Proposition 3.2 *Let a closed collection* $\mathcal{E}^*$ *(satisfying Condition* A*) and a set* $\mathcal{L}$ *as above be fixed. Then for every compact set* $X \subset \mathbb{R}^n$ *and for every* $\varepsilon \in (0, 1]$ *there exists* $\theta(\varepsilon) \in (0, \varepsilon]$ *with the following property: if* $\mathcal{E} \subset \mathcal{E}^*$ *is a viability domain for* $\mathcal{L}$*, then for every* $E \in \mathcal{E}$ *for which* $E \cap X \neq \emptyset$ *there exist* $l(\cdot) \in \mathcal{L}$ *and* $\tilde{E} \in \mathcal{E}$ *such that*

$$(I + \theta(\varepsilon) l(\cdot))(E) \subset \tilde{E} + \varepsilon \theta(\varepsilon) \mathbf{B}.$$

Notice the difference with the claim of Proposition 2.2: $\theta(\varepsilon)$ is the same for all $\mathcal{E} \subset \mathcal{E}^*$ and $E \in \mathcal{E}$ as in the formulation.

Proof. Fix arbitrarily a compact set $X \subset \mathbb{R}^n$ and $\varepsilon \in (0, 1]$. According to Condition A, there is a number R such that $E \subset R\mathbf{B}$ for each $E \in \mathcal{E}^*$ for which $E \cap (X + \mathbf{B}) \neq \emptyset$. Let L and M be a Lipschitz constant and a bound of the functions from $\mathcal{L}_{|(R+1)\mathbf{B}}$. We shall prove the claim of the Proposition for

$$\theta(\varepsilon) = \frac{\varepsilon}{(3 + LM)e^L + M}.$$

We denote for brevity $h = \theta(\varepsilon)$.

Take an arbitrary viability domain $\mathcal{E} \subset \mathcal{E}^*$ (without any restriction we may suppose that $\mathcal{E}$ is closed) and $E_0 \in \mathcal{E}$ for which $E_0 \cap X \neq \emptyset$.

By the definition of viability domain and Proposition 2.2, for any $E \in \mathcal{E}$ there exist $l_E(\cdot) \in \mathcal{L}$, $\sigma(E) \in (0, h^2]$ and $\tilde{E} \in \mathcal{E}$ such that

$$(I + \sigma(E) l_E(\cdot))(E) \subset \tilde{E} + h\sigma(E)\mathbf{B}. \tag{6}$$

For an arbitrary compact set $Z \subset \mathbb{R}^n$ we denote

$$\alpha(Z) \stackrel{def}{=} \sup\{\sigma(E) : \ E \subset \mathcal{E}, \ H^+(Z, E) \leq \mathrm{dist}(Z, \mathcal{E}) + h\sigma(E)\},$$

where

$$\mathrm{dist}(Z, \mathcal{E}) \stackrel{def}{=} \inf_{E' \in \mathcal{E}} H^+(Z, E').$$

Since $\alpha(Z) \geq \sigma(E^*) > 0$ for E^* at which the above infimum is attained (notice that E^* exists thanks to the closedness of $\mathcal{E}$), there is $\mathcal{F}(Z) \in \mathcal{E}$ such that

$$H^+(Z, \mathcal{F}(Z)) \leq \mathrm{dist}(Z, \mathcal{E}) + h\sigma(\mathcal{F}(Z))$$

and

$$\sigma(\mathcal{F}(Z)) \geq \frac{1}{2}\alpha(Z).$$

Define a sequence $E_0, E_1, \ldots$ of elements of $\mathcal{E}$ (E_0 is the already chosen set) by

$$\sigma_k = \sigma(E_k), \quad Z_k = (I + \sigma_k l_{E_k}(\cdot))(E_k), \quad E_{k+1} = \mathcal{F}(Z_k), \quad k = 0, 1, \ldots.$$

