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БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

Pliska Stud. Math. Bulgar. **12** (1998), 213-226

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VIABILITY, OPTIMALITY AND STABILITY OF DYNAMICAL SYSTEMS AND ESTIMATION OF CONVERGENCE OF NUMERICAL SCHEMES

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We present in this paper some recent developments dealing with dynamical controlled systems with state constraints. After recalling the basic frame of Viability Theory and it numerical aspects, we estimate the convergence of numerical schemes for computing the optimal time for target problem. We give also a relaxation result for decomposable problems. These properties are enhanced through the study of the Norvegian Fishermen problem arising in Dynamic of Population. Another interesting application of this approach is shortly presented when considering the approximation of the so-called the minimal time of crisis. This appears for problems where some constraints are "soft" (reversibility) and others are "hard" (irreversibility).

Keywords: Dynamical Systems, Viability Kernel, Approximation.

AMS subject classification: 49N35,49N55,65Lxx

1 Introduction

The recent developments of the Viability Theory have led to the emergence of the concept of the Viability Kernel associated with a constrained dynamic system.

This set appears to play a crucial role for studying qualitative and quantitative problems mainly in automatic, in economy, in biology or in demography, for instance. Moreover numerous characteristics of a dynamic system can be expressed in term of Viability Kernel or of Invariant Kernel. In this way we can characterize the set of all equilibria of a system. We can also characterize the Value function for optimal control problem or the Minimal time function for target problems.

On the other hand approximation theory for set-valued maps allows to compute numerical approximations of the Viability Kernel and we can give estimations of the convergence of algorithms. These methods allow calculus of feed-back laws ensuring the state both to remain in the constraint set – viability criteria – and to preserve the principles underlying these functions – optimality, stability or robustness criteria for instance.

Many linear or nonlinear dynamic models have been studied through this approach. Let us cite models in the economy of renewable resource (L. Doyen, D. Gabay [11]) or models in demography and population evolution (N. Bonneuil [6], N. Bonneuil & P.S.-P. [7]). For studying particular models, general results cannot be applied directly and it is necessary to state adapted results. We give here a relaxation theorem giving sufficient conditions for the Viability Kernel in non convex dynamic to coincide with the Viability Kernel for the relaxed problem.

2 Control Problems with state constraints

Let us first recall some basic results and definitions in Set-Valued Analysis. Let us consider a dynamic system where $X = \mathbb{R}^N$ denotes the state space and V the set of possible controls.

Let $f : X \times V \to X$ be the map describing the dynamic of the system

$$(1) \qquad \begin{cases} x'(t) & = f(x(t), u(t)), \quad p.p.t \geq 0 \\ u(t) & \in U, \quad \forall t \geq 0 \end{cases}$$

We assume that the state must remain in a closed set K: $x(t) \in K, \quad \forall t \geq 0$.

Let us denote $F(x) := \{f(x,u), \ u \in U\}$ and $S_F(x)$ the set of solutions to (1) starting from x at time $t = 0$.

2.1 The Viability Kernel

When constraints occur, we are interested in solutions to (1) which satisfy these constraints. The **Viability Kernel of** K **for** F – that we denote $Viab_F(K)$ – is precisely the subset of all points in K from which at least one viable solution starts, that is to say a solution which remains forever in K

$$Viab_F(K) := \{x \in K, \ \exists x(\cdot) \in S_F(x), \ \ x(t) \in K \ \ \forall t \geq 0\}$$

One can easily prove that any viable solution remains necessarily in the very Viability Kernel.

The Viability Kernel has an interest only when constraints occur since, for any unconstrained system, the whole space is trivially a viability domain. We refer to Viability Theory (J. P. Aubin [3]) and Set-Valued Analysis (J. P. Aubin, H. Frankowska, [1]) for a general discussion about this subject. The approximation of the Viability Kernel has first been studied in [14].

This set can be characterized by the mean of geometric conditions. Indeed, if K is closed and if $F : X \rightsquigarrow X$ is a Marchaud[1] map, then $Viab_F(K)$ is the largest closed subset

[1] A set-valued map is a Marchaud map if it is upper semicontinuous with convex compact nonempty values and with linear growth: $\exists c > 0, \ \ \forall x \in X, \ \ \|F(x)\| := \sup_{y \in F(x)} \|y\| \leq c(1 + \|x\|$

$D \subset K$ satisfying

$$\forall x \in D,\ \exists p \in \mathcal{NP}_K(x)\ \inf_{u \in U} < f(x,u), p > \leq 0$$

where $\mathcal{NP}_K(x) := \{p \in X \mid d_K(x+p) = \|p\|\}$ denotes the set of proximal normals to K at x.

