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WELL-POSEDNESS AND CONDITIONING OF OPTIMIZATION PROBLEMS

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For a given abstract optimization problem in a Banach space subject to data perturbations, conditions linking well-posedness to well-conditioning are obtained. Explicit estimates of the modulus of well-posedness allow to bound the condition number. Application to mathematical programming problems are presented.

Keywords: well-posed optimization problems, conditioning, stability analysis in optimization.

AMS subject classification: 49K40, 90C31

1 Introduction

An optimization problem is well-posed by perturbations if its unique solution attracts all approximate solutions corresponding to small perturbations of the given problem. This notion (firstly introduced in [1]) is relevant to the stability analysis of problems of the calculus of variations [1], optimal control [2] and mathematical programming.

A further property of optimization problems is that of conditioning, which is relevant to sensitivity analysis and the performance of numerical methods, see e.g [3]. In this paper we link the two notions in an abstract setting, obtaining conditions of both qualitative and quantitative nature which allows us to check well-conditioning from well-posedness and conversely.

Several results are known obtaining well-conditioning in mathematical programming problems from constraint qualification properties and some form of second-order optimality conditions (see Section 6). However, as far as we know, no result connecting explicitly well-posedness with well-conditioning in a general setting is available.

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In [4], [5] and [6] conditioning is meant in a non-technical fashion (except for strongly convex functions in [4]). The main emphasis in [4] and [5] is on stability estimates for the minimizers and the optimal value function by using the epi-distance.

In Section 4 we link well-conditioning to well-posedness by exploiting the modulus of well-posedness. In Section 5 we show how estimates of the epi-distance can be used to obtain well-conditioning or well-posedness. In Section 6 and 7 we consider such properties, in a global or local setting, for mathematical programming problems with data perturbations. We obtain estimates of the modulus of well-posedness by perturbations starting from the modulus of Tikhonov well-posedness, and then we estimate the condition number.

2 Definitions and notations

Throughout the paper we consider real Banach spaces X and P, a given point $p^* \in P$ and a closed ball L in P of center p^* and positive radius. We are given extended real-valued proper functions

$$f : X \to (-\infty, +\infty], F : X \times L \to (-\infty, +\infty]$$

such that $F(x, p^*) = f(x), x \in X$. Let

$$V(p) = \inf\{F(x, p) : x \in X\}, p \in L.$$

The global optimization problem (X, f), to minimize $f(x)$ subject to $x \in X$, is called *well-posed by perturbations* with respect to the embedding F, or *well-posed* for short, if the following hold:

(1) there exists a unique minimizer $u^* = \text{argmin}(X, f)$;

(2) the value function $V(p)$ is finite for every $p \in L$;

(3) for every sequences $p_n \in P, x_n \in X$ such that $p_n \to p^*$ and $F(x_n, p_n) - V(p_n) \to 0$ one has $x_n \to u^*$.

Sequences x_n as in (3) are called *asymptotically minimizing* corresponding to p_n. This definition was introduced in [1]; see also [7] for a partial survey and [8, 9, 10, 11, 12] for characterizations, extensions to problems (X, f) without uniqueness and applications.

In the following we shall write argmin(p) instead of argmin$[X, F(\cdot, p)]$, and problem (p) to denote the global optimization problem $[X, F(\cdot, p)]$

Problem (X, f) will be called *well-conditioned* with respect to the embedding F if (1) is true and the following hold:

(4) argmin$(p) \neq \emptyset$ for each $p \in L$;

(5) there exists a constant $c > 0$ such that for every $p \in L$ and $m(p) \in \text{argmin}(p)$ we have

$$\limsup_{p \to p^*} ([\|m(p) - m(p^*)\| / \|p - p^*\|) \leq c.$$

The infimum of those c such that (5) holds is called the *condition number* of problem (p^*). The above definition is standard, however is not the only one possible since uniqueness of the minimizer of problem (p) is not required.

A real-valued function

$$\alpha : [0, +\infty) \times L \to R$$

will be called *forcing* if

$$\text{for every sequences } p_n \to p^*, t_n \geq 0 \text{ such that } \limsup \alpha(t_n, p_n) \leq 0 \text{ we have } t_n \to 0. \tag{6}$$

The above definition is an extension of the one used in [9], since the conditions $\alpha(t, p) \geq 0$ and $\alpha(0, p^*) = 0$ are not required here (see also [9, remark 3.4,]), nor $\alpha(t, \cdot)$ is required to depend on $\|p - p^*\|$ only. $\mathrm{ind}(A, x)$ denotes the indicator function of the set A at x, i.e $= 0$ if $x \in A$ and $= +\infty$ elsewhere.