From the definition of $\mathcal{F}$

$$H^+(Z_k, E_{k+1}) \leq \mathrm{dist}(Z_k, \mathcal{E}) + h\sigma_{k+1} \tag{7}$$

and from (6)

$$\mathrm{dist}(Z_k, \mathcal{E}) \leq h\sigma_k.$$

Hence,

$$Z_k \subset E_{k+1} + h(\sigma_k + \sigma_{k+1})\mathbf{B}. \tag{8}$$

Obviously also

$$H(Z_i, E_i) \leq \sigma_i \max_{x \in E_i} |l_{E_i}(x)|. \tag{9}$$

We shall prove by induction that if

$$\sum_{i=0}^{k-1} \sigma_i \leq h, \tag{10}$$

then

$$E_i \subset R\mathbf{B}, \quad Z_i \subset (R+1)\mathbf{B} \tag{11}$$

for $i = 0, \ldots, k$.

Suppose that (11) is fulfilled for $i = 0, \dots, k$ and that (10) is fulfilled for k (instead of $k-1$). We have from (9) and (8)

$$E_0 \subset Z_0 + \sigma_0 M\mathbf{B} \subset E_1 + h(\sigma_0 + \sigma_1)\mathbf{B} + \sigma_0 M\mathbf{B} \subset \dots$$

$$\dots \subset E_k + M\sum_{i=0}^{k-1}\sigma_i\mathbf{B} + h\left(\sum_{i=0}^{k-1}\sigma_i + \sum_{i=1}^{k}\sigma_i\right)\mathbf{B}$$

$$\subset Z_k + M\sum_{i=0}^{k}\sigma_i\mathbf{B} + h\left(\sum_{i=0}^{k-1}\sigma_i + \sum_{i=1}^{k}\sigma_i\right)\mathbf{B}$$

$$\subset E_{k+1} + \left(M\sum_{i=0}^{k}\sigma_i + h(\sum_{i=0}^{k}\sigma_i + \sum_{i=1}^{k+1}\sigma_i)\right)\mathbf{B}$$

$$E_{k+1} + (Mh + h(h + h + \sigma_{k+1}))\mathbf{B} \subset E_{k+1} + h(M+3)\mathbf{B} \subset E_{k+1} + \mathbf{B},$$

because of $h = \theta(\varepsilon) \le \varepsilon \le 1$ and $\sigma_{k+1} \le h^2 \le 1$.

Since $E_0 \cap X \ne \emptyset$, this implies $E_{k+1} \cap (X + \mathbf{B}) \ne \emptyset$. Hence, $E_{k+1} \subset R\mathbf{B}$, and the first inclusion in (11) holds for $k+1$. The second one follows from (9) and the inequality $\sigma_{k+1}M \le h^2 M \le \theta(\varepsilon)M \le 1$.

We shall prove that there is a maximal number N such that (10) is fulfilled for $k = N$ (and thus (10) is not fulfilled for $k = N+1$). If such N does not exists, then $\sum_{i=0}^{\infty}\sigma_i$ is finite. In particular, $\sigma_N \to 0$ and $\alpha(Z_N) \le 2\sigma_N \to 0$. Moreover, from (11)

$$E_N \subset R\mathbf{B}, \quad Z_N \subset (R+1)\mathbf{B}.$$

Therefore, for a subsequence $Z_N \to \overline{Z}$, $E_N \to \overline{E} \in \mathcal{E}$ in Hausdorff sense.

Since $\alpha(Z_N) \to 0$, we have $\alpha(Z_N) < \sigma(\overline{E})$ for all sufficiently large N. Then, according to the definition of $\alpha(\cdot)$,

$$H^+(Z_N, \overline{E}) > \text{dist}(Z_N, \mathcal{E}) + h\sigma(\overline{E}).$$

Passing to the limit (for the convergent subsequence)

$$H^+(\overline{Z}, \overline{E}) \ge \text{dist}(\overline{Z}, \mathcal{E}) + h\sigma(\overline{E}).$$

On the other hand, passing to the limit in (7) we obtain

$$H^+(\overline{Z}, \overline{E}) \le \text{dist}(\overline{Z}, \mathcal{E}).$$

This contradiction implies existence of largest N such that (10) is fulfilled for $k = N$. Clearly

$$\sum_{i=0}^{N-1}\sigma_i \le h, \quad h < \sum_{i=0}^{N}\sigma_i \le h + h^2. \tag{12}$$