This geometric characterization provides numerical algorithms. For that purpose we associate with the system $x'(t) \in F(x(t))$ the explicit discrete system of the form

$$x^{n+1} \in G_\varepsilon(x^n) := x^n + \varepsilon F_\varepsilon(x) \tag{2}$$

The **discrete Viability Kernel of** K **for** G_ε denoted by $\overrightarrow{Viab}_{G_\varepsilon}(K)$ is the set of initial values belonging to K such that at least one sequence exists which is a solution to the discrete system (2) and which remains in K: $\forall n \in \mathbb{N},\ x^n \in K$.

Let us consider a family of approximations F_ε of F satisfying[2]:

$$\operatorname*{Limsup}_{\varepsilon \to 0}\ Graph(F_\varepsilon) \subset Graph(F)\ \text{ and }\ \forall \epsilon \in]0, \frac{1}{M}],\ \ F(x + \varepsilon M \mathcal{B}_X) \subset F_\varepsilon(x) \tag{3}$$

where $M := \max_{y \in K} \|F(y)\|$.

Theorem 1 *Let F be a Marchaud map satisfying (3) and let $K \subset X$ be closed. Then*

$$\operatorname*{Lim}_{\varepsilon \to 0}\ \overrightarrow{Viab}_{G_\varepsilon}(K) = Viab_F(K) \subset \overrightarrow{Viab}_{G_\varepsilon}(K)$$

The computation of the Viability Kernel is derived from the following approximation Theorem based on the construction of a sequence of sets K^n defined by

$$\begin{array}{lll} K_\varepsilon^0 & := & K \\ K_\varepsilon^{n+1} & := & \{x \in K_\varepsilon^n \mid G_\varepsilon(x) \cap K_\varepsilon^n \neq \emptyset\} \end{array}$$

Theorem 2 *Let F be a Marchaud map satisfying (3) and let $K \subset X$ be closed. Then*

$$K_\varepsilon^\infty = \operatorname*{Lim}_{n \to \infty}\ K_\varepsilon^n = \bigcap_n K_\varepsilon^n = \overrightarrow{Viab}_{G_\varepsilon}(K)$$

and

$$\operatorname*{Lim}_{\varepsilon \to 0}\ K_\varepsilon^\infty = Viab_F K$$

[2]The limits of sets are taken at the sense of Painlevé-Kuratovski lower limit or upper limit.

When projecting the discrete dynamic onto an integer lattice X_h of X we can define fully discrete Viability Kernels – denoted $\overrightarrow{Viab}_{G_{h,\varepsilon}}(K_h)$ – and we can approach numerically $Viab_F(K)$ precisely with a sequence of fully discrete Viability Kernels. For that purpose we apply the following Refinement Principle

Theorem 3 *Let F be a Marchaud map satisfying (3) and let $K \subset X$ be closed. Let X_h be an integer lattice of X and let $G_{\varepsilon h}$ be defined by $G_{\varepsilon,h}(x_h) := (G_\varepsilon(x_h) + h\mathcal{B}) \cap X_h$. Let be $h > 0$, $h' > 0$, $h' < h$ and $\varepsilon_h > 0$ such that $\lim_{h\to 0} \frac{\varepsilon_h}{h} = 0$. Then*

$$\overrightarrow{Viab}_{G_{\varepsilon_{h'},h'}}(K_{h'}) = \overrightarrow{Viab}_{G_{\varepsilon_{h'},h'}}((\overrightarrow{Viab}_{G_{\varepsilon_h,h}}(K_h) + 2h\mathcal{B}) \cap X_{h'})$$

where $\mathcal{B}$ denotes the unit ball of X, and

$$\operatorname*{Lim}_{h\to 0} \overrightarrow{Viab}_{G_{\varepsilon_h,h}}(K_h) = Viab_F(K)$$

2.2 The Target Problem with State Constraints

Let C be a target and K the constraint set. Let us consider the system:

$$x'(t) \in F(x(t)) := \{f(x(t), v), \ \ v \in V\}, \ a.e. \ t > 0 \tag{4}$$

A first question which arises when studying target problems is to find the set of points of K from which at least one solution starts, reaching C in a finite time while remaining in K until it reaches K. We denote this set $Vict_F(K, C)$. Let us define $\tilde{F}$ the set-valued map which coincides with F everywhere except on C and $\forall x \in C$, $\tilde{F}(x) = \overline{Co}(F(x) \cup \{0\})$.

Proposition 1 *Let F be a Marchaud map and let K and C be closed subsets of X. Let us assume that $Viab_F(K) = \emptyset$. Then*

$$Vict_F(K, C) = Viab_{\tilde{F}}(K)$$

A second question[3] is to determine the Minimal Time function with values in $\mathbb{R}^+ \cup \{+\infty\}$ defined on X by

$$\vartheta_C^K(x_0) := \inf_{x(\cdot)\in S_F(x_0)} \{\tau \mid x(\tau) \in C, \ \ x(t) \in K, \forall t \le \tau\}$$

Let us set $\mathcal{H} := \{(x, y) \in K \times \mathbb{R}^+\}$ and

$$\Phi(x, y) = \begin{cases} F(x) \times \{-1\} \text{ if } \ x \in X \backslash C \\ \overline{Co}((F(x) \times \{-1\}) \cup (0, 0)) \text{ otherwise} \end{cases}$$

The Minimal Time function enjoys the following properties

[3] Results dealing with this question are detailed in a joint work with P. Cardaliaguet and M. Quincampoix [9], [10].