3 Examples

In general, well-posedness and well-conditioning are quite independent properties, as the following examples (on the real line) show.

Example 3.1 Let $F(x, p) = x^4/4 - px, p^* = 0$. Here $u^* = 0, V(p) = -3p\sqrt[3]{p}/4$ and problem (0) is well-posed and ill-conditioned, since $m(p) = \sqrt[3]{p}$.

Example 3.2 Let $F(x, p) = xe^{-px^2}, p^* = 1, x \geq 0$. Then $m(p) = 0$ for every p and problem (1) is well- conditioned. However $V(p) = 0$ and $x_n = n$ is a minimizing sequence for problem (1), whence (Tikhonov) ill-posedness.

In the final example we consider linear perturbations of the Vajnberg example [1]3[ex. 18 p. 8]13 of a Tikhonov ill-posed problem with a unique minimizer.

Example 3.3 Let X be an infinite-dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and a countable orthonormal basis e_n. Consider

$$F(x, p) = \sum_{n=1}^{\infty} \frac{\langle x, e_n \rangle^2}{n^2} - \langle p, x \rangle, \ \|x\| \leq 1, \ p \in X = P, \ p^* = 0.$$

For every $p, F(\cdot, p)$ is strictly convex, continuous, Fréchet differentiable on $\|x\| < 1$, and there exists a unique $m(p) = \mathrm{argmin}(p)$, $p \in X$. Denote by $DF(\cdot, p)$ the Fréchet gradient of $F(\cdot, p)$ with respect to the first variable. If $\|m(p)\| < 1$ then $DF(m(p), p) = 0$ yielding

$$\langle p, e_n \rangle = 2 \frac{\langle m(p), e_n \rangle}{n^2}$$

hence

$$\sum_{n=1}^{\infty} n^4 \langle p, e_n \rangle^2 < +\infty.$$

This condition is violated by each

$$p_k = (1/k) \sum_{n=1}^{\infty} e_n/n^2, \ k = 1, 2, 3, \ldots.$$

It follows that $\|m(p_k)\| = 1$, hence $p_k \to 0$ and $m(p_k) \not\to u^*$, whence ill-posedness of problem (0). Ill-conditioning follows because $\|m(p_k)\|/\|p_k\| \to +\infty$.

4 An approach by the modulus of well-posedness

A characterization of well-posedness by perturbations, obtained in [9], is based on suitable estimates from below of $F(x, p) - V(p)$ as $p \to p^*$. By using such estimates, we show in this section that one can link well-posedness to well-conditioning. So we need an extension of the well-posedness criterion [9, th. 3.2] in order to allow more flexible estimates using forcing functions as defined in Section 2.

Theorem 4.1 *If problem (p^*) is well-posed, then there exist $u^* \in X$ and a forcing function α such that*

(7) $$F(x, p) \geq V(p) + \alpha(\|x - u^*\|, p) \text{ for every } x \in X \text{ and } p \in L.$$

Conversely, let V be finite on L and $F(\cdot, p^*)$ be lower semicontinuous at $u^* \in X$. If (7) holds with a forcing function α, then problem (p^*) is well-posed.

Proof. Let problem (p^*) be well-posed with solution u^*. Consider

$$\alpha(t, p) = \inf\{F(x, p) - V(p) : \|x - u^*\| = t\}, \quad p \in L.$$

Let t_n, p_n be as in (6). Then there exists a sequence $x_n \in X$ such that

$$\|x_n - u^*\| = t_n, 0 \leq \alpha(t_n, p_n) \leq F(x_n, p_n) - V(p_n) \leq \alpha(t_n, p_n) + 1/n$$

hence x_n is an asymptotically minimizing sequence corresponding to p_n. Well-posedness yields $x_n \to u^*$, whence $t_n \to 0$, thus α is forcing. Conversely, assume (7). Let $p_n \to p^*$ and x_n be asymptotically minimizing corresponding to p_n. Then by (7)

$$\limsup \alpha(\|x_n - u^*\|, p_n) \leq 0$$

hence $x_n \to u^*$ and by semicontinuity

$$V(p^*) = \liminf F(x_n, p^*) \geq F(u^*, p^*),$$

as required. □

The best estimate in (7) is obtained making use of the *modulus of well-posedness*

$$\beta(t, s) = \inf\{F(x, p) - V(p) : x \in X, \ p \in L, \ \|x - x^*\| = t, \ \|p - p^*\| = s\}$$

