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EFFICIENT CONTROL IN MULTISTAGE STOCHASTIC OPTIMIZATION PROBLEM

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An efficient control problem for bilinear multistage system with random perturbations is considered. The efficient solutions are choosen by two criteria: the first is maximization of a mean value, the second is minimization of a variance of utility function. Such approach has been suggested by Markovitz H. [13] to solve one-stage problem of the portfolio selection in financial analysis.

The existence conditions of the stationary efficient controls are obtained in case of incomplete information on the parameters of distributions. The randomization method for unknown parameters is used to construct a control problem solution. The concept of an adjoint stochastic optimization problem is introduced. The connection and separation problems of efficient control and observation are studied by means of adjoint problem solution.

Keywords: stochastic optimization, bilinear multistage system, efficient solution, adjoint problem, separation of control and observation.

AMS subject classification: 90C31, 90A09, 49K15, 49L20

1 Problem statement

Multistage bilinear control system with random and deterministic perturbations

$$x_{k+1} = A_k x_k + b_{k+1} + \xi_{k+1}, \quad k = 0, 1, \dots \tag{1.1}$$

$$w_{k+1} = r_k w_k + u_{k+1}^\top x_{k+1} + c_{k+1}^\top (u_{k+1} - u_k) \tag{1.2}$$

is considered. Here $x_k \in \mathbb{R}^n$ is a state vector, w_k is a scalar value connected with utility of control, ξ_k is an independent Gaussian random vector with known statistical moments:

$$E\xi_k = 0, \quad E\xi_k \xi_k^\top = R_k > 0. \tag{1.3}$$

It is supposed that $x_0 \in \mathbb{R}^n$, $u_0 \in \mathbb{R}^n$, $w_0 \in \mathbb{R}^1$, r_k and matrices A_k $[n \times n]$ are given, b_k, c_k are unknown deterministic disturbances given by their possible values domains:

$$c_k \in C_k, \quad b_k \in B_k, \tag{1.4}$$

where C_k, B_k are convex compacts sets in $\mathbb{R}^n$. The term $c_k^\top(u_{k+1} - u_k)$ represents the cost of control change.

Considered model (1.1) – (1.3) arises in particular in multistage portfolio selection problem in case of a linear regression model of the stock prices moving. In this case r_k is a riskless interest rate, $x_{k+1}^{(i)}$ is connected with a return on i-th stock, $x_{k+1}^{(i)} = s_{k+1}^{(i)} - r_k s_k^{(i)}$, where $s_k^{(i)}$ is a current price of i-th stock, $u_k^{(i)}$ is an amount of i-th stocks at k-th step, c_k is a transaction cost, w_k represents a current net wealth, w_0 is an initial capital. The similar problems were considered in papers [1, 5] for a geometrical Brounian model of the stock prices moving. The linear regression model may be more convenient for statistical identification and control especially in case of unstable money market.

Our purpose is to maximize a value $w_N = w_N(u, c, b, \xi)$ in a final moment N choosing a program control $u = \{u_1, \dots, u_N\} \in U$ for the whole time interval, here $c = \{c_1, \dots, c_N\}$, $b = \{b_1, \dots, b_N\}$, $\xi = \{\xi_1, \dots, \xi_N\}$ and $U \subset \mathbb{R}^{nN}$ is a convex set of the admissible controls. The problem may be considered as a multistage linear stochastic optimization problem with incomplete information about probabilistic distribution and with deterministic restrictions on the admissible solutions.The multistage stochastic optimization problems with incomplete information were considered in [6, 15] and others. On the other hand our problem (1.1) – (1.3) is a control problem for the bililear stochastic system [2]. The control problems in bilinear system with uncertainty were considered in [10, 18]. We study the problem of the efficient program control but a positional control may be obtained on this base (see Section 4) using the method of decomposition [3]. The value w_N is a random one and its distribution depends upon the chosen control and unknown parameters. A formulated problem may be solved on the base of the minimax stochastic approach developed by Kurzhanski A. B. [8, 12] in linear control problem under uncertainty. The control $u = \{u_1, \dots, u_N\}$ may be chosen by a criterion of maximization the least possible mean value:

$$f_1(u) \to \max, \quad u \in U,$$

$$f_1(u) = \min\{Ew_N(u, c, b, \xi) \mid c \in C,\ b \in B\},$$

where $C = C_1 \times \cdots \times C_N$, $B = B_1 \times \cdots \times B_N$. In case of bilinear control problem the variance of the utility function $w_N = w_N(u, c, b, \xi)$ depends on chosen control u so the risk of decision making connected with the variance should be taken into account.