Proposition 2 *If F is a Marchaud map and if K and C are closed, then*

the function $\vartheta_C^K(\cdot)$ is lower semicontinuous,

$\forall x_0 \in Dom(\vartheta_C^K)$ there exists a viable solution $x(\cdot)$ such that $x(\vartheta_C^K(x_0)) \in C$ and

$$Epigraph(\vartheta_C^K) = Viab_\Phi(\mathcal{H})$$

So we can compute the Minimal Time function in the lack of regularity or controllability assumptions on the boundary of the target or of the constraint set. The only condition is that K is closed. Let be suitable $\tau > 0$ and $\rho > 0$, depending on τ and let us consider the sequence of functions defined by $\vartheta_\tau^0 \equiv \mathcal{I}_K$, where $\mathcal{I}_K$ is the indicator function of K, and

$$\vartheta_\tau^n(x) := (1-\rho)\tau + \inf_{v \in V, |w| \leq 1} \vartheta_\tau^{n-1}(x + \tau(f(x,v) + \rho w)).$$

Proposition 3 *Under the previous assumptions, the functions $\vartheta_\tau^n(\cdot)$ are upper bounded by $\vartheta_C^K(\cdot)$ and the sequence $\vartheta_\tau^n(\cdot)$ converges pointwisely to $\vartheta_C^K(\cdot)$.*

The proof of this Proposition is a consequence of the Convergence Theorem 1.

2.2.1 T-Viability and upper stability

Let be $T > 0$ fixed and consider

$$\Phi_T(t,x) := \begin{cases} \{-1\} \times F(x) & \text{if } t < T \\ \overline{Co}((0,0), \{-1\} \times F(x)) & \text{if } t \geq T \end{cases}$$

Let us denote $Viab_F^T(K) := \Pi_X(Viab_{\Phi_T}(\mathbb{R}^+ \times K))$.

Proposition 4 *For any $\alpha > 0$ we have*

$$Viab_F^T(K) \subset Viab_{F+\alpha\mathcal{B}}^T(K) \subset Viab_F^T(K + \frac{\alpha}{\ell}(e^{\ell T} - 1)\mathcal{B})$$

Sketch of the proof.
– The first inclusion is obvious.
– Let be $x_0 \in Viab_{F+\alpha\mathcal{B}}^T(K)$ and $\tilde{x}(\cdot) \in S_{F+\alpha\mathcal{B}}(x_0)$ a T-viable solution in K.
From the Filippov Theorem, $\exists \overline{x}(\cdot) \in S_F(x_0)$ satisfying the following estimations: $\forall t \in [0,T]$

$$\begin{cases} i) & \|\tilde{x}(t) - \overline{x}(t)\| & \leq e^{\ell t}\int_0^t \alpha e^{-\ell s} ds \leq \frac{\alpha}{\ell}(e^{\ell t} - 1) \\ & & \leq \frac{\alpha}{\ell}(e^{\ell T} - 1) \\ ii) & \|\tilde{x}'(t) - \overline{x}'(t)\| & \leq \alpha e^{\ell t} \leq \alpha e^{\ell T} \end{cases}$$

Also, $\overline{x}(\cdot) \in \tilde{x}(\cdot) + \frac{\alpha}{\ell}(e^{\ell T} - 1)\mathcal{B}$ is T-viable in $K + \frac{\alpha}{\ell}(e^{\ell T} - 1)\mathcal{B}$ and

$$x_0 = \overline{x}(0) \in Viab_F^T(K + \frac{\alpha}{\ell}(e^{\ell T} - 1)\mathcal{B})$$

□

2.2.2 Relations between $\vartheta_\rho^K(\cdot)$ and $\Theta_C^K(\cdot)$.

The following Proposition give an estimation between the approched minimal time, the exact minimal time for some perturbation of the target and of the constaint set and the exact minimal time out of perturbation.

Proposition 5 *Assume the previous assumptions and moreover assume that F is ℓ Lipschitz and M bounded. At any point x_0 where $\Theta_C^{K_\rho}(x_0) \leq T$, we have*

$$\Theta_{C_\rho}^{K_\rho}(x_0) - \frac{\alpha(\rho)}{\ell}(e^{\ell T} - 1) \leq \vartheta_\rho(x_0) \leq \Theta_C^K(x_0)$$

where $K_\rho := K + (M\rho + \frac{\alpha_\rho}{\ell}(e^{\ell T} - 1))\mathcal{B}$ and $\Theta_{C_\rho}^{K_\rho}(x_0)$ is the minimal time for a solution to (4) starting from x_0 to reach C_ρ and remaining in K_ρ until the target is reached.