and

$$\alpha(t, p) = \beta(t, \|p - p^*\|)$$

(compare with the modulus of Tikhonov well-posedness, [13, p. 7]). Often, in a given optimization problem, the modulus is quite difficult to obtain, and we must rely on estimates making use of a suitable forcing function, as shown in the sequel. Given a forcing function α, consider

$$\omega(p) = \sup\{t \geq 0 : \alpha(t,p) \leq 0\}, \quad p \in L. \tag{8}$$

Since α is forcing, ω is finite for p sufficiently near to p^* (otherwise we could find sequences $p_n \to p^*, t_n \to +\infty$ such that $\limsup \alpha(t_n, p_n) \leq 0$.)

Theorem 4.2 *Problem* (p^*) *is well-conditioned if* $\operatorname{argmin}(p) \neq \emptyset$ *for every* $p \in L$ *and the following conditions hold:*

(9) *there exist a forcing function* α *fulfilling* (7) *and a constant* $K > 0$ *such that*
$$\omega(p) \leq K\|p - p^*\|, \quad p \in L,$$

where ω *is defined by* (8).

Proof. Let $m(p) \in \operatorname{argmin}(p), p \in L$. Then by (7), $\alpha(\|m(p) - u^*\|, p) \leq 0$ hence by (9) $\|m(p) - u^*\| \leq \omega(p) \leq K\|p - p^*\|$ so that well-conditioning follows (with condition number $\leq K$). $\square$

Example 4.1 uniformly convex functions under linear perturbations. Let X be reflexive and f be continuous and uniformly convex with modulus φ of uniform convexity, i.e.

$$f[tx + (1-t)y] \leq tf(x) + (1-t)f(y) - t(1-t)\varphi(\|x - y\|) \tag{10}$$

for every x, y and $t \in (0,1)$, with $\varphi \geq 0$ a given forcing function (as defined in [13, p. 5]). Without restriction we assume that φ is continuous and increasing ([13, p. 10]). Let $P = X^*$ be the dual space of X and fix a ball $L \subset X^*$ with center $0 = p^*$. Let

$$F(x,p) = f(x) - \langle p, x\rangle, \quad x \in X, \quad p \in X^*.$$

Since $F(\cdot, p)$ is again uniformly convex and continuous, there exists a global minimizer $m(p) = \operatorname{argmin}(p), p \in L$, see [14].

Proposition 4.1 *Problem* (0) *is well-posed if*

$$\liminf_{t \to +\infty} \varphi(t)/t \text{ is a positive real number or} + \infty. \tag{11}$$

Problem (0) *is well-conditioned if* (11) *holds and*

(12) *there exists* $K > 0$ *such that*
$$\sup\{t \geq 0 : \varphi(t) \leq ts\} \leq Ks \text{ for sufficiently small } s > 0.$$

Proof. From (10) with $u^* = m(0)$

$$f(x) \geq f(u^*) + (1-t)\varphi(\|x - u^*\|), x \in X, 0 < t < 1,$$

hence

$$f(x) \geq f(u^*) + \varphi(\|x - u^*\|) \tag{13}$$

which means that problem (p^*) is Tikhonov well-posed. By [15, th.3.8], we get

$$f(u^*) \geq f[m(p)] + \langle p, u^* - m(p)\rangle + \varphi(\|u^* - m(p)\|)$$

since $p \in \partial f[m(p)]$. Hence by (13)

$$F(x,p) \geq V(p) + \langle p, u^* - x\rangle + \varphi(\|x - u^*\|) \geq V(p) + \alpha(\|x - u^*\|, p) \tag{14}$$

where

$$\alpha(t,p) = \varphi(t) - t\|p\|, \quad t \geq 0, \quad p \in L, \tag{15}$$

and α is forcing by (11). Indeed, let $p_n \to 0$ and $t_n \geq 0$ be such that $\limsup(\varphi(t_n) - t_n\|p_n\|) \leq 0$. Then t_n is bounded, otherwise for a subsequence $t_n \to +\infty$ and $\limsup \varphi(t_n)/t_n \leq 0$, against (11). Well-posedness then follows from Theorem 4.1. Well-conditioning follows from Theorem 4.2. □

The particular case of a strongly convex function f has $\varphi(t) \geq \theta t^2$ for some $\theta > 0$. Then Proposition 4.1 applies, yielding a condition number $c \leq 1/\theta$. A better estimate of the condition number, making specific use of strong convexity, yields $c \leq 1/2\theta$ (as well known, see e.g [4, prop.5.7]).