Other approaches [10] are to optimize the least confidence level $w_\alpha(u)$ corresponding to a fixed probability α:

$$w_\alpha(u) = \min\{w_\alpha(u, c, b) \mid c \in C,\ b \in B\} \to \max,$$

where $P\{w_N(u, c, b, \xi) \geq w_\alpha(u, c, b)\} = \alpha$ or to optimize the least confidence probability corresponding to a given level w:

$$\alpha(u) = \min\{\alpha(u, c, b) \mid c \in C, b \in B\} \to \max,$$

where $\alpha(u, c, b) = P\{w_N(u, c, b, \xi) \geq w\}$. These appoaches take into account the whole information about probabilistic distributions but they lead to the complicated decision making algorithms. In this paper a bicriterial mean-variance appoach is used

$$\begin{cases} f_1(u) \to \max, \\ f_2(u) = \operatorname{var} w_N(u, c, b, \xi) \to \min, \quad u \in U. \end{cases} \tag{1.5}$$

Here a variance $f_2(u) = \operatorname{var} w_N(u, c, b, \xi) = E(w_N - Ew_N)^2$ does not depend on the unknown parameters c and b. It results from linearity of equations (1.1), (1.2) with respect to these parameters [8].

Definition 1.1 *A program control $u^* \in U$ is called* efficient *if it is the Pareto optimal solution in the bicriterial problem* (1.1) – (1.5)*, i.e. for any admissible control $u \in U$ at least one of the following conditions hold* [17]*:*
(i) $f_1(u) < f_1(u^*)$
(ii) $f_2(u) > f_2(u^*)$
(iii) $f_2(u) = f_2(u^*), f_1(u) = f_1(u^*)$.

It should be noted that optimization of the confidence level or quantile optimization leads to one of the efficient solutions since value $w_N(u, c, b, \xi)$ is Gaussian [9, 10]. With respect to our problem the equation (1.2) may be rewritten as

$$w_{k+1} = r_k w_k + u_{k+1}^\top x_{k+1} - \varphi_{k+1}(u_k - u_{k+1}),$$

where $\varphi_k(v) = \max\{v^\top c_k \mid c_k \in C_k\}$ is the support function of set C_k. In case of $C_k = [-\alpha_1; \alpha_1] \times \ldots \times [-\alpha_n; \alpha_n]$ the following equality holds:

$$\varphi(u_k - u_{k+1}) = \sum_{i=1}^{n} \alpha_i |u_{k+1}^{(i)} - u_k^{(i)}|.$$

2 Existence of stationary efficient solutions

Dynamic multistage problem (1.1) – (1.5) may be written as a bicriterial one-stage problem in $\mathbb{R}^{nN}$ space:

$$w_N(u) = u^\top \Phi x + c^\top G u,$$

where $x = \{x_1, \ldots, x_N\} \in \mathbb{R}^{nN}$ is Gaussian vector

$$Ex = \overline{x}, \quad \operatorname{cov} x = E(x - \overline{x})(x - \overline{x})^\top = P > 0.$$

Matrices Φ, P may be calculated from the equations (1.1) – (1.3). Values c and $\overline{x}$ are not known exactly and are given by

$$\overline{x} \in \overline{X}, \quad c \in C,$$

where C is a convex compact set in $\mathbb{R}^{nN}$, $\overline{X}$ is an information set [8] of phase vectors for the system (1.1) – (1.3). The criterion (1.5) has the form

$$\begin{cases} f_1(u) \to \max, \\ f_2(u) \to \min, \quad u \in U, \end{cases} \tag{2.1}$$

where $f_1(u) = \min\{u^\top(\Phi\overline{x} + G^\top c) \mid \overline{x} \in \overline{X}, c \in C\}$, $f_2(u) = u^\top \Phi P \Phi^\top u$.