Sketch of the proof. Let be $\overrightarrow{x} \in \overrightarrow{S}_{G_\rho}(x_0)$ which is viable in K and optimal for the function $\vartheta_\rho^K(\cdot)$: if $N_\rho = \vartheta_\rho(x_0)/\rho$, then $x_{N_\rho} \in C_\rho$.

Let us denote by $x_\rho(\cdot)$ the linear interpolation of the sequence $(x_n)_n$: for any $n < N_\rho$ and for any $t \in [n\rho, (n+1)\rho[$ we have

$$x_\rho(t) = x_n + \frac{t - n\rho}{\rho}(x_{n+1} - x_n)$$

It is clear that $x_\rho(t) \in (K + M\rho\mathcal{B})$ and for any $t \geq N_\rho\rho$, $x_\rho(t) = x_{N_\rho}$. We have

$$x'_\rho(t) = \frac{x_{n+1} - x_n}{\rho} \in F_\rho(x_n) \subset F_\rho(x_\rho(t)) + \ell\|x_\rho(t) - x_n\|\mathcal{B}$$

so $x'_\rho(t) \in F(x_\rho(t)) + \alpha_\rho\mathcal{B}$, with $\alpha_\rho = \frac{M\ell\rho}{2}(3 + \ell\rho)$.

Let us define

$$\tilde{\Gamma}_\rho(x_\rho, y_\rho) = \begin{cases} (F(x_\rho) + \alpha_\rho\mathcal{B}) \times \{-1\} & \text{if } \; x_\rho \notin C_\rho \\ \overline{Co}(((F(x_\rho) + \alpha_\rho\mathcal{B}) \times \{-1\}) \cup \{0\}) & \text{if } \; x_\rho \in C_\rho \end{cases}$$

and

$$\tilde{\Phi}_\rho(x, y) = \begin{cases} F(x) \times \{-1\} & \text{if } \; x \notin C_\rho \\ \overline{Co}((F(x) \times \{-1\}) \cup (\{0\} \times \{0\})) & \text{if } \; x \in C_\rho \end{cases}$$

We have $(x_0, \vartheta_\rho(x_0)) \in Viab_{\tilde{\Gamma}_\rho}(K \times \mathbb{R}^+)$. We derives from Theorem 1 the inequality $\vartheta_\rho(x_0) \leq \Theta_C^K(x_0) \leq T$. Consequently

$$(x_0, \vartheta_\rho(x_0)) \in Viab^T_{\tilde{\Gamma}_\rho}(K \times \mathbb{R}^+) \subset Viab^T_{\tilde{\Phi}_\rho + \alpha_\rho(\mathcal{B}\times\mathcal{B}_\mathbb{R})}((K + M\rho\mathcal{B}) \times \mathbb{R}^+)$$

From Proposition 4

$$(x_0, \vartheta_\rho(x_0)) \in Viab^T_{\tilde{\Phi}_\rho}(K_\rho \times (\mathbb{R}^+ - \{\frac{\alpha_\rho}{\ell}(e^{\ell T} - 1)\}\mathcal{B}_\mathbb{R})$$

where $K_\rho := K + (M\rho + \frac{\alpha_\rho}{\ell}(e^{\ell T} - 1))\mathcal{B}$

This holds true for any x_0 such that $\Theta_C^K(x_0) \leq T$. So

$$Graph(\vartheta_\rho^K) \cap (K_\rho \times [0,T]) \subset Viab_{\tilde{\Phi}_\rho}^T(K_\rho \times (\mathbb{R}^+ - \{\frac{\alpha_\rho}{\ell}(e^{\ell T} - 1)\}\mathcal{B}_\mathbb{R})$$

This inclusion means that at least there exists one trajectory remaining in a neighborhood of K of order ρ and reaching C_ρ in a time $\Theta_{C_\rho}^{K_\rho}(x_0)$ which satisfies

$$\Theta_{C_\rho}^{K_\rho}(x_0) - \frac{\alpha_\rho}{\ell}(e^{\ell T} - 1) \leq \vartheta_\rho^K(x_0)$$

From the very definition of the minimal time function, we always have

$$\Theta_{C_\rho}^{K_\rho}(x_0) \leq \Theta_{C_\rho}^K(x_0)$$

Let $x_\rho(\cdot) \in S_\Phi(x_0)$ the solution satisfying $x_\rho(\Theta_{C_\rho}^{K_\rho}(x_0)) \in C_\rho$. Since C and K are closed and since the set of solution is compact in $W^{1,1}([0,\infty],X)$ there exists a subsequence of solutions which converges to $\overline{x}(\cdot) \in S_\Phi(x_0)$ satisfying $\overline{x}(t) \in K \forall t \geq 0$ and $\overline{x}(\vartheta^\star) \in C$ where

$$\vartheta^\star = \liminf_{\rho \to 0} \Theta_{C_\rho}^{K_\rho}(x_0)$$

This implies that

$$\Theta_C^K(x_0) \leq \vartheta^\star \leq \vartheta_\rho(x_0) + \frac{\alpha_\rho}{\ell}(e^{\ell T} - 1) + o(\rho) \qquad \square$$