The approach followed in this example is based on obtaining a forcing function α (fulfilling (14)) based on an estimate of the modulus of Tikhonov well-posedness of problem (p^*). This approach will be extended in Section 6 to mathematical programming problems.

A further link between well-posedness and well-conditioning can be obtained as follows. Suppose that $u^* = \text{argmin}(p^*)$,

$$\text{argmin}(p) \neq \emptyset, \quad p \in L, \tag{16}$$

and consider, for any selection $m(p) \in \text{argmin}(p), p \in L$,

$$c(m) = \limsup_{p \to p^*} \|m(p) - m(p^*)\|/\|p - p^*\|;$$

$$k(m,t,p) = \inf\{F(x,p) - V(p) : x \in X,\ \|x - m(p)\| = t\},\ t \geq 0,\ p \in L.$$

The condition $c(m) < +\infty$ can be viewed as a weak form of well-conditioning of problem (p^*). Well-conditioning as defined in Section 2 by (5) means that $\sup\{c(m) : m(p) \in \text{argmin}(p) \text{ for every } p \in L\} < +\infty$.

Theorem 4.3 *Let* (16) *hold. If $k(m,\cdot,\cdot)$ is forcing and $c(m) < +\infty$ for some selection m, then problem (p^*) is well-posed. Conversely, if problem (p^*) is well-posed and $c(m) < +\infty$ for some m, then $k(m,\cdot,\cdot)$ is forcing.*

Proof. Let $p_n \to p^*$ and x_n be asymptotically minimizing coresponding to p_n. Then $k(m, p_n, \|x_n - m(p_n)\|) \to 0$ hence $\|x_n - m(p_n)\| \to 0$. Since $c(m) < +\infty$ we have $m(p_n) \to u^*$ yielding $x_n \to u^*$, whence well-posedness. Conversely , let $t_n \geq 0, p_n \to p^*, k(m,t_n,p_n) \to 0$ and $m(p) \in$ arg min $(p), p \in L$. Let $x_n \in X$ be such that $\|x_n - m(p_n)\| = t_n$ and

$$F(x_n,p_n) \leq V(p_n) + k(m,t_n,p_n) + 1/n.$$

Then x_n is asymptotically minimizing corresponding to p_n, hence $x_n \to u^*$ by well-posedness. Moreover $\|m(p_n) - u^*\| \leq$ (constant) $\|p_n - p^*\| \to 0$ hence $t_n \to 0$ as required. □

A sufficient condition to both well-posedness and well-conditioning, making use of the approximate solutions to problem (p), is obtained as follows. Let $V(p)$ be finite, and write

$$\epsilon - \text{argmin}(p) = \{u \in X : F(u,p) \leq V(p) + \epsilon\},$$

$$\gamma(\epsilon, p) = \sup\{\|u^* - x\|/\|p - p^*\| : p \in P,\ p \neq p^*,\ x \in \epsilon - \text{argmin}(p)\}.$$

Proposition 4.2 *If* $\limsup_{(\epsilon,p)\to(0,p^*)} \gamma(\epsilon, p) < +\infty$ *and problem* (p^*) *is Tikhonov well-posed, then problem* (p^*) *is well-posed and well-conditioned.*

Proof. For every selection $m(p) \in \text{argmin}(p), p \in L$ we have

$$\|m(p) - u^*\|/\|p - p^*\| \leq \gamma(\epsilon, p), \quad \epsilon > 0,$$

whence well-conditioning. Now let $p_n \to p^*$ and x_n be asymptotically minimizing corresponding to p_n. Then there exists a positive sequence $\epsilon_n \to 0$ such that $x_n \in \epsilon_n - \text{argmin}(p_n)$. Let $p_n \neq p^*$ for every sufficiently large n. Then

$$\|u^* - x_n\| \leq \text{ (constant) } \|p_n - p^*\|,$$

hence $x_n \to u^*$. If otherwise $p_n = p^*$ for infinitely many n, a subsequence fulfills $x_n \in \epsilon_n - \text{argmin}(p^*)$ yielding $x_n \to u^*$ by Tikhonov well-posedness. The previous argument shows that the original sequence x_n converges toward u^*, yielding well-posedness. □

Example 4.2 The assumption of Proposition 4.2 is not necessary for well-posedness. Let $F(x, p) = |lnx| + p|x - 1|, p \geq 0 = p^*$. Then $m(p) = 1$ for every p, hence well-conditioning. Moreover problem (p^*) is well-posed, however

$$\epsilon - \text{argmin}(p) = [e^{-\epsilon}, e^{\epsilon}],$$

and $\gamma(\epsilon, p) \geq e^{-\epsilon}(e^{\epsilon} - 1)/p$.