If a domain U of admissible controls u is defined by linear restrictions then the problem (2.1) is reduced to a piecewise linear quadratic bicriterial problem. An algorithm suggested in [16] may be used to solve the problem.

An existence of a stationary efficient solutions is important in many applications. For example, one of the disadvantages of the geometrical growth model [1, 5] is an absense of stationary efficient solutions in the multistage portfolio selection problem: one has to sell or to buy stocks at every step even in case of constant statistical parameters of the return distribution.

As usual a program efficient control $u = \{u_1, \ldots, u_N\}$ is called *stationary* if it is does not depend on time, i.e. $u_k = u_1,\ k = 2, \ldots, N$. The conditions of existence of stationary efficient solution may be obtained using the Pareto optimality conditions.

Let us consider a simple case of independent phase vectors x_k with no uncertainty in their distributions parameters and no restrictions on admissible controls.

Theorem 2.1 *Let $A_k = 0$, $B_k = \{b_k\}$ for all $k = 0, 1, \ldots, N$, $U = \mathbb{R}^{nN}$. If a condition*

$$S = \bigcap R_k^{-1}(C_k - b_k) \neq \emptyset \tag{2.2}$$

holds then problem (1.1) – (1.5) *has a stationary nonzero efficient solution.*

Proof. In the considered case the criterion (1.5) may be rewritten as

$$\sum_{k=1}^{N} [u_k^\top b_k - \varphi_k(u_{k-1} - u_k)] \to \max$$
$$\sum_{k=1}^{N} u_k^\top R_k u_k \to \min, \quad u_k \in \mathbb{R}^N, \quad k = 1, \ldots, N.$$

The sufficient Pareto optimality conditions have the form [16]:

$$0 \in \lambda R_k u_k^* - b_k + \partial\varphi_k(u_{k-1}^* - u_k^*),$$

where $\partial\varphi_k(v)$ is subdifferential of function $\varphi_k(v)$. This function is a support function of the set C_k so $\partial\varphi_k(0) = C_k$ [14].

Pareto optimality conditions is rewritten as $0 \in \lambda R_k u_k^* - b_k + C_k,\ k = 1, \ldots, N$.

Let condition (2.2) holds. Denote a vector $u_1^* \in -S$ and consider the stationary control $u^* = \{u_1^*, \ldots, u_1^*\}$. The relation $R_k u_1^* \in -C_k + b_k$ or $0 \in R_k u_1^* - b_k + C_k$ holds for $k = 1, \ldots, N$. Therefore sufficient Pareto optimality conditions hold with $\lambda = 1$ and control u^* is a stationary efficient one. □

Corollary 2.1 *If $U = \mathbb{R}^{nN}$, $A_k = 0$, $R_k = R$, $B_k = \{b_k\}$ and $b_k \in b^* + C_k$ for all $k = 1, \ldots, N$ then nonzero efficient solution of* (1.1) – (1.5) *exists.*

Theorem 2.2 *If $A_k = 0$, $k = 0, \ldots, N-1$, $U = \mathbb{R}^{nN}$, and a condition*

$$S = \bigcap_k R_k^{-1}(C_k \overset{*}{-} B_k) \neq \emptyset$$

holds then nonzero efficient control in (1.1) – (1.5) *exists. Here $C_k \overset{*}{-} B_k$ is geometrical difference of two sets: $C_k \overset{*}{-} B_k = \{v \in \mathbb{R}^n : v + B_k \subset C_k\}$.*

PROOF. In this case criterion (1.5) may be rewritten as

$$-(\sum_{k=1}^{N} \psi_k(u_k) + \varphi_k(u_{k-1} - u_k)) \to \max$$
$$\sum_{k=1}^{N} u_k^\top R_k u_k \to \min, \quad u_k \in \mathbb{R}^N, \quad k = 1, \ldots, N.$$

Here $\psi_k^{(v)}$ is the support function of the set $(-B_k)$,

$$\partial\psi_k(v) = \{b \in -B_k \mid b^\top v = \psi_k(v)\} \subset -B_k$$

[14]. The sufficient Pareto optimality conditions have a form $0 \in \lambda R_k u_k^* + \partial\psi_k(u_k^*) + \partial\varphi_k(u_{k-1}^* - u_k^*)$, $\lambda \geq 0$. Denote $u_1^* \in -S$, $u_1^* \in -R_k^{-1}(C_k \overset{*}{-} B_k)$ for all k. It results in $-R_k^{-1}u_k^* + B_k \subset C_k$ and $-R_k^{-1}u_k^* - \partial\psi_k(u_k) \subset C_k$. For a stationary control $u^* = \{u_1^*, \ldots, u_1^*\}$ the Pareto optimality conditions hold so this control is stationary efficient one. $\square$

In general case of $A_k \neq 0$ the similar results may be obtained using an concept of adjoint problem (see Sect. 3).