Corollary 1 *Let us assume that the following controlability assumption holds true:*

(A1) $\exists \rho_0 > 0$ such that for any $x_0 \in C + \rho_0\mathcal{B}$ a trajectory exists that reaches C in a time of order $o(d_C(x_0))$. Then

$$\Theta_C^{K_\rho}(x_0) - \frac{\alpha_\rho}{\ell}(e^{\ell T} - 1) - o(\rho) \leq \vartheta_\rho^K(x_0)$$

Moreover, if $K = X$, then for any x_0 such that $\Theta_C^K(x_0) \leq T$, we have

$$\Theta_C(x_0) - \frac{\alpha_\rho}{\ell}(e^{\ell T} - 1) - o(\rho) \leq \vartheta_\rho(x_0) \leq \Theta_C^K(x_0)$$

Let us mention a second relation, due to P. Cardaliaguet, between $\vartheta_\rho(\cdot)$ and $\Theta_C(\cdot)$ which states that at any point x_0 where $\Theta_C(x_0) \leq T$, we have

$$\inf_{\|y - x_0\| \leq r_\rho^T} \Theta_C(y) - r_\rho^T \leq \vartheta_\rho(x_0) \leq \Theta_C(x_0)$$

where $r_\rho^T := (2(1+\ell)e^{\ell T} - 1)M\rho$.

The proof is also based on Filippov Theorem but applied to the backward dynamical system. It implies namely the existence of some optimal solution to the continuous problem starting close to x_0 and remaining in a tube – in the *timexstate* space – around the computed piecewise linear optimal solution to the discrete problem starting exactly from x_0.

In other words, these results estimate the rate of convergence of the approached minimal time function to the exact one of order ρ at points where the minimal time function is continuous. At points od discontinuity of the minimal time function, convergence of order ρ still occurs but the epigraphic sense.

All these results can be extended to Bolza problem or to infinite horizon control problem.

2.2.3 The Norvegian Fishermen, an example of Target Problem without convexity

As an example let us now describe the following Norvegian Fishermen model studied by F. Barth ([4]). This problem deals with the behavior of the population of fishermen. N. Bonneuil ([5]) has proposed to exhibit what are the "good decisions" that the captains must choose, between following the group of fishermen who exploits a known site and taking risk for finding new site, so as to assure the survival of the population.

From a mathematical point of view, the main interest of this problem lies in the fact that the right hand side of the differential inclusion is not convex valued and not Lipschitz at points $y = 0$. Our aim is to prove that in this case the viability kernels for the initial problem and for the relaxed problem coincide so that the Viability Kernel Algorithm can be implemented. We state that this system in fact is relevant to a class of uncoupled dynamical systems.

Let us briefly summarize the model:

– the wealth of the population at time t is denoted by $z(t)$.

– the known level of resource of fish at time t is known through a density variable $y(t)$.

– the probability p "not to discover somewhere else a new site which density is higher than the known one" y is given through a repartition law of the form $p = 1 - e^{-\lambda y}$. Its evolution depends on the ratio $1 - u$ of captains who exploit together the known site. The complement u represents the ratio of thoses captains who explore the sea in order to discover some new abundant sites

Following N. Bonneuil in [5], we consider the dynamical system

$$\left\{ \begin{array}{lcl} z'(t) & = & c - (1 - u(t))y(t) \\ y'(t) & = & -\alpha(1 - u(t))y(t) \\ p'(t) & = & p(t)[(1 - e^{-\lambda y(t)})^{u(t)} - 1] \end{array} \right. \tag{5}$$

where c is the flow of irreductible expenses, for instance to keep boats in repair.

Considering the two first equations it necessarily comes out that the population becomes ruined in a finite time. That is to say that the Viability Kernel is necessarily empty. Also the question is to choose a suitable regulation u such that, at bankrupt $x = 0$, the population is "sure" (say with a probability greater than 0.95) to be able to discover a more abundant site somewhere else and so to restart a new fishing campaign.

Solving this problem amounts first to find the set of initial situations from which at least one trajectory starts remaining in the constraint set

$$K = \mathbb{R}^+ \times \mathbb{R}^+ \times [0,1]$$

until it reaches the target

$$C = \{0\} \times \mathbb{R}^+ \times [0, 0.05]$$

and second to find the feedback law $(z, y) \to u(z, y)$ assuring success.