5 An approach by epidistance

Here we link Tikhonov well-posedness of problem (p^*) with well-conditioning under any perturbation which is Lipschitz stable at p^* with respect to the epigraphical distance. We make use of the stability results of Lipschitz type in [4]. Following concepts introduced in [16] (see also [4]) , for a given $\rho > 0$ denote by $d_\rho(g, h)$ the ρ-epi-distance between two extended real-valued functions g, h defined on X. We shall consider Tikhonov well-posed problems (X, f) with an associate forcing function $\alpha = \alpha(t) \geq 0$ (independent of p). Consider

$$\alpha^*(t) = \inf\{\alpha(s) + |t - s| : s \geq 0\}, \quad t \geq 0.$$

Lemma 5.1 *If* α *is forcing, then* α^* *is.*

Proof. Let $t_n \geq 0$ be such that $\alpha^*(t_n) \to 0$. Then for some sequences $s_n \geq 0$ we have $\alpha(s_n) \leq 1/n + \alpha^*(t_n) \to 0$ hence $s_n \to 0$, moreover $|t_n - s_n| \leq 1/n + \alpha^*(t_n) \to 0$, whence $t_n \to 0$. □

If problem (p^*) is Tikhonov well-posed, then there exists a forcing function $\alpha \geq 0$ such that

$$f(x) \geq f(u^*) + \alpha(\|x - u^*\|), \quad x \in X \tag{17}$$

([13, th. 12 p. 6]).

Theorem 5.1 *Let problem* (p^*) *be Tikhonov well-posed with* α *fulfilling* (17) *and* $\mathrm{argmin}(p) \neq \emptyset$, $p \in L$. *Then problem* (p^*) *is well-conditioned provided the following hold:*

(18) *there exists* $A > 0$ *such that*
$$\sup\{t \geq 0 : \alpha^*(t) \leq s\} \leq As, \quad s > 0;$$

(19) $\mathrm{argmin}(p)$ *and* $V(p)$ *are uniformly bounded on* L;

(20) *for every sufficiently large* $\rho > 0$ *there exists* $K > 0$ *such that*

$$d_\rho[f, F(\cdot, p)] \leq K\|p - p^*\|, \quad p \in L.$$

Proof. By (19), it follows from [4, th.3.8] that

$$\alpha^*(\|u^* - m(p)\|) \leq 4d_\rho[Tf, TF(\cdot, p)]$$

for every sufficiently large ρ; here

$$(Tg)(x) = g(x + u^*) - f(u^*).$$

Then by (18)

$$\|u^* - m(p)\| \leq 4Ad_\rho[Tf, TF(\cdot, p)]. \tag{21}$$

Elementary computations show that

$$d_\rho(Tf, TF(\cdot, p)] \leq d_\sigma[f, F(\cdot, p)]$$

where $\sigma = \rho + \|u^*\| + |f(u^*)|$. Then by (21) $\|u^* - m(p)\| \leq 4AK\|p - p^*\|$ yielding well-conditioning. □

Well-posedness in the form of convergence of bounded asymptotically minimizing sequences requires weaker conditions than those of Theorem 5.1, as follows.

Theorem 5.2 *Let* $p_n \to p^*$ *and* x_n *be an asymptotically minimizing sequence corresponding to* p_n *such that* x_n *and* $F(p_n, x_n)$ *are bounded. Then* $x_n \to u^*$ *provided that the following hold:*

(22) *problem* (p^*) *is Tikhonov well-posed;*

(23) $d_\rho[f, F(\cdot, p)] \to 0$ *as* $p \to p^*$ *for every* ρ *sufficiently large .*

