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CODIFFERENTIABLE MAPPINGS WITH APPLICATIONS TO VECTOR OPTIMALITY

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Codifferentiable mappings are defined as the ones which can be locally approximated by a particular type of difference convex mappings, adapting an analogous notion recently introduced for scalar functions. Some calculus rules are proved and some applications to vector optimization problems described by codifferentiable criteria and constraints are given.

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1 Introduction

The introduction of generalized differential objects to adequately describe first order properties of nonsmooth operators has proved to be a difficult task for many reasons. When trying to extend known concepts of generalized derivative of a scalar function, the main problem to overcome is that, instead of the natural (complete) order of the real axis, one faces a partial order strucure which is unavoidably poorer. Among the efforts made in this direction we limit ourselves to mention the contributions made by Penot [14], Thibault [19], Demyanov and Rubinov [5], whose analysis is naturally based on previous results on convex operators (see e.g. [20, 16, 2, 11]). Other authors approach the problem from a different point of view, which leaves aside the order structure of the image space: besides Clarke's generalized Jacobian for functions taking values in a finite dimensional space [3], we refer to the work of Ioffe [7], Mordukhovich [13], Aubin [1] to cite just a few.

With this picture in mind, it is not surprising that the extension to the vector valued case of all the achievements of scalar nonsmooth analysis is not completely satisfactory. The least we can say is that this subject needs further investigation.

Our starting point is an idea, which proved to be fruitful in the scalar case. The class of codifferentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$ has been recently introduced by V.F.

Dem'yanov (see [6] and references therein): a function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be codifferentiable at x if there exist a pair of convex, compact sets of $\mathbb{R}^{n+1}$, $(\underline{d}f(x), \overline{d}f(x))$, called codifferential, such that:

$$f(x+v) - f(x) = \max_{(a,u)\in\underline{d}f(x)} [a + \langle u, v\rangle] + \min_{(b,w)\in\overline{d}f(x)} [b + \langle w, v\rangle] + o(v),$$

where $\lim_{t\to 0^+} t^{-1}o(tv) = 0$. Thus the main idea behind the definition is to approximate some function around a point x by the difference of two convex functions which are not necessarily homogeneous.

This class is found to be equivalent to the one of quasidifferentiable functions (the ones whose directional derivative can be seen as the difference of two sublinear functions), but the new definition allows to single out an important subclass: f is said to be continuously codifferentiable at x if a codifferential can be found for all points y in a neighbourhood of x such that the mapping $y \mapsto (\underline{d}f(y), \overline{d}f(y))$ is Hausdorff continuous at x. For instance all convex functions and $\mathcal{C}^1$ functions are continuously codifferentiable.

The main reason to introduce an approximation which is not homogeneous relies in a sort of trade-off between algebraic and topological properties of local approximations; generalized differential objects defined by means of the directional derivative are positively homogeneous with respect to the direction, but lack continuity properties. Indeed the directional derivative is continuous with respect to the initial point x only if it is linear with respect to the direction, i.e. when the function is differentiable; analogously upper semicontinuity at x implies sublinearity in the direction [17, 4]. To analyse nonsmooth functions the Clarke generalized derivative and generalized gradient have often proved to be useful tools; they are upper semicontinuous (the latter as a set valued map) and thus they meet part of the requirement, but they fail to give a complete approximation of the increment: the Clarke derivative $f^\circ(x, v)$ is an upper sublinear approximation, i.e. it holds $f(x + v) - f(x) \le f^\circ(x, v) + o(v)$.

The concept of codifferentiability allows to obtain continuity within the framework of first order approximations; the price we pay for this is positive homogeneity with respect to directions.

In Section 2 we give a coherent extension of the main definitions to the abstract case and prove some calculus rules for them, including formulas for computing the codifferential of a composition. Section 3 is aimed at applications to vector optimization problems described by codifferentiable mappings. To this purpose, we exploit some ideas from [10] where necessary vector optimality conditions were developed on the basis of the axiomatic concept of upper convex approximation; codifferentiability proves to be a convenient tool to obtain a constructive example of this notion.

We close this section by fixing some notations and recalling some basic results on ordered spaces and on convex and conjugate functions, which will be useful later.

Throughout the paper X is a Banach space, Y and Z are Banach lattices, which are assumed to be complete with respect to the ordering relation induced by the closed, convex cones $K \subseteq Y$, $S \subseteq Z$ (in other words Y and Z are Banach K-spaces). Order completeness means that every subset which is bounded above admits a supremum (i.e. a least upper bound). This assumption allows to extend to the vector case many important results of the theory of convex and sublinear functions, including Hahn-Banach theorem on majorized extension of linear operators. The requirement that Y (and similarly for Z) is a Banach lattice also entails that any pair of elements of Y is upper bounded and that the norm is compatible with the order structure, i.e. it is monotone on K. These assumptions are strong and could be weakened, but are nevertheless enjoyed by important classes of functional spaces such as $B[0,1]$ of bounded functions and L_p, $1 < p < \infty$.