2.2.4 Decomposable inclusion systems

Let us consider the differential inclusion system

$$x'(t) \in F(x(t) := \{f(x(t), u),\ \ u \in U(x)\} \tag{6}$$

Let us assume that a decomposition $X = X_1 \times X_2$ exists such that $K := K_1 \times K_2$ and that the differential inclusion system (6) can be written under the form

$$\begin{cases} i) & x_1' = f_1(x_1) + g_1(x_1)u \\ ii) & x_2' = f_2(x_1, x_2, u) \end{cases} \tag{7}$$

Let us consider $\overline{F}(\cdot)$ the relaxed set-valued map associated with F

$$\overline{F}(x) = \overline{F}(x_1, x_2) = \overline{Co}\{(f_1(x_1, u), f_2(x_1, x_2, u)),\ \ u \in U\}$$

Let us define the relaxed differential inclusion system

$$x'(t) \in \overline{F}(x(t) := \overline{Co}\{f(x(t), u),\ \ u \in U(x)\} \tag{8}$$

Proposition 6 *Assume that $\forall x$, $U(x)$ is convex and that F is a set-valued map such that K_2 is invariant*[4] *for f_2 when (x_1, u) covers $K_1 \times U$.*

Then we have

$$\mathrm{Viab}_{\overline{F}}(K) = \mathrm{Viab}_F(K)$$

and

$$\forall x \in \mathrm{Viab}_{\overline{F}}(K),\ F(x) \cap T_{\mathrm{Viab}_{\overline{F}}(K)}(x) \neq \emptyset.$$

Proof. a) Since $F(x) \subset \overline{F}(x)$, we always have $\mathrm{Viab}_F(K) \subset \mathrm{Viab}_{\overline{F}}(K)$. It is sufficient to state the converse inclusion.

Let be $x^0 = (x_1^0, x_2^0) \in \mathrm{Viab}_{\overline{F}}(K)$ and $\overline{x}(\cdot) \in S_{\overline{F}}(x_0)$ a solution of (8) viable in K. From the very definition of $\overline{F}$, there exists $n+1$ measurable functions $\overline{u}_i(\cdot) \in U(\overline{x}(\cdot))$ and $n+1$ positive measurable real valued functions $\alpha_i(\cdot)$ such that for any $t : \sum_{i=1}^{n+1} \alpha_i(t) = 1$ and

$$\begin{cases} \overline{x}_1'(t) & = f_1(\overline{x}_1(t)) + g_1(\overline{x}_1(t)) \sum_{i=1}^{n+1} \alpha_i(t)\overline{u}_i(t) \\ \overline{x}_2'(t) & = \sum_{i=1}^{n+1} \alpha_i(t) f_2(\overline{x}_1(t), \overline{x}_2(t), \overline{u}_i(t)) \end{cases} \tag{9}$$

[4] that is to say that for any absolutely continuous function $x_(\cdot)$ with values in K_1 and for any measurable function $u(\cdot)$, the solution to the equation $x_2'(t) = f_2(x_1(t), x_2(t), u(t))$ starting from any initial point $x_2(0) \in K_2$ remains in K_2 forever.

Let be $\overline{u}(\cdot) = \sum_{i=1}^{n+1} \alpha_i(t)\overline{u}_i(t)$ and let us consider the solution $\tilde{x}(\cdot)$ of the differential system

$$(10) \qquad \begin{cases} i) & \tilde{x}_1'(t) = f_1(\tilde{x}_1(t)) + g_1(\tilde{x}_1(t))\overline{u}(t), \quad x_1(0) = x_1^0 \\ ii) & \tilde{x}_2'(t) = f_2(\tilde{x}_1(t), \tilde{x}_2(t), \overline{u}(t)), \quad x_2(0) = x_2^0 \end{cases}$$

It is clear that $\tilde{x}_1(\cdot) = \overline{x}_1(\cdot)$. Then if we consider any solution $\tilde{x}_2(\cdot)$ of equation

$$x_2'(t) = f_2(\overline{x}_1(t), x_2(t), \overline{u}(t))$$

the pair $\tilde{x}(\cdot) := (\overline{x}_1(\cdot), \tilde{x}_2(\cdot))$ is a solution to the initial system (6) viable in K. So $x^0 = (x_1^0, x_2^0) \in \mathrm{Viab}_F(K)$. □

2.2.5 Application to the Norvegian Fishermen problem

We can apply this result to system (5). Let be $X_1 = \mathbb{R}^2$ and $X_2 = \mathbb{R}$, $x_1 = (z, y)$ and $x_2 = p$, $f_1(x_1) = (c - y, -\alpha y)$, $g_1(x_1) = (y, \alpha y)$ and $f_2(x_1, x_2, u) = x_2[(1 - e^{-\lambda y})^u - 1]$. Then we define

$$(11) \qquad F(x) := \begin{cases} \{f_1(x_1) + g_1(x_1)u, f_2(x_1, x_2, u), u \in U(x)\} & \text{if } (x_1, x_2) \notin C \\ (0, 0) & \text{if } (x_1, x_2) \in \overset{o}{C} \end{cases}$$

Let $K = \mathbb{R}^+ \times \mathbb{R}^+ \times [0, 1]$. Since for any $x_1 = (z, y) \in K_1$, $f_2(x_1, x_2, u) \leq 0$, it is easy to check that $K_2 = [0, 1]$ is an invariant set with respect to f_2 whatever are $x_1 \in K_1$ and $u \in U(x)$.