Proof. Denote by $\mathrm{epi}h$ the epigraph of $h : X \to R \cup \{+\infty\}$. By (23), there exists a sequence $u_n \in X$ such that

$$\|x_n - u_n\| + |F(x_n, p_n) - f(u_n)| \to 0. \tag{24}$$

Then

$$V(p^*) \leq f(u_n) - F(x_n, p_n) + F(x_n, p_n) - V(p_n) + V(p_n)$$

yielding

$$V(p^*) \leq \liminf V(p_n). \tag{25}$$

For sufficiently large $\rho > 0$ one has $\mathrm{dist}[(u^*, V(p^*)), \mathrm{epi}F(\cdot, p_n)] \leq e[(epif)_\rho, epiF(\cdot, p_n)]$, where $\mathrm{dist}[(x, a), (y, b)] = \max\{\|x - y\|, |a - b|\}$ for x, y in X, a, b in R, e denotes the Hausdorff excess within the normed space $X \times R$, and $(epi)_\rho$ denotes the intersection of epi$f$ with the closed ball of center 0 and radius ρ. By (23) there exists a sequence $y_n \in X$ such that

$$\|y_n - u^*\| + |V(p_n) - F(y_n, p_n)| \to 0.$$

Then, remembering (25), we see that $V(p_n) \to V(p^*)$, hence $f(u_n) \to V(p^*)$. Tikhonov well-posedness yields $u_n \to u^*$, and by (24) we get $x_n \to u^*$. □

6 Application to mathematical programming

Stability properties and well-conditioned behavior of mathematical programming problems are of the utmost importance for theoretical and practical reasons, as well known. In this section we take X a real Hilbert space and consider global optimization problems with more specific structure than previously treated. In addition to p^* and the unperturbed objective function f, we are given a multifunction

$$G : L \longrightarrow X$$

with nonempty values, modeling the perturbations acting on the feasible set $G(p^*)$. Then we take

$$F(x, p) = f(x) + \text{ ind } [G(p), x].$$

Consider the excess

$$e[G(p), G(p^*)] = \sup\{\text{dist}[z, G(p^*)] : z \in G(p)\}, \quad p \in L. \tag{26}$$

In the next result we extend the approach of Example 4.1, Section 4. We obtain explicitly a forcing function α as in (7) in terms of an estimate of the modulus of Tikhonov well-posedness of problem p^*, of the excess defined by (26) and the value function. From such explicit estimates, sufficient conditions for well-conditioning can be derived. The modulus of Tikhonov well-posedness of problem (p^*) (see [13, p.7]), given by

$$\beta(t) = \inf\{f(x) - f(u^*) : x \in G(p^*), \|x - u^*\| = t\}, t \geq 0,$$

is called *superquadratic* if there exists $Q > 0$ such that

$$\beta(t) \geq Qt^2, \quad t \geq 0 \text{ sufficiently small .} \tag{27}$$

Condition (27) is sometimes referred to as the growth condition of order 2.

Among the several results dealing with Lipschitz stability of perturbed minimizers, hence well-conditioning, obtained as a consequence of constraint qualification and second-order optimality conditions , we mention [17, 18, 19, 20, 21, 22, 23, 24, 25]. The approach presented in this section is however different, and independent of such conditions (see also [26]).

Theorem 6.1 *Problem (p^*) is wellposed if $G(p^*)$ is closed and the following conditions hold:*

$$f \textit{ is Lipschitz on } G(L); \tag{28}$$

$$V \textit{ is upper semicontinuous at } p^*; \tag{29}$$

(30) $$\varphi(p) = e[G(p), G(p^*)] \to 0 \text{ as } p \to p^*;$$

(31) *problem* (p^*) *is Tikhonov well-posed with a superquadratic modulus.*

In such a case,

(32) $$\alpha(t, p) = Qt^2 - (2Qt + H)(\varphi(p) + \|p - p^*\|) + V(p^*) - V(p)$$

is a forcing function verifying (7)*, where* Q *is given by* (27) *and* H *is a Lipschitz constant of* f*.*

Proof. For every $p \neq p^*$ and $x \in G(p)$ there exists $y \in G(p^*)$ such that

$$\|x - y\| \leq \varphi(p) + \|p - p^*\| = \overline{\varphi}(p) \text{ say }.$$

Then by (28) and (31) we get

$$f(x) \geq f(y) - H\|x - y\| \geq V(p^*) + Q\|y - u^*\| - H\|x - y\| \geq$$

$$\geq V(p^*) + Q\|x - u^*\|^2 - \overline{\varphi}(p)(2Q\|x - u^*\| + H).$$

Hence $V(p)$ is finite, $p \in L$, and α given by (32) fulfills (7). The proof will be ended, by Theorem 4.1, showing that α is forcing. Let $p_n \to p^*, t_n \geq 0$ be such that

(33) $$\limsup \alpha(p_n, t_n) \leq 0.$$

If for some subsequence $t_n \to +\infty$, then for every $\epsilon > 0$ we obtain

$$Qt_n \leq 2Q\overline{\varphi}(p_n) + [V(p_n) - V(p^*)]/t_n + \epsilon$$

for every sufficiently large n. However by (30) this contradicts (29). It follows that t_n is bounded. For a subsequence we have $t_n \to T, T \geq 0$. Then by (33) we get $T = 0$ because of (29), (30) and this shows that α is forcing. □

