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PROBLEMS FOR P-MONGE-AMPÈRE EQUATIONS

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Dedicated to P. Popivanov on occasion of his 65th birthday

Abstract. We consider the homogeneous Dirichlet problem for a class of equations which generalize the p-Laplace equations as well as the Monge-Ampère equations in a strictly convex domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. In the sub-linear case, we study the comparison between quantities involving the solution to this boundary value problem and the corresponding quantities involving the (radial) solution of a problem in a ball having the same η_1-mean radius as Ω. Next, we consider the eigenvalue problem for such a p-Monge-Ampère equation and study a comparison between its principal eigenvalue (eigenfunction) and the principal eigenvalue (eigenfunction) of the corresponding problem in a ball having the same η_1-mean radius as Ω. Symmetrization techniques and comparison principles are the main tools used to get our results.

1. Introduction. In the paper [9], G. Talenti established sharp a priori estimates of quantities involving solutions to boundary value problems of second order elliptic linear PDE's via Schwarz symmetrization. In the subsequent paper

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[10], he introduced a suitable symmetrization to get a priori estimates of quantities involving solutions to boundary value problems of a Monge-Ampère equation in dimension 2.

These papers have inspired the use of similar methods in several investigations involving both linear and non-linear elliptic problems. In the recent paper [2], B. Brandolini and C. Trombetti extended some results of Talenti to more general Monge-Ampère equations in dimension 2. In the paper [14], K. Tso generalized the symmetrization introduced in [10] for obtaining isoperimetric inequalities of quantities involving solutions to k-Hessian equations in convex domains of arbitrary dimension. The work in [14] provides a direct generalization of the results of Talenti [10] for $n = k = 2$. In the paper [13], N.S. Trudinger extended the results of Tso to the case of $(k-1)$-convex domains. We also refer the reader to [1, 5, 6, 11] and references therein for works that use symmetrization methods to study sharp a priori estimates of solutions to elliptic Dirichlet problems. Results of existence, uniqueness and regularity of solutions to Hessian equations can be found in [3, 4, 15, 16].

The p-Laplace operator with $p > 1$ generalizes in a natural way the classical Laplace operator. Although it is nonlinear (for $p \neq 2$), many features true for the linear case extend to the p-Laplace case. We refer to the paper of P. Tolksdorf [12] for a theoretical investigation on this operator. The definition of (p, k)-Hessian appears in [4]. This operator, with $p > 1$ and $1 \leq k \leq n$ extends in a natural way both the p-Laplace operator as well as the k-Hessian operator. As far as we know, this operator has not received good attention yet. In the present work we investigate the case $k = n$, that is the (p, n)-Hessian operator, which we prefer to denote as p-Monge-Ampère operator. We employ similar methods as in [2, 10, 14] to get isoperimetric inequalities of various quantities that involve solutions of Dirichlet problems related to a p-Monge-Ampère operator with $p \geq 2$. Since the precise statement of our results needs several definitions and preliminary results, we postpone them to the next section.

In the present paper we extend the case $p = 2$, $k = n$ investigated in the recent work [6], to the case $p \neq 2$, $k = n$. The results that we find give information on the solutions of our problem in a general domain Ω as soon as we have the solutions of a corresponding problem in a suitable symmetric domain.

The paper is organized as follows. In Section 2, we provide notations and basic definitions as well as the statement of our main results. In Section 3, we find various estimates related to our problem in the sublinear case. In Section 4, we investigate isoperimetric inequalities involving the eigenvalues and appropriately normalized eigenfunctions associated with the p-Monge-Ampère operator.