The notation we use is standard and recalled here for completeness. The continuous dual space to X, endowed with the weak* topology is denoted by X'. The (positive) polar of the cone $K \subseteq Y$ is the cone $K^+ = \{\theta \in Y' : \theta(k) \geq 0, \ \forall k \in K\}$. We will say that the linear functional θ is strictly positive ($\theta \in K^{+i}$) if $\theta k > 0$ for all $k \in K\backslash\{0\}$.

The mapping $g : X \to Z$ is *S-convex* if for every u, $v \in X$ and every $t \in [0, 1]$ it holds:

$$g(tu + (1-t)v) - tg(u) - (1-t)g(v) \in -S;$$

g is *S-sublinear* if it is S-convex and positively homogeneous. We will also say that the mapping g is *difference convex* (DC) (resp. *difference sublinear* (DSL)) if it can be written as the difference of two convex (resp. sublinear) mappings.

For a set $D \subset X$, we shall denote the *closure* and *interior* of D by cl D and $\operatorname{int} D$ respectively.

Let $h : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous (l.s.c.) convex function. Then, the *conjugate* function of h, $h^* : X' \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$h^*(v) = \sup\{v(x) - h(x) \mid x \in \text{ dom } h\}$$

where the domain of h is given by dom $h = \{x \in X \mid h(x) < +\infty\}$. The epigraph of h is defined by epi $h = \{(x,r) \in X \times \mathbb{R} \mid x \in \text{ dom } h, \ h(x) \leq r\}$. If $\tilde{h}(x) = h(x) - k, x \in X, k \in \mathbb{R}$, then epi $\tilde{h}^* =$ epi $h^* + (0,k)$. For a continuous S-convex mapping $g : X \to Y$ it is easy to show that the set $\bigcup_{\lambda \in S^+} \text{epi}\,(\lambda g)^*$ is a convex cone [9].

Given two closed, convex subsets A and B of a metric space Y, the *excess* of A over B is given by

$$e(A,B) = \sup_{a \in A} \inf_{b \in B} d(a,b)$$

and the *Hausdorff distance* between A and B is given by

$$haus(A,B) = \max\{e(A,B), e(B,A)\}.$$

We say that a set-valued map $\mathcal{A} : X \rightrightarrows Y$ is Hausdorff continuous if it is continuous with respect to this distance.

2 Codifferentiable Mappings

Let X be a Banach space, (Y,K) an order complete Banach lattice and consider a mapping $f : X \to Y$.

Definition 2.1 *The mapping $f : X \to Y$ is* codifferentiable *at $x \in X$ if there exists a pair of continuous mappings $F_x^\cup$, $F_x^\cap : X \times \mathbb{R} \to Y$, with $F_x^\cup$ sublinear and $F_x^\cap$ superlinear, such that for every $v \in X$ it holds:*

$$f(x+v) - f(x) = F_x^\cup(v,1) + F_x^\cap(v,1) + o(v),$$

where $\lim_{t\to 0^+} t^{-1}o(tv) = 0$.

We say that f *is Hadamard codifferentiable at* x *for each fixed* $v \in X$ *if the following condition holds for the remainder* $o(\cdot)$*:*

$$\lim_{\substack{t\to 0^+\\ w\to v}} t^{-1}o(tw) = 0.$$

To the mappings $F^\cup$ and $F^\cap$ we can associate their subdifferential $\underline{\partial}F^\cup$ and, respectively, superdifferential $\overline{\partial}F^\cap$:

$$\underline{\partial}F^\cup = \{(A,a) \in L(X,Y)\times Y :\ Av + ar \le F^\cup(v,r),\ \forall (v,r)\in X\times\mathbb{R}\}$$

and

$$\overline{\partial}F^\cap = \{(B,b) \in L(X,Y)\times Y :\ Bv + br \ge F^\cap(v,r),\ \forall (v,r)\in X\times\mathbb{R}\}.$$

We will denote the pair $(\underline{\partial}F^\cup, \overline{\partial}F^\cap)$ by $Df(x) = (\underline{d}f(x), \overline{d}f(x))$ and call it codifferential of f at x.