Figure 1 shows the "victory domain" for the Norvegian Fishermen problem.

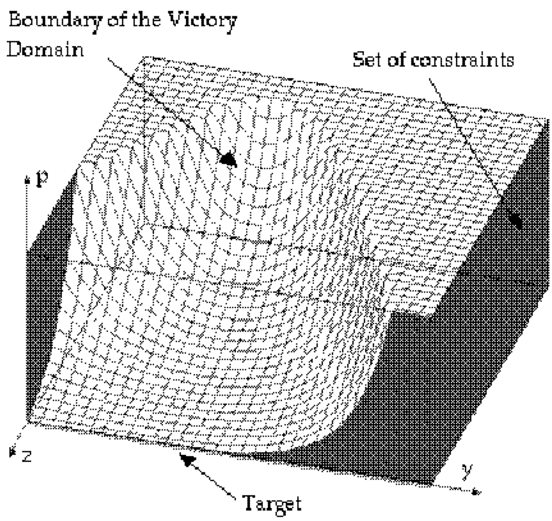

Fig. 1. Victory Domain for $F : \mathrm{Vict}_F(K, C)$. From any point in this set, there exists at least one trajectory which reaches the stage of ruin. $x = 0$ with a probability lower than 5 %.

2.3 The Minimal Time of Crisis

Let us consider the reference system (1). Now we admit that the state of the system can violate "temporarily" the constraint K. In such situation we say that a "crisis" occurs. We want to determine the Minimal Time of Crisis function[5] defined by

$$\mathcal{C}_F^K(x_0) := \inf_{x(\cdot)\in S_F(x_0)} \mu(t \mid x(t) \notin K) \;=\; \inf_{x(\cdot)\in S_F(x_0)} \int_0^{+\infty} \mathcal{X}_{K^c}(x(s))ds$$

where μ denotes the Lebesgue's measure in $\mathbb{R}$ and $\mathcal{X}_{K^c}(\cdot)$ denotes the characteristic function of the complement of K

$$\mathcal{X}_{K^c}(x) := \begin{cases} 0 & \text{if } \; x \in K \\ 1 & \text{if } \; x \notin K \end{cases}$$

Let us introduce the upper semicontinuous set-valued map:

$$\mathcal{X}^\natural_{K^c}(x) := \begin{cases} [0,1] & \text{if } \; x \in \partial K \\ 1 & \text{if } \; x \notin K \\ 0 & \text{otherwise} \end{cases}$$

and let us consider the extended system

$$\begin{cases} x'(t) & \in F(x(t)) \\ y'(t) & \in -\mathcal{X}^\natural_{K^c}(x). \end{cases}$$

We denote $\tilde{F} = F \times -\mathcal{X}^\natural_{K^c}$.

Proposition 7 *If $F : X \rightsquigarrow X$ is a Marchaud map and if K is closed in X, then*

the function $\mathcal{C}_F^K(\cdot)$ is lower semicontinuous,

$\forall x_0 \in Dom(\mathcal{C}_F^K)$, *there exists $x^\star(\cdot) \in S_F(x_0)$ such that* $\mathcal{C}_F^K(x_0) = \int_0^{+\infty} \mathcal{X}_{K^c}(x^\star(s))ds$, *and*

$$Epi(\mathcal{C}_F^K) = Viab_{\tilde{F}}(X \times \mathbb{R}^+)$$

In particular, as for the Minimal Time function, the Minimal Time of Crisis function can be approached by an increasing sequence of functions defined on successively refined grids X_h of X.

2.3.1 A numerical example

We consider the following controlled non linear equation

$$\begin{cases} \dot{x} = x(1 - x/10) - yx \\ \dot{y} = u \in [-1, 1] \end{cases}$$

[5]The following results are presented in a joint paper with L. Doyen [12].

and the domain of constraints K defined by

$$y(x-1) \geq 1.$$

The figure 2 represents the crisis map associated with this problem.