From the definition of Z_k and (8) we have

$$(I + \sigma_k l_{E_k}(\cdot))(E_k) \subset E_{k+1} + h(\sigma_k + \sigma_{k+1})\mathbf{B}.$$

Now we shall apply Lemma 5.2 from the Appendix for the sets E_k and $S = R\mathbf{B}$. It gives

$$\left(I + \sum_{i=0}^{N-1} \sigma_i l_{E_i}(\cdot)\right)(E_0) \subset E_N + e^{L\sum_{i=0}^{N-1}\sigma_i}\left(h\left(\sum_{i=0}^{N-1}\sigma_i + \sum_{i=1}^{N}\sigma_i\right) + LM\left(\sum_{i=0}^{N-1}\sigma_i\right)^2\right)\mathbf{B}$$

$$\subset E_N + h^2 e^L(3 + LM)\mathbf{B}.$$

Denote

$$l_\varepsilon(\cdot) = \frac{1}{\sum_{i=0}^{N-1}\sigma_i}\sum_{i=0}^{N-1}\sigma_i l_{E_i}(\cdot) \in \mathcal{L}.$$

Then

$$\left(I + \left(\sum_{i=0}^{N-1}\sigma_i\right) l_\varepsilon(\cdot)\right)(E_0) \subset E_N + h^2 e^L(3 + LM)\mathbf{B}.$$

Using (12) we obtain

$$(I + h l_\varepsilon(\cdot))(E_0) \subset E_N + h^2 e^L(3 + LM)\mathbf{B} + \left(h - \sum_{i=0}^{N-1}\sigma_i\right) M\mathbf{B}$$

$$\subset E_N + (h^2 e^L(3 + LM) + \sigma_N M)\mathbf{B} \subset E_N + h(e^L(3 + LM) + M)h\mathbf{B} = E_N + h\varepsilon\mathbf{B},$$

according to the definition of $h = \theta(\varepsilon)$. This completes the proof, since $E_N \in \mathcal{E}$. □

Proposition 3.3 *Let a closed collection $\mathcal{E}^*$ (satisfying Condition* A*) and a set $\mathcal{L}$ (as in the definition of viability domain) be given. Let Ω be an arbitrary set of collections $\mathcal{E} \subset \mathcal{E}^*$ each of which is a viability domain for $\mathcal{L}$. Then*

$$\hat{\mathcal{E}} = \mathrm{cl} \bigcup_{\mathcal{E}\in\Omega} \mathcal{E}$$

is a viability domain for $\mathcal{L}$.

Proof. First of all $\hat{\mathcal{E}}$ is a collection (in the sense of Condition A) since it is a subset of the collection $\mathcal{E}^*$. We shall prove that it is a viability domain. Take arbitrarily $\hat{E} \in \hat{\mathcal{E}}$ and $\varepsilon \in (0, 1]$. Let $\theta(\varepsilon)$ be the number corresponding to the collection $\mathcal{E}^*$, ε and the compact set $X = \hat{E} + \mathbf{B}$ according to Proposition 3.2. By the definition of $\hat{\mathcal{E}}$ there are $\mathcal{E} \in \Omega$ and $E \in \mathcal{E}$ such that $H(E, \hat{E}) \le \varepsilon\theta(\varepsilon)$. Since $\mathcal{E}$ is a viability domain for $\mathcal{L}$ and $\mathcal{E} \subset \mathcal{E}^*$, and since $E \cap X = E \cap (\hat{E} + \mathbf{B}) \ne \emptyset$, Proposition 3.2 implies existence of $l_\varepsilon(\cdot) \in \mathcal{L}$

and $\tilde{E} \in \mathcal{E}$ such that

$$(I + \theta(\varepsilon) l_\varepsilon(\cdot))(E) \subset \tilde{E} + \varepsilon\theta(\varepsilon)\mathbf{B}.$$