If Definition 2.1 holds with $F^\cap = 0$, the mapping f is said *hypodifferentiable* at x and analogously *hyperdifferentiable* if $F^\cup = 0$. If one between $F^\cup$ and $F^\cap$ is 0 and the other is a linear mapping, f is Gâteaux differentiable. If, for all points y in a neighbourhood of x, $F_y^\cup$ and $F_y^\cap$ can be choosen in a way that the mapping $Df(\cdot)$ is Hausdorff continuous, f is said to be *continuously codifferentiable* at x.

It is immediate to see that this definition extends the one in the scalar case, thanks to the duality between convex, compact subsets of $\mathbb{R}^n$ and sublinear functions.

If f is hypodifferentiable at x, $F_x^\cup(0,1) = 0$ and consequently, if $(A,a) \in \partial F_x^\cup$, then $a \le 0$. Moreover since (Y,K) is order complete, then the max formula holds:

$$F^\cup(v,r) = \max_{(A,a)\in\underline{\partial}F^\cup} Av + ar$$

and thus $\max\{a \in Y : (A,a) \in \underline{\partial}F^\cup\}$ is attained and equals 0. Analogously, when f is codifferentiable, it holds $F^\cup(0,1) + F^\cap(0,1) = 0$ and consequently $\max_{(A,a)\in\underline{\partial}F^\cup} a + \min_{(B,b)\in\overline{\partial}F^\cap} b = 0$.

As one may suspect, there is a strong link between the classes of codifferentiable and quasidifferentiable mappings [5] (we remind that a mapping f is quasidifferentiable at x if it is directionally differentiable and the derivative is difference sublinear with respect to directions). Indeed it can be proved by the same reasoning as in the scalar case [6] that the two classes coincide and that from the knowledge of one between the quasi- and the codifferential it is possible to derive the other.

More precisely if it holds $f'(x,v) = \phi_1(v) + \phi_2(v)$ with ϕ_1 sublinear and ϕ_2 superlinear, we call quasidifferential of $f : X \to Y$ at x a pair of subsets $\underline{\partial}f(x)$, $\overline{\partial}f(x) \subseteq L(X,Y)$, which are respectively the subdifferential of ϕ_1 and the superdifferential of ϕ_2 and the following relation can be easily derived: if $(\underline{\partial}f(x), \overline{\partial}f(x))$ is a quasidifferential for f, then a codifferential is given by $\underline{d}f(x) = \underline{\partial}f(x)\times\{0\}$ and $\overline{d}f(x) = \overline{\partial}f(x)\times\{0\}$; if conversely $(\underline{d}f(x), \overline{d}f(x))$ is a codifferential and $\overline{a} = \max_{(A,a)\in\underline{\partial}F^\cup} a$, $\overline{b} = \min_{(B,b)\in\overline{\partial}F^\cap} b$, then we have $\underline{\partial}f(x) = \{A \in L(X,Y) : (A,\overline{a}) \in \underline{d}f(x)\}$ and $\overline{\partial}f(x) = \{B \in L(X,Y) : (B,\overline{b}) \in \overline{d}f(x)\}$.

By slightly changing the notation, we can rewrite Definition 2.1 in order to emphasize the dependence on the point x and the direction v; set $F_x^\cup(v,1) = f^\cup(x,v)$, $F_x^\cap(v,1) =$

$f^{\cap}(x,v)$ and $f^{\cup}(x,v) + f^{\cap}(x,v) = f^{\odot}(x,v)$; we obtain then:

$$f(x+v) - f(x) = f^{\odot}(x,v) + o(v),$$

where the bifunction $f^{\odot} : X \times X \to Y$ satisfies the following: $f^{\odot}(x,\cdot)$ is difference convex (w.r.t. K) with $f^{\odot}(x,0) = 0$. If $Df(\cdot)$ is Hausdorff continuous at x, then $f^{\odot}$ is jointly continuous at (x,v) for every $v \in X$ (see [18]). We will say that the mapping $f^{\odot}(x,\cdot)$ is a (continuous) approximator of f at the point x. In [18] the concept of continuous approximator as a bifunction depending continuously on the point and the direction is the starting point to analyse local properties of nonsmooth mappings. This notion does not require any order structure on the image space and is therefore more general. Nevertheless the class of continuously codifferentiable mappings represents an important example in this direction.

We emphasize that the most relevant feature of codifferentiability is the possibility of choosing a codifferential which is Hausdorff continuous and this comes from the fact that we define codifferentials in the product space $L(X,Y) \times Y = L(X \times \mathbb{R}, Y)$. On the other hand, if we have a continuous codifferential, then the approximator will not be linear with respect to directions at points of differentiability which are near to a point where the mapping f is not differentiable; indeed at nonsmooth points the codifferential will not be a singleton and its Hausdorff continuity prevents it to be singleton in a small neighbourhood.