2. Notations and Preliminaries. Throughout this work, we suppose that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded strictly convex domain with a C^2 boundary $\partial\Omega$. Let

$$\Phi(\Omega) := \{u \in C^2(\Omega) \cap C^{0,1}(\overline{\Omega}) : u < 0 \ \text{ and } \ u \text{ convex in } \Omega,\ u = 0 \text{ on } \partial\Omega\}.$$

If $u \in \Phi(\Omega)$, from Sard's theorem it follows that for almost all $t \in (m_0, 0)$, $m_0 = \min u$, the sub-level set

$$\Omega_t = \{x \in \Omega : u(x) < t\}$$

is convex and will have a smooth boundary Σ_t given by the level surface

$$\Sigma_t = \{x \in \Omega : u(x) = t\}.$$

In what follows, p will be a real number with $p \geq 2$. If $u \in \Phi(\Omega)$, at points where $|Du| > 0$ we define the matrix

$$Q(Du) = |Du|^{p-2}\left(I + (p-2)\frac{Du \bigotimes Du}{|Du|^2}\right).$$

Let D^2u be the Hessian matrix of u. For $0 \leq q < (p-1)n$, we consider the Dirichlet problem

$$\det[Q(Du)D^2u] = (-u)^q, \quad u < 0 \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega. \tag{1}$$

For $p = 2$, we have $\det[Q(Du)D^2u] = \det[D^2u]$, the usual Monge-Ampère operator. On the other hand, if $p > 1$, the trace of $[Q(Du)D^2u]$ yields the p-Laplace operator. For this reason, we call (1) a p-Monge-Ampère equation. The eigenvalues of the matrix $Q(Du)$ are $|Du|^{p-2}$ (with multiplicity $n-1$) and $(p-1)|Du|^{p-2}$ (with multiplicity 1). It follows that

$$\det[Q(Du)D^2u] = (p-1)|Du|^{(p-2)n}\det[D^2u]. \tag{2}$$

We can also motivate the definition of the p-Monge-Ampère operator as follows. Define the $n \times n$ matrix

$$(|Du|^{p-2}u_{x_i})_{x_j} \equiv |Du|^{p-2}\left(\delta_{i\ell} + (p-2)\frac{u_{x_i}}{|Du|}\frac{u_{x_\ell}}{|Du|}\right)u_{x_\ell x_j},$$

where $\delta_{i\ell}$ is the usual Kronecker symbol. The trace of this matrix is the p-Laplace operator, whereas, its determinant is our p-Monge-Ampère operator. From now on we shall write u_i for u_{x_i}.

It is possible to write this operator in a variational form. Indeed, if $T^{(n-1)}(D^2u)$ is the adjoint matrix of D^2u, that is, if

$$T_{ij}^{(n-1)}(D^2u) = \frac{\partial \det[D^2u]}{\partial u_{ij}},$$

then

$$T_{ij}^{(n-1)}(D^2u)u_{ij} = n \det[D^2u],$$

$$\left(T_{ij}^{(n-1)}(D^2u)\right)_j = 0, \quad i = 1, \ldots, n,$$

and

$$u_{j\ell}T_{ij}^{(n-1)}(D^2u) = \delta_{\ell i} \det[D^2u],$$

where the summation convention over repeated indexes is in effect. For a detailed proof of these equalities we refer to [7, 8].

In view of the equalities in above we have

$$\begin{aligned}
&\frac{1}{n}\left(|Du|^{(p-2)n}T_{ij}^{(n-1)}(D^2u)u_i\right)_j \\
&= (p-2)|Du|^{(p-2)n-2}u_{j\ell}u_\ell T_{ij}^{(n-1)}(D^2u)u_i + \frac{1}{n}|Du|^{(p-2)n}T_{ij}^{(n-1)}(D^2u)u_{ij} \\
&= (p-2)|Du|^{(p-2)n-2}u_iu_i \det[D^2u] + |Du|^{(p-2)n} \det[D^2u] \\
&= (p-1)|Du|^{(p-2)n} \det[D^2u].
\end{aligned}$$

Therefore, recalling (2) we find

$$\begin{aligned}
\det[Q(Du)D^2u] &= (p-1)|Du|^{(p-2)n} \det[D^2u] \\
&= \frac{1}{n}\left(|Du|^{(p-2)n}T_{ij}^{(n-1)}(D^2u)u_i\right)_j,
\end{aligned} \tag{3}$$

that is, our p-Monge-Ampère in a variational form.