Let Y be a compact set containing all sets $E' + \mathbf{B}$, where $E' \in \mathcal{E}^*$ and $E' \cap X \neq \emptyset$ (such exists, according to Condition A). Let L be a Lipschitz constant for the functions in $\mathcal{L}_{|Y}$. Then

$$(I + \theta(\varepsilon) l_\varepsilon(\cdot))(\hat{E}) \subset (I + \theta(\varepsilon) l_\varepsilon(\cdot))(E) + (1 + L)\varepsilon\theta(\varepsilon)\mathbf{B} \subset \tilde{E} + \varepsilon\theta(\varepsilon)(2 + L)\mathbf{B}.$$

Now take $\varepsilon = \varepsilon_k = 1/k$ and let $l(\cdot) \in \mathcal{L}_{|Y}$ be the limit of a subsequence of $l_{\varepsilon_k}(\cdot)$ on Y. Then for $h_k = \theta(\varepsilon_k) \leq \varepsilon_k \to 0$ we have

$$(I + h_k l(\cdot))(\hat{E}) \subset \tilde{E} + h_k \left((2 + L)\varepsilon_k + \|l(\cdot) - l_{\varepsilon_k}(\cdot)\|_{C(Y)} \right) \mathbf{B},$$

which implies that $l(\cdot) \in \mathcal{T}_{\hat{\mathcal{E}}}(\hat{E})$ according to Proposition 2.2. □

From Proposition 3.3 we conclude that given a closed collection $\mathcal{E}^*$ and $\mathcal{L}$, there is a maximal viability domain for $\mathcal{L}$ that is contained in $\mathcal{E}^*$. As in the single valued case we call it the *viability kernel* of $\mathcal{E}^*$ for $\mathcal{L}$, and denote it by $\mathrm{Viab}_{\mathcal{L}}(\mathcal{E}^*)$.

Let us return to the control system (1) under the following standing supposition.

Condition B.

1. The mapping $f(\cdot, \cdot) : \mathbb{R}^n \times U \mapsto \mathbb{R}^n$ has the form

$$f(x, u) = f_0(x) + G(x)u,$$

where $f_0(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $G(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^{n \times r}$ are locally Lipschitz mappings and U is a convex compact subset of $\mathbb{R}^r$;

2. for any compact set $S \subset \mathbb{R}^n$ and for any $T \geq 0$ there is a compact $S' = S'(S, T)$ such that every solution of (1) starting from S at $t = 0$ (and corresponding to some measurable $u(\cdot) : [0, T] \mapsto U$) is contained in S' for $t \in [0, T]$.

Remark 3.4 *Obviously Condition* B.2 *is fulfilled under the standard growth condition:*

$$|f(x, u)| \leq \text{const.}(1 + |x|) \qquad \forall x \in \mathbb{R}^n,\ u \in U.$$

From Condition B.1 *it follows that for any compact set S, all solutions of the differential inclusion*

$$\dot{x} \in f(x, U), \quad x(0) \in S$$

are extendible to infinity and the set of solutions is compact in $C[0, T]$ for each finite T.

Clearly the set

$$\mathcal{L} = \{f(\cdot, u) :\ u \in U\}$$

satisfies the assumptions for $\mathcal{L}$. In this case the definition of viability domain reads

$$\mathcal{E} \text{ is a viability domain for } (f, U) \quad \text{if} \quad \mathcal{T}_{\mathcal{E}}(E) \cap f(\cdot, U) \neq \emptyset \qquad \forall E \in \mathrm{cl}\,(\mathcal{E}).$$

The viability kernel of a given collection $\mathcal{E}^*$ will be denoted in this case by $\mathrm{Viab}_f(\mathcal{E}^*)$, *or merely by* $\mathrm{Viab}(\mathcal{E}^*)$ *if this cannot lead to ambiguities.*