To see how large the class of continuously codifferentiable mappings is, we can start by noting that it trivially contains all continuously differentiable mappings, since the assumption of Hausdorff continuity reduces to norm continuity when the mapping Df is single valued. We show now that convex continuous operators are continuously codifferentiable at least under some restriction on the image space; we say that Y has the *Dini property* if every increasing and bounded above sequence has a topological limit (necessarily equal to its supremum). We need also the following extension of convexity for subsets of a space of linear mappings $L(X,Y)$, where Y is an ordered space: a set $U \subset L(X,Y)$ is said *operator convex* if, for all A, $B \in L(X,Y)$, and for all $\alpha \in L^{+}(Y)$, $0 \le \alpha \le I_Y$ it holds:

$$\alpha A + (I_Y - \alpha)B \in U,$$

where I_Y is the identity operator in Y and $L^{+}(Y) = \{\alpha \in L(Y,Y) : \alpha k \in K, \ \forall k \in K\}$ is the set of positive operators from Y to itself. The operator convex hull of a set $U \subseteq L(X,Y)$ is the intersection $coo\, U$ of all operator convex sets containing U.

In Theorem 2.2 we will prove a property of convex operators which is actually stronger than continuous hypodifferentiability and can be described in terms of a Lipschitz behaviour of the codifferential.

Theorem 2.2 *If Y has the Dini property and the mapping $f : X \to Y$ is convex and continuous, then its hypodifferential $\underline{d}f(x)$ satisfies, for every $x \in X$, the following condition: there exist a neighbourhood $U(x)$ and a constant $M > 0$ such that*

$$haus\ (\underline{d}f(x_1), \underline{d}f(x_2)) \le M\|x_1 - x_2\|, \quad \forall x_1, x_2 \in U.$$

Proof. Consider a closed bounded set $\Omega \subseteq X$ with $x \in \operatorname{int}\Omega$. For every $z \in \Omega$ choose a fixed $A_z \in \partial f(z)$; it holds $f(x) \ge f(z) + A_z(x-z)$ for every $z \in \Omega$ and

$$f(x) = \max_{z\in\Omega}\left[f(z) + A_z(x-z)\right],$$

since $x \in \Omega$. Thus for $v \in X$ such that $x+v \in \operatorname{int}\Omega$, we have:

$$\begin{aligned} f(x+v) &= f(x) + \max_{z\in\Omega}\left[f(z) - f(x) + A_z(x-z) + A_z v\right] \\ &= f(x) + \max_{(A,w)\in H(x)} Av + w, \end{aligned}$$

where

$$H(x) = \{(A,w) \in L(X,Y)\times Y : A = A_z \in \partial f(z),\, w = f(z) - f(x) + A_z(x-z),\, z\in\Omega\}.$$

By the results in [16] the hypodifferential of f is given by:

$$\underline{d}f(x) = \overline{\text{coo}}\, H(x),$$

where the closure is taken in the weak operator topology of $L(X\times\mathbb{R}, Y)$. To prove Lipschitz continuity of the hypodifferential around x, take x_1 and x_2 and fix any $(A_1, w_1) \in H(x_1)$. Choose $z_1 \in \Omega$ such that $A_1 = A_{z_1}$ and $w_1 = f(z_1) - f(x_1) - A_1(x_1 - z_1)$. As, for $w_2 = f(z_1) - f(x_2) - A_1(x_2 - z_1)$, one has $(A_1, w_2) \in H(x_2)$, one may write:

$$\begin{aligned} \inf_{(A_2,w_2)\in H(x_2)} \|(A_1,w_1) - (A_2,w_2)\| &\le \|f(x_1) - f(x_2) - A_z(x_1 - x_2)\| \\ &\le \|f(x_1) - f(x_2)\| + \|A_z(x_1 - x_2)\|, \end{aligned}$$

for some $z \in \Omega$ and $A_z \in \partial f(z)$; the thesis follows by Lipschitz continuity of convex continuous operator [15] and local boundedness of its subdifferential [2]. □

We note that Theorem 2.2 can be proved for more general classes of spaces using results in [11].