Remark 6.1 *If more realistic estimates are available such that*

(34) $$\delta(p) \leq V(p^*) - V(p), \varphi_1(p) > \varphi(p) \text{ if } p \neq p^*$$

$$\text{and } \liminf \delta(p) \geq 0, \varphi_1(p) \to 0 \text{ as } p \to p^*,$$

then

(35) $$\alpha_1(t, p) = Qt^2 - (2Qt + H)\varphi_1(p) + \delta(p)$$

is still a forcing function verifying (7)*.*

A sufficient condition for upper semicontinuity (29) (equivalent to continuity under the assumptions of Theorem 6.1) can be obtained making use of the gap

$$\theta(p, \epsilon) = \inf\{\|y - z\| : y \in \epsilon - \text{argmin}(p^*),\ z \in G(p)\},\ \epsilon > 0,\ p \in L.$$

Theorem 6.2 *Let* f *be Lipschitz on* $G(L)$*. Then* V *is upper semicontinuous at* p^* *provided*

(36) $$\theta(p, \epsilon) \to 0 \text{ as } p \to p^* \text{ for every sufficiently small } \epsilon > 0.$$

Proof. For every $z \in F(p)$ and $y \in \epsilon - \text{argmin}(p^*)$ we get, for some constant H

$$V(p^*) + \epsilon \geq f(y) \geq f(z) - H\|y - z\| \geq V(p) - H\|y - z\|.$$

By taking the infimum with respect to y and z we obtain

$$V(p) \leq H\theta(p,\epsilon) + \epsilon + V(p^*).$$

The conclusion comes from (36). $\square$

Example 6.1 Condition (36) with $\epsilon = 0$ only does not imply upper semicontinuity of V. Consider $N = 2, 0 < p < 1, p^* = 1/2$,

$$G(p) = \{(p, x_2) \in R^2 : p^2 + (x_2 - 1)^2 \leq 1 \text{ or } 4p^2 + 4(x_2+1)^2 \leq 1\}, f(x_1, x_2) = x_2.$$

Remark 6.2 If (36) holds and $\epsilon - \text{argmin}(p)$ is bounded, $\epsilon > 0$, then the conclusion of Theorem 6.1 obtains assuming f Lipschitz continuous on bounded sets. Indeed, in the proof we can assume $\theta(p,\epsilon)$ bounded for sufficiently small $\|p-p^*\|$, and $\|z-y\| \leq q+\theta(p,\epsilon)$ for any prescribed $q > 0$. Then $\|z-u^*\| \leq V(p^*)+\epsilon+h[q+\theta(p,\epsilon)]$, hence the conclusion.

Theorem 6.3 *Problem* (p^*) *is well-posed and well-conditioned if the assumptions of Theorem 6.1 hold,* δ *and* φ_1 *are as in* (34)*, and there exist constants* Q_1, Q_2 *such that*

$$\varphi_1(p) \leq Q_1\|p\|, \qquad \delta(p) < H\varphi_1(p) \leq \delta(p) + Q_2\|p\|^2.$$

The elementary proof is based on applying Theorem 4.2 with α_1 given by (35).

Remark 6.3 Let $g_1, ..., g_M$ be given real-valued functions on $X = R^N$, let $P = R^M, p^* = 0$ and let $x \in G(p)$ iff

$$g_1(x) \leq p_1, \cdots, g_M(x) \leq p_M.$$

This model encompasses the standard mathematical programming problem with data perturbations, see [27, p.33-34], (and well-posedness is invariant under the corresponding transformation allowing us to consider only constraint perturbations). Then (27) holds provided $f, g_1, \cdots, g_M$ are smooth and the weak version [27, p.29] of the second order sufficient conditions are fulfilled at u^* (compare [28, th.5.2], [29] and the previously listed references for more general results). However, Theorems 6.1 and 6.3 can be applied without requiring second-order conditions or smooth data.

Example 6.2 Let $X = R^2, f(x_1, x_2) = x_1$, and $x \in G(p)$ iff

$$-x_1^3 \leq x_2 \leq x_1^3 + p^2 e^{-px_1},$$

$p \geq 0 = p^*$. Here problem (p^*) does not fulfill the Mangasarian-Fromovitz constraint qualification condition, and the second order sufficient conditions fail. Theorems 6.1 and 6.3 apply with $Q = 1, \delta(p) = 0, \varphi_1(p) = 2p^2$. Problem (p^*) is both well-posed and well-conditioned.