For the collection $\mathcal{E}^* = \mathcal{E}^s_K$ of all singletons from K we obviously have

$$\mathrm{Viab}(\mathcal{E}^s_K) = \mathrm{Viab}(K)$$

where the usual viability kernel [2] *stays in the right-hand side.*

4 The viability problem with uncertain initial condition

We consider the control system (1) under Condition B and a collection $\mathcal{E}^*$ satisfying Condition A. The main problem investigated in this paper, as formulated in the introduction, is to find the maximal subcollection $\hat{\mathcal{E}} \subset \mathcal{E}^*$ with the property:

for each $E_0 \in \hat{\mathcal{E}}$ there is a measurable $u(\cdot) : [0, +\infty) \mapsto U$ such that

$$x[E_0, u(\cdot)](t) \subset E(t) \in \hat{\mathcal{E}} \qquad \forall t \geq 0. \tag{13}$$

Below we show that this maximal collection $\hat{\mathcal{E}}$ exists and coincides with $\mathrm{Viab}_f(\mathcal{E}^*)$.

Let $\mathcal{E}^*$ be the collection of all closed subsets of the constraint K. Consider the maximal collection of sets $\tilde{\mathcal{E}} \subset \mathcal{E}^*$ such that for each $E_0 \in \tilde{\mathcal{E}}$

$$x[E_0, u(\cdot)](t) \subset E(t) \in \mathcal{E}^* \qquad \forall t \geq 0. \tag{14}$$

It is obvious that for this particular choice of the collection $\mathcal{E}^*$ we have $\tilde{\mathcal{E}} = \hat{\mathcal{E}}$. However, if $\mathcal{E}^*$ is specified in a more restrictive way (for example, the set of all ellipsoids, boxes, etc., in K) we have only $\hat{\mathcal{E}} \subset \tilde{\mathcal{E}}$, the second collection being strictly bigger. The reason is that for $E_0 \in \tilde{\mathcal{E}}$ and a control $u(\cdot)$, the set $x[E_0, u(\cdot)](t)$ is not obliged to be an element of $\tilde{\mathcal{E}}$.

From the point of view of the numerical realization we prefer to deal with sets from $\mathcal{E}^*$ only. Therefore, the set $x[E_0, u(\cdot)](t)$ (which is the guaranteed estimation of the state at the moment t) has to be embedded in some $E(t) \in \mathcal{E}^*$, from which viable control is still possible, that is $E(t) \in \hat{\mathcal{E}}$ has to be fulfilled. Thus (13) is the relevant requirement for the evolution determined by the initial set E_0 and the control $u(\cdot)$.

Below we describe the evolution of the set $E(t)$ that satisfies (13).

Definition 4.1 *We call a* tube *in* $\mathcal{E}^*$ *any mapping* $E(\cdot) : [0, +\infty) \mapsto \mathcal{E}^*$ *satisfying the following condition:*

Condition C. *The mapping* $E(\cdot)$ *is upper semicontinuous and locally Lipschitz from the right: for every compact interval* $[0, T]$ *there is a constant* L *such that*

$$E(s) \subset E(t) + L(t - s)\mathbf{B} \qquad \text{if } 0 \leq s \leq t \leq T.$$

We shall describe the evolution of the estimation $E(t)$ of $x[E_0, u(\cdot)](t)$ by the following set-dynamic equation

$$\lim_{h \to 0+} \frac{1}{h} H^+((I + hf(\cdot, u(t)))(E(t), E(t + h)) = 0. \tag{15}$$

Definition 4.2 *A mapping* $E(\cdot) : [0, +\infty) \mapsto \mathrm{comp}(\mathbb{R}^n)$ *is a solution of* (15) *in* $\mathcal{E}^*$ *if* $E(\cdot)$ *is a tube in* $\mathcal{E}^*$ *for which* (15) *is fulfilled for a.e.* $t > 0$.

A similar equation is introduced and studied in a different context (for linear systems with $\mathcal{E}^* = \text{comp}(\mathbb{R}^n)$) by Kurzhanski and Nikonov [7], see also [6, 8, 12, 13].

The main results are formulated in the following two theorems.