Various calculus rules can be given for codifferentiable mappings. It is easily seen that the sum of two codifferentiable mappings is codifferentiable and that the product of a real number by a codifferentiable mapping is codifferentiable. Thus we can see that the space of codifferentiable mappings is a vector space. We can also describe the codifferential as follows: let the mappings $f, g : X \to Y$ be codifferentiable at x with codifferentials $Df(x) = (\underline{d}f(x), \overline{d}f(x))$ and $Dg(x) = (\underline{d}g(x), \overline{d}g(x))$ and consider $\alpha \in \mathbb{R}$. Then immediately from Definition 2.1 we see that the codifferential of the mappings $h = f+g$ and $l = \alpha f$ are given by $Dh(x) = (\underline{d}f(x) + \underline{d}g(x), \overline{d}f(x) + \overline{d}g(x))$ and $Dl(x) = (\alpha\underline{d}f(x), \alpha\overline{d}f(x))$ if $\alpha \ge 0$ and $Dl(x) = (\alpha\overline{d}f(x), \alpha\underline{d}f(x))$ is $\alpha < 0$. These results together with Theorem 2.2 show that any difference convex operator (whose image space has the Dini property) is codifferentiable.

More importantly we can show that the class of codifferentiable operators is closed under composition. For this purpose, consider three Banach spaces X, Y and Z, with Y and Z order complete Banach lattices. A linear mapping Λ defined between Y and Z is *regular* if it can be seen as the difference of two positive linear mappings; a sublinear mapping P between Y and Z is said to be *order bounded* if there exist a pair of regular mappings Λ_1 and Λ_2 such that:

$$\Lambda_1(y) \le -P(-y) \le P(y) \le \Lambda_2(y), \quad \forall y \ge 0.$$

Since $P(y) = \max\{Ay,\ A \in \underline{\partial} P\}$, we can say that P is regular when:

$$\Lambda_1(y) \leq \min_{A \in \underline{\partial} P} Ay \leq \max_{A \in \underline{\partial} P} Ay \leq \Lambda_2(y), \ \ \forall y \geq 0,$$

that is if and only if the subdifferential of P consists of regular operators and it is order bounded in the space of regular operators.

Lemma 2.3 *Consider the sublinear operators $P_1, P_2 : X \to Y$ and $Q_1, Q_2 : Y \to Z$ and let Q_1 and Q_2 be order bounded. Then the operator $R = (Q_1 - Q_2) \circ (P_1 - P_2)$ can be expressed as the difference of two sublinear operators $R_1, R_2 : X \to Z$. Moreover for the subdifferentials $\underline{\partial} R_1$ and $\underline{\partial} R_2$ it holds:*

$$\underline{\partial} R_1 = \bigcup_{C \in \underline{\partial} Q_1} P_C \qquad \textit{and} \qquad \underline{\partial} R_2 = - \bigcup_{C \in -\underline{\partial} Q_2} P_C,$$

where $P_C = (C - \Lambda_1)P_1 + (\Lambda_2 - C)P_2$, Λ_1 and Λ_2 are common lower and upper bounds for the sets $\underline{\partial} Q_1$ and $-\underline{\partial} Q_2$.

Theorem 2.4 *Let the mapping $f : \Omega \subseteq X \to Y$ be (continuously) codifferentiable at x and the mapping $g : Y \to Z$ be Hadamard (continuously) codifferentiable at $y = f(x)$. Assume that there exists a codifferential of g at y such that $\underline{d}g(y)$ and $\overline{d}g(y)$ belong to the space of regular operators and are bounded there. Then the mapping $h = g \circ f$ is (continuously) codifferentiable at x.*

Proof. By hypothesis there exist DSL operators F and G (between the appropriate spaces) such that: $f(x+v) - f(x) = F(v,1) + o(v)$ and $g(y+w) - g(y) = G(w,1) + o'(w)$. Setting $p(v) = F(v,1)$ and $q(w) = G(w,1)$ we obtain DC approximations for f at x and for g at $y = f(x)$; since g is Hadamard codifferentiable, it is possible to approximate the operator h at the point x by means of the mapping $s(v) = q(p(v))$. To show that s is a DC mapping, define the mapping $\Phi : X \times \mathbb{R} \to Y \times \mathbb{R}$ as $\Phi(v,r) = (F(v,r), r)$ and set $H(v,r) = G(\Phi(v,r))$. It holds $H(v,1) = G(\Phi(v,1)) = G(F(v,1),1) = G(p(v),1) = q(p(v))$. The operator H is DSL as the composition of DSL operators and q is DC since it is a restriction (at $r = 1$) of a DSL operator. This shows that the mapping h is codifferentiable at x.