Example 6.3 Let $X = R^2, f(x_1, x_2) = x_1$, and $x \in G(p)$ iff

$$-\sqrt{x_1} - p \leq x_2 \leq \sqrt{x_1} + p,$$

$0 \leq x_1 \leq 1, p \geq 0 = p^*$. Problem's data are not smooth. Theorem 6.1 applies with $Q = 1/2, \delta(p) = 0, \varphi_1(p) = p$. Problem (p^*) is well-conditioned since for every $m(p) \in \text{argmin}(p)$ we have $|m(p) - u^*| \leq p$ (however Theorem 6.3 is not applicable).

7 Local well-posedness

The notion of local solution of mathematical programming problems is often more significant than the global one. Accordingly, a definition of local well-posedness, similar to the one of [10], is appropriate in such a context. We limit ourselves to the finite-dimensional framework of Section 6 with $X = R^N$. Problem (p^*) will be called *locally Tikhonov well-posed* with local solution u^* if $u^* \in G(p^*)$, there exists a closed ball B in X centered at u^* of positive radius such that $f(u^*) = \inf f[B \cap G(p^*)]$, and $x_n \to u^*$ for every sequence $x_n \in B \cap G(p^*)$ verifying $f(x_n) \to \inf f[B \cap G(p^*)]$. The point u^* is a *strict local minimizer* of problem (p^*) if $u^* \in G(p^*)$ and there exists a closed ball B centered at u^* such that

$$f(y) > f(u^*) \text{ for every } y \neq u^* \text{ and } y \in G(p^*) \cap B.$$

Proposition 7.1 *Let $G(p^*)$ be closed and f be lower semicontinuous. Then problem (p^*) is locally Tikhonov well-posed with solution u^* iff u^* is a strict local minimizer.*

The proof is trivial (owing to compactness of $G(p^*) \cap B$ and semicontinuity).

Problem (p^*) is *locally well-posed* by perturbations with solution u^* if $u^* \in G(p^*)$ and there exists a ball B centered at u^*, with positive radius, such that

$$u^* \text{ is the unique global minimizer of } f \text{ on } G(p^*) \cap B;$$

$$V(p) = \inf\{f(x) : x \in G(p) \cap B\} \text{ is finite, } p \in L;$$

$$p_n \to p^* \text{ in } P \text{ and } x_n \in G(p_n) \cap B \text{ fulfilling } f(x_n) - V(p_n) \to 0 \text{ imply } x_n \to u^*.$$

This definition is slightly more general than that of [10] (uniqueness of local minimizers is not required here).

Proposition 7.2 *Let f be continuous and G be continuous at p^* with closed values. If f has a strict local minimizer on $G(p^*)$, then problem (p^*) is locally well-posed.*

PROOF. By assumption, G is simultaneously upper and lower semicontinuous at p^*, and $(G(p^*) \cap B, f)$ is Tikhonov well-posed for some compact ball B centered at u^*, [13, th. 23 p.13]. Then the conclusion will follow by checking the assumptions required by [7, prop. 5.1 p. 234]. Let

$$F(x,p) = f(x) + \text{ ind } (G(p) \cap B, x), V(p) = \inf\{F(x,p) : x \in R^N\}.$$

We need to show that

$$F \text{ is lower semicontinuous at } R^N \times \{p^*\}; \tag{37}$$

$$V \text{ is finite on } L \text{ and upper semicontinuous at } \{p^*\}. \tag{38}$$

To prove (37) let $x_n \to x$ in $R^N, p_n \to p^*$. If $x_n \notin G(p_n) \cap B$ for every n sufficiently large, then $\liminf F(x_n, p_n) = +\infty \geq F(x, p^*)$. If $x_n \in G(p_n) \cap B$ for infinitely many n, then for some subsequence y_n of x_n we have $\liminf F(x_n, p_n) = \lim F(y_n, p_n) = \lim f(y_n)$. By compactness, for some further subsequence $y_n \to x \in G(p^*) \cap B$ because of upper semicontinuity. Then $\liminf F(x_n, p_n) \geq f(x) = F(x, p^*)$, proving (37). The (local) value function V fulfills (38) by standard results, [13, prop.2 p. 335]. □

We plan to show elsewhere that Theorems 6.1 and 6.3 may be extended to the local setting again making use of quantitative estimates.

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