Define

$$\begin{aligned}
&S_{p,q}(\Omega) \\
&= \inf\left\{\int_\Omega(-v) \det[Q(Dv)D^2v]dx : \quad v \in \Phi(\Omega), \quad \int_\Omega(-v)^{q+1}dx = 1\right\}.
\end{aligned} \tag{4}$$

Note that we have

$$S_{p,q}(\Omega) = \inf \left\{ \frac{\displaystyle\int_\Omega (-v) \det[Q(Dv)D^2v]dx}{\left(\displaystyle\int_\Omega (-v)^{q+1}dx\right)^{\frac{1+(p-1)n}{q+1}}} : \quad v \in \Phi(\Omega) \right\}.$$

Our first result asserts that a minimizer $v \in \Phi(\Omega)$ of (4) satisfies

$$\det[Q(Dv)D^2v] = S_{p,q}(\Omega)(-v)^q, \quad v < 0 \ \text{ in } \ \Omega, \quad v = 0 \ \text{ on } \ \partial\Omega. \tag{5}$$

It is easy to check that if v satisfies (5) then

$$u = \left(S_{p,q}(\Omega)\right)^{\frac{1}{q-(p-1)n}} v$$

satisfies (1).

To state our next results we need the definition of a suitable rearrangement. Let $\kappa_1, \ldots, \kappa_{n-1}$, be the principal curvatures of $\partial\Omega$. Since Ω is strictly convex, we have $\kappa_i > 0$, $i = 1, \ldots, n$. If $n > 2$, we define the $(n-2)$-th mean curvature of $\partial\Omega$ by

$$\mathcal{H}_{n-2}(\partial\Omega) = S_{n-2}(\kappa_1, \ldots, \kappa_{n-1}),$$

where S_{n-2} denotes the elementary symmetric function of order $n-2$ of $\kappa_1, \ldots, \kappa_{n-1}$. If $n = 2$ we put $\mathcal{H}_0(\partial\Omega) = 1$. We also put $\mathcal{H}_{n-1}(\partial\Omega) = \kappa_1 \cdots \kappa_{n-1}$, the usual total curvature.

The *quermassintegral* $V_1(\Omega)$ is defined by

$$V_1(\Omega) = \frac{1}{n(n-1)} \int_{\partial\Omega} \mathcal{H}_{n-2}(\partial\Omega) d\sigma,$$

where $d\sigma$ denotes the $(n-1)$-dimensional Hausdorff measure.

Following [13] and [14] we define the 1-mean radius of Ω, denoted by $\eta_1(\Omega)$, as

$$\eta_1(\Omega) = \frac{V_1(\Omega)}{\omega_n},$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$. In case Ω is a ball, $\eta_1(\Omega)$ is the usual radius of Ω. For a general Ω we denote with Ω^*_{n-1} the ball with radius $\eta_1(\Omega)$.

The following isoperimetric inequality is well known for convex domains Ω

$$\left(\frac{|\Omega|}{\omega_n}\right)^{\frac{1}{n}} \le \frac{V_1(\Omega)}{\omega_n}. \tag{6}$$

It follows that
$$|\Omega| \le |\Omega^*_{n-1}|.$$
In Section 3 we shall prove that, if $p \ge 2$ and $0 \le q < (p-1)n$ then
$$S_{p,q}(\Omega) \ge S_{p,q}(\Omega^*_{n-1}),$$
where $S_{p,q}$ is defined as in (4).