If we write $F = F_1 - F_2$ and $G = G_1 - G_2$, with F_i, G_i sublinear, $i = 1, 2$, we have $\underline{d}f(x) = \underline{\partial} F_1$, $\overline{d}f(x) = -\underline{\partial} F_2$, $\underline{d}g(y) = \underline{\partial} G_1$ and $\overline{d}g(y) = -\underline{\partial} G_2$; consequently, for the codifferential $Dh(x) = \big(\underline{d}h(x), \overline{d}h(x)\big) = (\underline{\partial} H_1, -\underline{\partial} H_2)$, where $H = H_1 - H_2$, we set $\Lambda_1 = (L_1, l_1)$ and $\Lambda_2 = (L_2, l_2))$ as common upper and lower bounds for $\underline{d}g(y)$ and $-\overline{d}g(y)$ and use Lemma 2.3 to obtain:

$$\begin{aligned} \underline{d}h(x) &= \bigcup_{C \in \underline{d}g(y)} \underline{\partial} \left[(C - \Lambda_1)\Phi_1 + (\Lambda_2 - C)\Phi_2 \right] \\ &= \bigcup_{(B,b) \in \underline{d}g(y)} \underline{\partial} \left[(B - L_1)F_1 + b - l_1 + (L_2 - B)F_2 \right] \end{aligned}$$

and

$$\overline{d}h(x) = \bigcup_{C \in \overline{d}g(y)} \underline{\partial} \left[(C - \Lambda_1)\Phi_1 + (\Lambda_2 - C)\Phi_2 \right]$$

$$= \bigcup_{(B,b)\in\overline{d}g(y)} \underline{\partial}\,[(B-L_1)F_1 + b - l_1 + (L_2 - B)F_2] \qquad \square$$

Remark 2.5 Consider the particular case where the outer mapping is a linear functional, i.e. $g = \lambda \in Y'$, and Y has weakly compact order intervals. Then it holds $Y' = K^+ - K^+$ and we can write any linear functional λ as the difference of two positive linear functional $\lambda_1 - \lambda_2$. Moreover for every continuous sublinear operator $F : X \to Y$ and every positive $\lambda \in Y'$ it holds $\underline{\partial}(\lambda F)(x) = \lambda \underline{\partial} F(x)$. Since $-\lambda_2 \leq \lambda \leq \lambda_1$, for $h(x) = \lambda f(x)$ we obtain:

$$(\underline{d}h(x), \overline{d}h(x)) = (\lambda_1 \underline{d}f(x) + \lambda_2 \overline{d}f(x),\ \lambda_1 \overline{d}f(x) + \lambda_2 \underline{d}f(x)).$$

3 Optimality Conditions

Consider the following vector optimization problem

$$(P) \qquad \text{K-Minimize} \quad f(x)$$
$$\text{subject to} \quad g(x) \in -S.$$

where $f : X \to Y$ and $g : X \to Z$, X is a Banach space, (Y, K) and (Z, S) are order complete Banach lattices; note here that we write "K-minimize" to emphasize that the image space is (only) partially ordered by the closed, convex, pointed cone K, i.e. we face a vector minimization problem. We shall briefly consider the different types of solution to a vector optimization problem, where $A = \{x \in X : \ g(x) \in -S\}$.

Definition 3.1 *Given the mapping $f : X \to Y$, the point $a \in A \subseteq X$ is said to be:*

(i) *(Pareto) minimal point ($a \in M(A)$) if*

$$(f(A) - f(a)) \cap (-K) = \{0\};$$

(ii) *weakly minimal point ($a \in W(A)$) if $int K \neq \emptyset$ and*

$$(f(A) - f(a)) \cap -int K = \emptyset;$$

(iii) *properly minimal point ($a \in Pr(A)$) if there exists a convex cone K' such that $K\backslash\{0\} \subseteq int K'$ and*

$$f(A) - f(a) \cap -K' = \{0\}.$$

From Definition 3.1, it is clear that the inclusions $Pr(A) \subseteq M(A) \subseteq W(A)$ hold true; simple examples in $\mathbb{R}^2$ show that both can be strict. Though the concepts of minimal and weakly minimal solutions are those most usually found in the literature on vector optimization, note that the assumption of a nonempty interior for the ordering cone which is required for weak minimality poses problems in infinite dimensions since cones of nonnegative functions have empty interior in most common spaces (e.g. L^p spaces), while no such assumption is required for proper minimality. Moreover under weak assumptions, the set of properly minimal points is dense in the one of minimal points (see [8, 12]).

The next result gives a necessary condition for proper optimality under the assumption that the cone S has nonempty interior. Theorem 3.6 will deal with situations in which this requirement is not satisfied. Note that Theorem 3.2 can be reworded in terms of quasidifferentials of f and g and that the method of proof can be used to give an analogous result for weakly minimal points, which is omitted here.