Theorem 4.3 *Let the control system* (1) *satisfy Condition* B *and let* $\mathcal{E}^*$ *be a collection for which Condition* A *is fulfilled.*

1. *If* $E(\cdot) : [0, +\infty) \mapsto \mathcal{E}^*$ *is a solution of* (15) *corresponding to some measurable* $u(\cdot) : [0, +\infty) \mapsto U$*, then*

a. $x[E(t), u(t + \cdot)](s) \subset E(t + s)$ *for every* $t, s \geq 0$*;*

b. $\text{cl}\,\{E(t) :\ t \geq 0\}$ *is a viability domain;*

c. *if* $\mathcal{E} \subset \mathcal{E}^*$ *is a viability domain containing* $E(t)$ *for all* $t \in [0, +\infty)$*, then*

$$u(t) \in U(E(t)) \qquad \text{for a.e. } t \geq 0, \tag{16}$$

where

$$U(E) = \{u \in U :\ f(\cdot, u) \in \mathcal{T}_{\mathcal{E}}(E)\}.$$

2. *If* $\mathcal{E} \subset \mathcal{E}^*$ *is a closed viability domain, then for every* $E_0 \in \mathcal{E}$ *there exists a measurable* $u(\cdot) : [0, +\infty) \mapsto U$ *such that* (15) *has a solution in* $\mathcal{E}$ *on* $[0, +\infty)$*, with initial condition* $E(0) = E_0$*.*

Theorem 4.4 *Under the conditions of Theorem* 4.3 *the following statements are equivalent:*

1) $E_0 \in \text{Viab}(\mathcal{E}^*)$*;*

2) *there is a measurable* $u(\cdot) : [0, +\infty) \mapsto U$ *such that* (15) *has a solution* $E(\cdot)$ *in* $\mathcal{E}^*$ *on* $[0, +\infty)$*, with initial condition* $E(0) = E_0$*. In this case*

$$x[E_0, u(\cdot)](t) \subset E(t) \in \text{Viab}(\mathcal{E}^*)$$

and (16) *is satisfied.*

Theorem 4.4 is a direct consequence of Theorem 4.3. The proof of Theorem 4.3 is given in [10].

5 Approximation of the viability kernel

In this section we propose an iterative scheme for approximation of the viability kernel of a given collection of sets. In the case of a collection consisting only of single points it is a particular case of the approximation techniques developed in [11, 9]. In the general case of an arbitrary collection the iterative approximation could be hard for computer realization. However, for collections consisting of finitely parametrized sets the method can be elaborated to the level of a numerical algorithm.

Condition B.2 implies existence of a compact set S' containing all trajectories of (1) on $[0, 1]$ starting from S. Let L and M be a Lipschitz constant and a bound of $f(\cdot, u)$ in $S' + \mathbf{B}$ (uniformly in $u \in U$).

Definition of the Algorithm

For $h > 0$ we set

$$\mathcal{E}_0^h = \mathcal{E}$$

and inductively

$$\mathcal{E}_{k+1}^h = \Big\{ E \in \mathcal{E}_k^h : \exists u \in U, \ \exists \tilde{E} \in \mathcal{E}_k^h \text{ such that}$$

$$(I + hf(\cdot, u))(E) \subset \tilde{E} + \frac{LM}{2} h^2 \mathbf{B} \Big\}, \quad k = 0, 1, \dots .$$

Clearly $\mathcal{E}_{k+1}^h \subset \mathcal{E}_k^h$ for all k, therefore one can define

$$\mathcal{E}^h = \bigcap_{k \geq 0} \mathcal{E}_k^h$$

(which could be empty, since some of the sets $\mathcal{E}_k^h$ could be empty).

This algorithm converges to the Viability kernels as it is stated in the following Theorem proved in [10]:

Theorem 5.1 *Let $\mathcal{E}$ be compact in* $\text{comp}(\mathbb{R}^n)$*. Suppose conditions* A *and* B*. Then* $\text{Viab}(\mathcal{E})$ *is nonempty if and only if $\mathcal{E}^h$ is nonempty for each $h \in (0, 1]$. In this case*

$$\text{Viab}(\mathcal{E}) = \operatorname*{Lim}_{h \to 0+} \mathcal{E}^h,$$

where the limit is in the Hausdorff metric between the compact subsets in the space $\text{comp}(\mathbb{R}^n)$*.*

Appendix

Here we state a technical lemma used above. The proof is easy and is given in [10].