3 Connection of efficient control and observation problems

Let us consider in detail a simple nondegenerate stochastic optimization problem (A) with a random utility function $w(u) = u^\top x$ and no restrictions on admissible controls $u \in \mathbb{R}^n$. Here x is n-dimensional Gaussian random vector with known moments $Ex = \overline{x}$, $E(x - \overline{x})(x - \overline{x})^\top = P > 0$. The corresponding bicriterial problem is

$$\begin{cases} Ew(u) = u^\top \overline{x} \to \max, \\ \operatorname{var} w(u) = u^\top P u \to \min, \quad u \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

The vector x is called a random purpose vector of the problem (A).

Efficient solutions set of (3.1) is written as

$$U^* = \{\lambda u^* \mid \lambda \geq 0\}, \quad u^* = P^{-1}\overline{x}. \tag{3.2}$$

Definition 3.1 *The efficient solution $u^* = P^{-1}\overline{x}$ is called a* **base efficient solution** *of the problem (A).*

We introduce the notion of adjoint stochastic optimization problems for solving of the dynamic optimization problem. The notion of the adjoint stochastic optimization problem is closely connected with the adjoint relations for linear systems in the control theory [4, 11, 12]. Percularities of the considered consept are bilinearity of the dynamic system and mean-variance approach to the control choosing. So obtained result has a similar form (e.g. (4.7)) as the classical equations of the adjoint dynamic problem but they have a special properties.

Definition 3.2 *Stochastic optimization problem* $(\tilde{A})$ *with random utility function* $\tilde{w}(x) = u^\top x$, $x \in \mathbb{R}^n$, *is* adjoint *to the problem* (A) *if* u *is Gaussian random* n*-dimensional vector with known moment* $Eu = \overline{u}$, $E(u-\overline{u})(u-\overline{u})^\top = D > 0$ *and*

$$\overline{u} = P^{-1}\overline{x}, \quad D = P^{-1}. \tag{3.3}$$

Let us consider the properties of the adjoint problems.

Property 3.1 *The problem adjoint to adjoint one coincides with the initial problem.*

It results from the definition of adjoint problem.

Property 3.2 *Let problem* (A_i), $i = 1, 2, 3$ *have the random purpose vectors* x_i*;* x_1, x_2 *are statistically independent and* $x_3 = x_1 + Gx_2$ *with* $n \times n$ *matrice* G*. Then for the purpose vectors* u_i, $i = 1, 2, 3$ *in the adjoint problem* (A_i) *equations hold:*

$$\begin{aligned} \overline{u}_3 &= D_3(D_1^{-1}\overline{u}_1 + GD_2^{-1}\overline{u}_2), \\ D_3 &= (D_1^{-1} + GD_2^{-1}G^\top)^{-1}, \end{aligned} \tag{3.4}$$

where $\overline{u}_i = Eu_i$, $D_i = \operatorname{cov} u_i = E(u_i - \overline{u}_i)(u_i - \overline{u}_i)^\top$*, i.e. distribution of* u_3 *coincides with a posteriori distribution of unknown vector* u_3 *after two observations* $u_1 = u_3$, $u_2 = G^\top u_3$*.*

Proof. Denote $\overline{x}_i = Ex_i$, $P_i = \operatorname{cov} x_i$, $P_i > 0$. Relation $x_3 = x_1 + Gx_2$ implies

$$\overline{x}_3 = \overline{x}_1 + G\overline{x}_2, \quad P_3 = P_1 + GP_2G^\top.$$