Theorem 3.2 *Let $a \in X$ be a properly minimal solution of (P) with $\operatorname{int} S \neq \emptyset$ and the mappings f and g be codifferentiable at a. Then for every $(M,m) \in \overline{d}f(a)$ with $m = \min\{c |\, (C,c) \in \overline{d}f(a)\}$ and every $(D,d) \in \overline{d}g(a)$ with $d = \min\{b |\, (B,b) \in \overline{d}g(a)\}$, there exist $\theta \in K^{+i} \cup \{0\}$ and $\lambda \in S^{+}$ such that $\lambda g(a) = 0$ and:*

$$(0,0) \in \underline{d}(\theta f)(a) + \theta(M,m) + \underline{d}(\lambda g)(a) + \lambda(D,d).$$

Proof. By the assumption of codifferentiability of f and g, we have:

$$f(x) - f(a) = F^{\cup}(x-a,1) + F^{\cap}(x-a,1) + o(x-a)$$

$$g(x) - g(a) = G^{\cup}(x-a,1) + G^{\cap}(x-a,1) + o'(x-a)$$

where $F^{\cup}$, $G^{\cup}$ are sublinear and $F^{\cap}$, $G^{\cap}$ are superlinear. Set $p(v) = F^{\cup}(v,1)$ and $q(v) = G^{\cup}(v,1)$. We can approximate the mappings f and g at the point a as follows: take $m = \min\{c \in Y,\ (C,c) \in \overline{\partial} F^{\cap}\}$ and $d = \min\{b \in Z,\ (B,b) \in \overline{\partial} G^{\cap}\}$ and take M, D such that $(M,m) \in \overline{\partial} F^{\cap}$, $(D,d) \in \overline{\partial} G^{\cap}$. Setting $\phi(x) = f(a) + p(x-a) + M(x-a) + m$ and $\psi(x) = g(a) + q(x-a) + D(x-a) + d$ it holds

$$f(x) \leq_K \phi(x) + o(x-a) \qquad f(a) = \phi(a)$$

and

$$g(x) \leq_S \psi(x) + o'(x-a) \qquad g(a) = \psi(a).$$

Proper minimality of a implies that, for every convex cone K' such that $K \backslash \{0\} \subseteq \operatorname{int} K'$, the following system is impossible:

$$\begin{cases} f(x) - f(a) & \in & -\operatorname{int} K' \\ g(x) & \in & -\operatorname{int} S \end{cases}$$

and consequently also

$$\begin{cases} \phi(x) - \phi(a) & \in & -\operatorname{int} K' \\ \psi(x) & \in & -\operatorname{int} S \end{cases}$$

is impossible. By cone-convexity of ϕ and ψ, the sets $[\phi(X) - \phi(a) + K']$ and $[\psi(X) + S]$ are convex and their cartesian product is disjoint from the cone $-\operatorname{int}(K' \times S)$. Thus there exist linear functionals $\theta \in Y'$ and $\lambda \in Z'$ such that $\lambda \in S^{+}$, $\theta \in K'^{+} \subseteq K^{+i} \cup \{0\}$ and

$$\theta\left(\phi(x) - \phi(a) + k'\right) + \lambda\left(\psi(x) + s\right) \geq 0 \quad \forall x \in X,\, k' \in K',\, s \in S.$$

Since $0 \in K'$, $0 \in S$, the previous inequality yields

$$\theta\left(\phi(x) - \phi(a)\right) + \lambda \psi(x) \geq 0 \quad \forall x \in X$$

and also, for every $x \in X$

$$\theta\left(p(x-a) + M(x-a) + m\right) + \lambda\left(g(a) + q(x-a) + D(x-a) + d\right) \geq 0.$$

Setting $x = a$ in the previous inequality we obtain the complementarity condition $\lambda g(a) = 0$ and the rest of the thesis follows. $\square$

Remark 3.3 Theorem 3.2 gives a necessary condition of Fritz John type, since the multiplier of the objective function can take zero value. We can obtain a Kuhn-Tucker type condition by requiring that there exists $z \in X$ such that $\psi(z) \in -\mathrm{int}\, S$ where ψ is the approximation of g at a as in the proof of the above result. This constraints qualification implies now that θ is strictly positive, which is an important feature of proper minimality.

Remark 3.4 If the spaces Y and Z have weakly compact order intervals, the thesis of Theorem 3.2 becomes

$$(0,0) \in \theta\left(\underline{d}f(a) + (M,m)\right) + \lambda\left(\underline{d}g(a) + (D,d)\right).$$

In [10] a continuous, K-convex mapping $\phi : X \to Y$ was called an upper convex approximation of the mapping $f : X \to Y$ at the point a, if $\phi(a) = f(a)$ and for every $x \in X$ there exists a mapping $o : \mathbb{R} \to Y$, with:

$$\lim_{t \to 0} o(t)/t = 0 \quad \text{and } \phi(a + t(x-a)) \geq_K f(a + t(x-a)) + o(t).$$

Such concept has there been used in an axiomatic way to give necessary vector optimality conditions for some nonconvex problems, but the only constructive examples were the linear approximation available for Gâteaux differentiable mappings or the ones obtained by means of some generalized sublinear vectorial derivatives. In this context, note that any codifferentiable operator trivially admits a convex upper approximation as in the proof of Theorem 3.2.