Recall that if $u \in \Phi(\Omega)$, the sub-level set
$$\Omega_t = \{x \in \Omega : u(x) < t\}$$
is convex. We define the rearrangement of u with respect to the quermassintegral V_1 as
$$u^{\bigstar}_{n-1}(s) = \sup\big\{t \le 0 : V_1(\Omega_t) \le s, \quad 0 \le s \le V_1(\Omega)\big\}.$$
The function $u^{\bigstar}_{n-1}(s)$ is negative, non decreasing, and satisfies $u^{\bigstar}_{n-1}(0) = \min_\Omega u(x)$, $u^{\bigstar}_{n-1}\big(V_1(\Omega)\big) = 0$. We also define
$$u^*_{n-1}(x) = u^{\bigstar}_{n-1}\big(\omega_n|x|\big), \quad 0 \le |x| \le \eta_1(\Omega).$$
The function $u^*_{n-1}(x)$ is called the $(n-1)$-symmetrand of u (see [13]), and can also be defined by
$$u^*_{n-1}(x) = \sup\big\{t \le 0 : \eta_1(\Omega_t) \le |x|, \quad 0 \le |x| \le \eta_1(\Omega)\big\}.$$
Since $u^*_{n-1}(x)$ is radially symmetric we often write $u^*_{n-1}(x) = u^*_{n-1}(r)$ for $|x| = r$. We have $u^*_{n-1}(0) = \min_\Omega u(x)$ and $u^*_{n-1}\big(\eta_1(\Omega)\big) = 0$.

By definition we have
$$\eta_1(\Omega_t) = \eta_1(\{u^*_{n-1}(x) < t\}).$$
By using this equation and (6) for Ω_t, we find
$$\eta_n(\Omega_t) \le \eta_1(\{u^*_{n-1}(x) < t\}) = \eta_n(\{u^*_{n-1}(x) < t\}),$$
where $\eta_n(E)$ is the radius of the ball with measure $|E|$. Recalling that Ω^*_{n-1} is the ball with radius $\eta_1(\Omega)$, it follows that
$$|\{x \in \Omega : u(x) < t\}| \le |\{x \in \Omega^*_{n-1} : u^*_{n-1}(x) < t\}|. \tag{7}$$
By (7) we find, for $\alpha \ge 1$,
$$\|u\|_{L^\alpha(\Omega)} \le \|u^*_{n-1}\|_{L^\alpha(\Omega^*_{n-1})}. \tag{8}$$

Let v be a super-solution of problem (1). We prove the isoperimetric inequality

$$\left(\frac{dv^*(r)}{dr}\right)^{(p-1)n} \leq n\int_0^r t^{n-1}(-v^*(t))^q dt, \quad 0 \leq r \leq \eta_1(\Omega)$$

where $v^*(r) := v^*_{n-1}(x)$ for $r = |x|$, $v^*_{n-1}(x)$ being the $(n-1)$-symmetrand of v. Next, we prove the inequality

$$v^*(r) \geq z(r), \quad 0 \leq r \leq \eta_1(\Omega),$$

where $z(r)$ is a sub-solution of problem (1) in Ω^*_{n-1}.

A related isoperimetric inequality concerns the Hessian integral

$$I(\Omega, u) := \int_\Omega (-u)\det[Q(Du)D^2u]dx.$$

We prove that

$$I(\Omega, v) \leq I(\Omega^*_{n-1}, z),$$

where v is a super-solution of problem (1) in Ω, and $z(r)$ is a sub-solution of problem (1) in Ω^*_{n-1}.

All the previous results will be proved in Section 3 for $p \geq 2$ and $0 \leq q < (p-1)n$. In Section 4, we consider the case $q = (p-1)n$. Let

$$\lambda(\Omega) = \inf\left\{\int_\Omega (-v)\det[Q(Dv)D^2v]dx : \quad v \in \Phi(\Omega), \quad \int_\Omega (-v)^{(p-1)n+1}dx = 1\right\}.$$

We show that a minimizer $u \in \Phi(\Omega)$ satisfies the eigenvalue problem

$$\det[Q(Du)D^2u] = \lambda(\Omega)(-u)^{(p-1)n}, \quad \lambda(\Omega) > 0, \quad u < 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Such a solution is called principal eigenfunction. We shall prove the inequality

$$\lambda(\Omega) \geq \lambda(\Omega^*_{n-1}).$$

Next, we will prove two inequalities involving appropriately normalized eigenfunctions u.

We collect here some non trivial results proved, for example, in [13].