For adjoint problems (A_3) we have by definition

$$\overline{u}_3 = P_3^{-1}\overline{x}_3 = (P_1 + GP_2G^\top)^{-1}(\overline{x}_1 + G\overline{x}_2),$$

and $D_i^{-1} = P_i^{-1}$, so (3.4) holds. □

Property 3.3 *Let* x_i, $i = 1, 2$ *be* n*-dimensional random purpose vectors in stochastic optimization problems* (A_i) *and* $x_2 = Gx_1$, $\det G \neq 0$*. Then for purpose random vectors* u_i *in adjoint problems* $(\tilde{A}_i)$ *the following relation holds:* $u_2 = (G^\top)^{-1}u_1$*.*

Proof. The statistical moments of x_2 are $Ex_2 = G\overline{x}_1$, $\operatorname{cov} x_2 = GP_1G^\top$. From the definition of an adjoint problem the relations follow:

$$\overline{u}_2 = (GP_1G^\top)^{-1}G\overline{x}_1 = (G^\top)^{-1}P_1^{-1}G^{-1}G\overline{x}_1 = (G^\top)^{-1}P_1^{-1}\overline{x}_1$$

It results in $\overline{u}_2 = (G^\top)^{-1}\overline{u}_1$. □

Property 3.4 *Let* x_i, $i = 1, 2, 3$*, be the random purpose vectors in stochastic optimization problems* (A_i) *and* x_1, x_2 *be independent Gaussian vectors with known distributions. Information on* x_3 *is given by two observations:* $x_1 = x_3$ *and* $x_2 = Gx_3$*. Then purpose random vectors* u_i *in adjoint problems* $(\tilde{A}_i)$ *are connected by relation*

$$u_3 = u_1 + G^\top u_2.$$

Proof. A posteriori statistical moments of x_3 are described by relations [4, 7]:

$$Ex_3 = \overline{x}_3 = P_3(P_1^{-1}\overline{x}_1 + G^\top P_2^{-1}\overline{x}_2), \mathrm{cov}(x_3) = P_3 = (P_1^{-1} + G^\top P_2^{-1}G)^{-1}$$

For the adjoint problem it follows from definition $\overline{u}_3 = P_3^{-1}\overline{x}_3 = P_1^{-1}\overline{x}_1 + G^\top P_2^{-1}\overline{x}_2 = \overline{u}_1 + G^\top\overline{u}_2$. $\square$

Theorem 3.1 *Let x_i, $i = 1, 2, 3$, be the random purpose vectors in stochastic optimization problems (A_i); x_1, x_2 are independent Gaussian vectors with known distributions. Information on x_3 is given by two relations:*

$$x_1 = x_3, \quad x_2 = Gx_3.$$

Then efficient controls set for (A_3) equals

$$U_3^* = \{\lambda u_3^* \mid u_3^* = u_1^* + G^\top u_2^*\},$$

where u_i^* is a base efficient control in problem (A_i).

Theorem 3.1 immediately follows from property 3.4 and relation (3.2). This allows us to correct easily control u if an additional information is obtained on a random state vector x is obtained.

4 Construction of the efficient solutions

Let us consider our dynamic optimization problem (1.1) – (1.5). Assume that there are no restrictions on admissible controls and there are no deterministic perturbations in dynamic equation (1.1). We use the method of substitution of unknown parameters c_k to Gaussian random perturbations η_k [11, 12]. Equations are obtained

$$x_{k+1} = A_k x_k + b_{k+1} + \xi_{k+1}, \quad k = 0, \dots, N-1, \tag{4.1}$$

$$w_{k+1} = r_k w_k + u_{k+1}^\top x_{k+1} + \eta_{k+1}^\top(u_{k+1} - u_k) \tag{4.2}$$

in place of (1.1), (1.2). Here ξ_k, η_k are independent Gaussian random vectors with known moments

$$E\xi_k = E\eta_k = 0, \quad E\xi_k\xi_k^\top = R_k > 0, \quad E\eta_k\eta_k^\top = Q_k > 0, \tag{4.3}$$

values $r_k \geq 0$, b_k, x_0, w_0, u_0 and matrices A_k are given. There is a bicriterial problem in space $\mathbb{R}^{nN}$ with complete information on distributions of the random parameters. The criterion (1.5) is written as

$$\begin{cases} E(w_N(u)) \to \max, \\ \mathrm{var}(w_N(u)) \to \min, \quad u \in \mathbb{R}^n, \end{cases} \tag{4.4}$$

here $f_2(u) = \mathrm{var}\, w_N(u, \xi, \eta)$ depends on covariance matrices R_k and Q_k [4]. Problem (4.1) – (4.4) may be solved by means of the adjoint problem formulation.