In the case where $\mathrm{int}\, S = \emptyset$, some optimality condition can be given in an asymptotic form assuming difference-convexity of the constraint mapping g and making use of appropriate alternative theorems; the one we present was derived in [10].

Lemma 3.5 *Let $T \subseteq Y$ and $S \subseteq Z$ be closed convex cones, with $intT \neq \emptyset$; let $f : X \to Y$ and $g : X \to Z$ be continuous, T-convex and respectively S-convex mappings. Suppose that the system $g(x) \in -S$ is consistent. Then exactly one of the following statements holds:*

(i) $\exists x \in X : f(x) \in -intT, \quad g(x) \in -S;$

(ii) $\exists \theta \in T^+ \backslash \{0\} : \; 0 \in \mathrm{epi}\,(\theta f)^* + \mathrm{cl}\,\left[\bigcup_{\lambda \in S^+} \mathrm{epi}\,(\lambda g)^*\right].$

Lemma 3.5 was used in [10] to obtain a characterization of weakly minimal and properly minimal solutions of a convex vector optimization problems without any regularity conditions; it will be used here to derive asymptotic necessary optimality conditions for codifferentiable mappings. We prove completely only the simplified version in which the objective is hypodifferentiable and the constraint mapping is convex. To obtain greater generality, one can construct upper convex approximation for f and g following the same reasoning as in Theorem 3.2; note that in this case (following the terminology of [10]) the approximation of g is tight, i.e. the remainder o is identically 0.

Theorem 3.6 *Let $a \in X$ be a properly minimal solution of (P) with f hypodifferentiable and g continuous and convex with respect to S. Then there exist $\theta \in K^{+i}$ and $s \in$*

$\partial(\theta f)(a)$ *such that:*

$$-(s, s(a)) \in \mathrm{cl}\left[\bigcup_{\lambda \in S^+} \mathrm{epi}\,(\lambda g)^*\right].$$

Proof. In this case proper minimality of a implies infeasibility of the system

$$\begin{cases} f(x) - f(a) & \in & -\mathrm{int}\,K' \\ g(x) & \in & -S \end{cases}$$

for some convex cone K' such that $K\backslash\{0\} \subseteq \mathrm{int}\,K'$. If we set $p(v) = F^{\cup}(v, 1)$, then $\phi(x) = f(a) + p(x - a)$ is an upper convex approximation of f at a. Thus we deduce impossibility of

$$\begin{cases} \phi(x) - \phi(a) & \in & -\mathrm{int}\,K' \\ \psi(x) & \in & -S \end{cases}$$

and this means that a is a properly minimal solution of the convex problem of minimizing the mapping ϕ in the feasible region $A = \{g(x) \in -S\}$. This is true if and only if there exists $\theta \in K^{+i}$ such that a is the optimal solution over A for the scalar valued objective function $\theta\phi(x)$; since the set A is convex, using standard results in Convex Analysis, this holds exactly when there exists $s \in \partial(\theta\phi)(a)$, such that $s(a) \leq s(x), \quad \forall x \in A$; this is equivalent to the implication that:

$$(\forall \lambda \in S^+)\;\; \lambda g(x) \leq 0 \Rightarrow s(x) \geq s(a)$$

or equivalently, to the inconsistency of the system:

$$(\forall \lambda \in S^+)\;\; \lambda g(x) \leq 0, \quad s(x) - s(a) < 0.$$

Letting $h(x) = s(x) - s(a)$ and applying Lemma 3.5, with $T = \mathbb{R}^+$, to the the previous system, we obtain:

$$0 \in \mathrm{epi}\,h^* + \mathrm{cl} \bigcup_{\lambda \in S^+} \mathrm{epi}\,(\lambda g)^*,$$

which gives the desired result, recalling that $\mathrm{epi}\,h^*$ in this case is the set $(s,\, s(a)) + (0, \mathbb{R}^+)$. $\square$

The closure operation appearing in the dual condition shows that optimality is characterized in an asymptotic way; this allows to obtain a nonzero multiplier for the objective function without requiring any regularity conditions on the mapping g, such as Slater or Robinson constraints qualification, which can often fail both in the finite and infinite dimensional case. It can be seen in [10] that, when regularity of the constraints is verified, our conditions include the "qualified" cases as particular ones. In this case optimality can be described in a more familiar way by means of subdifferentials of the scalarized functions instead of conjugate functions.

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