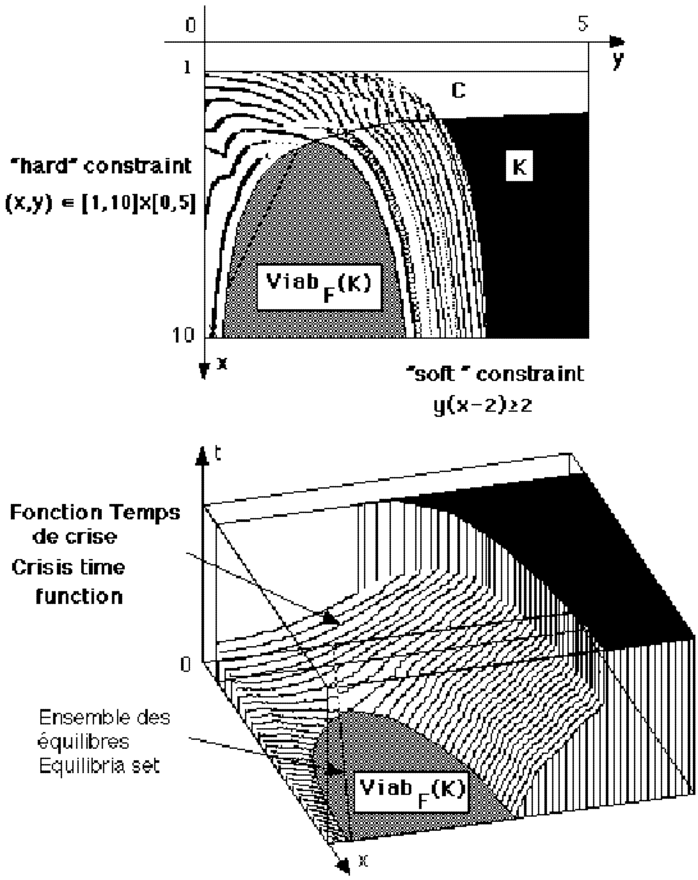

Fig. 2. Approximation of the graph of the crisis map for the dynamic $\dot{x} = x(1 - x/10) - yx,\ \dot{y} \in B$ and the constraint $y(x-1) \geq 1$.

2.3.2 Equilibria and Stability

Let be $F : X \rightsquigarrow Y$. Let us denote $Equi_F(K) := \{x \in K \mid 0 \in F(x)\}$ the set of all equilibria contained in a given set K.

We first give a characterization of $Equi_F(K)$ by the mean of the viability kernel of an extended dynamic system

Proposition 8 *Let F be a Marchaud map satisfying assumption (3) and let us consider the set-valued map $\Phi : X \times \mathbb{R} \rightsquigarrow X \times \mathbb{R}$ defined by*
$\Phi(x, y) := (0, \inf_{u \in F(x)} \|u\|_Y)$. Then Φ is a Marchaud map and

$$Equi_F(K) \times 0 = Viab_\Phi(K \times 0)$$

From this result and from the convergence Theorem 1 we can deduce a numerical method for finding either all the root of a polynomial $P(x) = 0$ or of piecewise lipschitz function $0 \in [\liminf_{x' \to x} f(x'), \limsup_{x' \to x} f(x')]$.

We are looking in second to the problem of finding the set of initial points from which a solution to (1) starts converging asymptotically to an equilibria. The following approach[6] is deeply related with continuity methods defining some paths following the graph of the given function.

Let us denote $\mathcal{F} := Graph(F)$.

A simple way to follow $\mathcal{F}$ is given by the differential inclusion

$$\begin{cases} (x'(t), y'(t)) & \in \mathcal{B}_{X \times Y} \\ (x(t), y(t)) & \in \mathcal{F} \end{cases}$$

We can explore the graph of F in such a way that $y(t) \to 0$. Thus we obtain an algorithm for finding all equilibria of the dynamic.

Let us then consider the following dynamic system

$$\begin{cases} i) & x'(t) \in & \mathcal{B}_X \\ ii) & y'(t) = & -ay(t) \end{cases}$$

Then the Viability Kernel of $\mathcal{F}$ for this dynamic is the graph (closed) of a set-valued map F^∞ containing all trajectories $(x(\cdot), y(\cdot))$ which now converges exponentially in y to an equilibrium. The set-valued maps F and F^∞ have the same equilibria. The following algorithm allows to approach F^∞ :

Let be F a closed graph set-valued map. Let us set $F_\rho^0 := F$ and let us consider the sequence of maps F_ρ^n defined by

$$F_\rho^n(x) := F_\rho^{n-1}(x) \cap \left(\frac{1}{1 - ah} \bigcup_{u \in \mathcal{B}} F_\rho^{n-1}(x + \rho u) \right)$$

Then we get

$$F_\rho^\infty(x) := \operatorname*{Lim}_{n \to \infty} F_\rho^n(x)$$

and $\forall (x_0, y_0) \in Graph(F_\rho^\infty)$ there exists $(x_n, y_n) \in Graph(F_\rho^\infty)$ such that

$$y_n = (1 - a\rho)^n y_0 \in F(x_n).$$

[6]We refer to a joint work with J.P. Aubin [2]

The knowledge of F_{ρ}^{∞} lead to the certitude that starting from any initial value in its graph, any trajectory which remain in the graph converges to an equilibrium.

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