Theorem 4.1 *Let $A_k = 0$, $r_k = 1$ for all $k = 0, \dots, N-1$, then a base efficient control $u^* = \{u_1^*, \dots, u_N^*\}$ of (4.1) – (4.4) coincides with a posteriori mean value of a phase* *vector for the following system with observation:*

$$v_k = v_{k-1} + \tilde{\eta}_k, \quad k = 1, \ldots, N,$$
$$d_k = v_k + \tilde{\xi}_k, \quad v_0 = u_0. \tag{4.5}$$

Here $d_k = R_k^{-1} b_k$, $\tilde{\xi}_k$, $\tilde{\eta}_k$ *are independent Gaussian vectors with known moments:*

$$E\tilde{\eta}_k = E\tilde{\xi}_k = 0, \quad E\tilde{\xi}_k\tilde{\xi}_k^\top = \tilde{R}_k, \quad E\tilde{\eta}_k\tilde{\eta}_k^\top = \tilde{Q}_k,$$
$$\tilde{R}_k = R_k^{-1}, \quad \tilde{Q}_k = Q_k^{-1}. \tag{4.6}$$

Proof. In case of $A_k = 0$, $r_k = 1$, $k = 0, \ldots, N-1$ the random function w_N of a control utility is written as

$$w_N = \sum_{k=1}^{N} u_k^\top (b_k + \xi_k) + \sum_{k=0}^{N-1} (u_{k+1} - u_k)^\top \eta_{k+1}.$$

The purpose vector for the problem is a sum of independent Gaussian vectors. From property 3.2 it is clear that the distribution of the purpose vector in adjoint problem coincides with a posteriori distribution of vector $u = \{u_1, \ldots, u_N\}$ after observations

$$u_k = d_k + \tilde{\xi}_k, \quad u_{k+1} - u_k = \tilde{\eta}_{k+1}.$$

Here $\tilde{\xi}_k$, $\tilde{\eta}_k$ are independent Gaussian vectors with known moments (4.6). Hence the theorem statement is obtained. □

Corollary 4.1 *If* $A_k = 0$, $r_k = 1$ *for all* $k = 0, \ldots, N-1$ *then a base efficient control in problem* (4.1) – (4.4) *is described by the Kalman equations of the filtration:*

$$u_{k+1}^* = u_k^* + \Lambda_{k+1}(d_{k+1} - u_k^*), \quad \Lambda_k = P_k R_k,$$

$$P_{k+1} = (P_k + Q_{k+1}^{-1})^{-1} + R_{k+1}, \quad k = 0, \ldots, N-1, \quad u_0^* = u_0, \quad P_0 = 0.$$

This results from theorem 4.1 and the standard equations for linear system states estimation [7].

Theorem 4.2 *In case of* $r_k = 1$, $k = 0, \ldots, N-1$, *a base efficient control* $u^* = \{u_1^*, \ldots, u_N^*\}$ *in problem* (4.1) – (4.4) *coincides with a posteriori mean value of a phase vector for the following system in reverse time*

$$v_N = d_N + \tilde{\xi}_N,$$
$$v_{k-1} = -A_{k-1}^\top v_k + d_{k-1} + \tilde{\xi}_{k-1}, \quad k = N, \ldots, 1 \tag{4.7}$$

with observation

$$v_k = v_{k-1} + \tilde{\eta}_k, \quad v_0 = u_0, \tag{4.8}$$

where $d_k = R_k^{-1} b_k$, $k = 2, \ldots, N$; $d_1 = R_1^{-1}(b_1 + A_0 x_0)$, $\tilde{\xi}_k$, $\tilde{\eta}_k$ *are independent Gaussian vectors with known moments* (4.6).