Lemma 5.2 *Let S be a subset of $\mathbb{R}^n$, let δ be a positive real number and let $l_i : S + \delta\mathbf{B} \mapsto \mathbb{R}^n$, $i = 0, \dots, N-1$, be Lipschitz functions with a Lipschitz constant L and a bound M. Let $E_0, \dots, E_N$ be subsets of S that satisfy*

$$(I + \sigma_i l_i(\cdot))(E_i) \subset E_{i+1} + \rho_i \mathbf{B}, \quad i = 0, \dots, N-1,$$

where $\sigma_0, \dots, \sigma_{N-1}$ and $\rho_0, \dots \rho_{N-1}$ are nonnegative numbers. Suppose that

$$\gamma = e^{L \sum_{i=0}^{N-1} \sigma_i} \left(\sum_{i=0}^{N-1} \rho_i + LM (\sum_{i=0}^{N-1} \sigma_i)^2 \right) < \delta \quad \textit{and} \quad M \sum_{i=0}^{N-1} \sigma_i < \delta.$$

Then

$$(I + \sum_{i=0}^{N-1} \sigma_i l_i(\cdot))(E_0) \subset E_N + \gamma \mathbf{B}.$$

REFERENCES

[1] J.-P. Aubin. Mutational and Morphological Analysis. Tools for Shape Evolution and Regulation.
[2] J.-P. Aubin. Viability Theory. Birkhäuser, Boston, Basel, Berlin, 1991.
[3] J.-P. Aubin, H. Frankowska. Set-valued Analysis. Birkhäuser, Boston-Basel-Berlin, 1990.
[4] L. Doyen. Filippov and invariance theorems for mutational equations for tubes. *Set-Valued Analysis* **1**, (1993), 289-303.
[5] A. Gore. Doctorat de mathématiques. PhD thesis, Université Paris-Dauphine, 1997.
[6] A.B. Kurzhanski, T.F. Filippova. On the theory of trajectory tubes – a mathematical formalism for uncertain dynamics, viability and control. In: Advances in Nonlinear Dynamics and Control: a Report from Russia, **17** of Progress in Systems and Control Theory. Birkhäuser, Boston, 1993, 122-188.
[7] A.B. Kurzhanski, O.I. Nikonov. Funnel equations and multivalued integration problems for control synthesis. In: Perspectives in Control Theory, vol. 2 of Progress in Systems and Control, (Eds. B. Jakubczyk, K. Malanowski, W. Respondek). Birkhäuser, Boston-Basel-Berlin, 1990, 143-153.
[8] A.B. Kurzhanski, I. Vályi. Ellipsoidal Calculus for Estimation and Control. Birkhäuser, Boston, 1997.
[9] M. Quincampoix, P. Saint-Pierre. An algorithm for viability kernels in holderian case: Approximation by discrete dynamic systems. *J. Math. Systems, Estimation and Control*, **5**, *1* (1995), 115-128.
[10] M. Quincampoix, V. Veliov. Viability under uncertain initial state. Prépublications du Laboratoire de Mathématiques de l'Université de Bretagne Occidentale (Brest), 1998, preprint.
[11] P. Saint-Pierre. Approximation of the viability kernel. *Appl. Math. Optim.* **29**, (1994), 187-209.
[12] V. Veliov. Guaranteed control of uncertain systems: Funnel equations and existence of regulation maps. Working Paper WP-92-62, International Institute for Applied Systems Analysis, Laxenburg, Austria, 1992.
[13] V. Veliov. Funnel equations and regulation of uncertain systems. In: Set-Valued Analysis and Differential Inclusions (Eds. A. B. Kurzhanski, V. M. Veliov), Birkhäuser, Boston-Basel-Berlin, 1993, 183-199.

Marc Quincampoix
Département de Mathématiques,
Faculté des Sciences et Techniques
Université de Bretagne
Occidentale, 6 Avenue Victor Le Gorgeu
BP 809, 29285 Brest, France
e-mail:Marc.Quincampoix@univ-brest.fr

Vladimir Veliov
Institute of Mathematics & Informatics
Bulgarian Academy of Sciences,
1113 Sofia, Bulgaria