Proof. A phase vector of system (4.1) on k-th step may be written as

$$x_k = \sum_{i=1}^{k-1} \Phi_{ki} y_i + y_k, \quad \Phi_{ki} = A_i \cdots A_{k-1},$$

where $y_1 = A_0x_0+b_1+\xi_1$, $y_k = b_k+\xi_k$, $k = 2, \ldots, N$. We obtain the linear relation $x = \Phi y$ for nN-dimensional random vectors $x = \{x_1, \ldots, x_N\}$, $y = \{y_1, \ldots, y_N\}$. Function $w_N(u)$ of the control utility has the form

$$w_N(u) = u^\top \Phi y + \eta^\top G u$$

and random purpose vector in problem (4.1) – (4.4) is $z = \Phi y + G^\top \eta$. Property 3.2 implies that distribution of the purpose vector in the adjoint problem coincides with a posteriori distribution of vector $u = \{u_1, \ldots, u_N\}$ after two observations:

$$u = (\Phi^{-1})^\top \tilde{y}, \quad Gu = \tilde{\eta}, \tag{4.9}$$

where $\tilde{y} = \{d_1 + \tilde{\xi}_1, \ldots, d_N + \tilde{\xi}_N\}$, $d_1 = R_1^{-1}(A_0x_0 + b_1)$, $d_k = R_k^{-1}b_k$, $\tilde{\xi}_k$, $\tilde{\eta}_k$ are independent Gaussian vectors with known moments (4.6). By direct calculation of an inverse matrix $(\Phi^{-1})^\top$ we obtain equations (4.7). $\square$

We can not write recurrent equations for $u_1^*, \ldots, u_N^*$ as in the simple case of $A_k \equiv 0$. But we may calculate a base efficient control at the first step. It is enough for constructing an adaptive control in the problem (4.1) – (4.4).

The standard estimation of a posteriori mean value for system (4.7), (4.8) results in the following statement.

Corollary 4.2 *The base efficient control u_1^* on the first step in case of $r_k \equiv 1$ is defined by the equations:*

$$\overline{u}_N = d_N,$$

$$\overline{u}_{k-1} = -A_{k-1}^\top \overline{u}_k + d_{k-1}, \quad k = N, \ldots, 2, \tag{4.10}$$

$$P_{k-1} = A_{k-1}^\top P_k A_{k-1} + R_k^{-1}, \quad P_N = R_N^{-1}, \tag{4.11}$$

$$u_1^* = \overline{u}_1 + \Lambda_1(u_0 - \overline{u}_1), \quad \Lambda_1 = (P_1^{-1} + Q_1)^{-1}R_1. \tag{4.12}$$

Theorem 4.3 *Control u_1^* defined by equations* (4.10) – (4.12) *is an efficient control on the first step for problem* (4.1) – (4.4) *with arbitrary coefficients $r_k > 0$, $k = 0, \ldots, N-1$.*

Proof. In case of arbitrary $r_k > 0$ the utility function $w_N(v)$ has the form:

$$w_N(v) = \sum_{k=1}^{N} l_k x_k^\top v_k + \sum_{k=0}^{N-1} \eta_{k+1}^\top (v_{k+1} - v_k) l_{k+1},$$

where $l_k = r_k \cdots r_{N-1}, l_N = 1$. We may write $w_N = v^\top L(\Phi y + G^\top \eta)$, where L is a diagonal matrix, v is unknown control.

From property 3.3 a relation $v = L^{-1}u$ is obtained for a purpose vector v in the problem adjoint to (4.1) – (4.4) and a purpose vector v in this problem in case of $r_k = 1$, $k = 0, 1, \ldots, N-1$, considered in theorem 4.2. As a result we have for a base efficient control $v^* = \{v_1^*, \ldots, v_N^*\}$ in (4.1) – (4.4) a following representation $v^* = L^{-1}u^*$, or $v_k^* = \tilde{l}_k u_k^*$, $\tilde{l}_k = l_k^{-1}$, $k = 1, \ldots, N$, where $u_k^* = \{u_1^*, \ldots, u_N^*\}$ is a base efficient control in (4.1) – (4.4) in case of $r_k = 1$, $k = 0, \ldots, N-1$. The set of all efficient controls in the problem is $U^* = \{\lambda v^* \mid \lambda \geq 0\}$, therefore $u_1^* = l_1 v_1^* = r_1 \cdots r_{N-1} v_1^*$ is an efficient control at the first step for system (4.1) – (4.4) with arbitrary positive values r_k. $\square$

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