Lemma 1. *Let Ω be convex and $u \in \Phi(\Omega)$. If $m_0 = \inf_\Omega u$, for almost every $t \in (m_0, 0)$ we have*

$$\frac{d}{dt}\int_{\Sigma_t} \mathcal{H}_{n-2}d\sigma = (n-1)\int_{\Sigma_t} \frac{\mathcal{H}_{n-1}}{|Du|}d\sigma, \tag{9}$$

$$T_{ij}^{(n-1)}(D^2u)u_iu_j = |Du|^{n+1}\mathcal{H}_{n-1} \quad on \quad \Sigma_t. \tag{10}$$

Furthermore, if $\zeta := u_{n-1}^*$, *for* $\alpha \geq n+1$ *we have*

$$\int_\Omega T_{ij}^{(n-1)}(D^2u)u_iu_j|Du|^{\alpha-n-1}dx \geq \int_{\Omega_{n-1}^*} T_{ij}^{(n-1)}(D^2\zeta)\zeta_i\zeta_j|D\zeta|^{\alpha-n-1}dx. \tag{11}$$

Proof. For the proof of (9), (10) and (11) see (2.20), (2.28) and (4.1) respectively of [13]. □

3. The case $p \geq 2,\ 0 \leq q < (p-1)n$.

Proposition 2. *Let* $S_{p,q}(\Omega)$ *be defined as in* (4). *If* $u \in \Phi(\Omega)$ *is a minimizer of* $S_{p,q}(\Omega)$ *then* u *satisfies* (5).

Proof. By (4) and (2), we find

$$S_{p,q}(\Omega) = (p-1)\int_\Omega (-u)|Du|^{(p-2)n}\det[D^2u]dx.$$

If $v \in \Phi(\Omega)$ and $t > 0$ we have

$$\begin{aligned} J &= \frac{d}{dt}\int_\Omega (-u-tv)|Du+tDv|^{(p-2)n}\det[D^2u+tD^2v]dx\Big|_{t=0} \\ &= \int_\Omega (-v)|Du|^{(p-2)n}\det[D^2u]dx \\ &+ (p-2)n\int_\Omega (-u)|Du|^{(p-2)n-2}u_\ell v_\ell \det[D^2u]dx \\ &+ \int_\Omega (-u)|Du|^{(p-2)n}T_{ij}^{(n-1)}(D^2u)v_{ij}dx. \end{aligned}$$

Integrating by parts, recalling the $T^{(n-1)}(D^2u)$ is divergence free and that

$$u_{\ell j}T_{ij}^{(n-1)}(D^2u) = \delta_{\ell i}\det[D^2u],$$

we find

$$\int_\Omega (-u)|Du|^{(p-2)n} T_{ij}^{(n-1)}(D^2u) v_{ij} dx$$

$$= \int_\Omega |Du|^{(p-2)n} T_{ij}^{(n-1)}(D^2u) v_i u_j dx + (p-2)n \int_\Omega u |Du|^{(p-2)n-2} u_{\ell j} u_\ell T_{ij}^{(n-1)}(D^2u) v_i dx$$

$$= \int_\Omega |Du|^{(p-2)n} T_{ij}^{(n-1)}(D^2u) v_j u_i dx + (p-2)n \int_\Omega u |Du|^{(p-2)n-2} u_\ell v_\ell \det[D^2u] dx.$$

Hence,

$$J = \int_\Omega (-v)|Du|^{(p-2)n} \det[D^2u] dx + \int_\Omega |Du|^{(p-2)n} T_{ij}^{(n-1)}(D^2u) v_j u_i dx.$$

Integrating by parts and using (3) we find

$$\begin{aligned} \int_\Omega |Du|^{(p-2)n} T_{ij}^{(n-1)}(D^2u) v_j u_i dx &= \int_\Omega (-v) \left(|Du|^{(p-2)n} T_{ij}^{(n-1)}(D^2u) u_i \right)_j dx \\ &= n(p-1) \int_\Omega (-v)|Du|^{(p-2)n} \det[D^2u] dx. \end{aligned}$$

Therefore,

$$J = \big(1 + n(p-1)\big) \int_\Omega (-v)|Du|^{(p-2)n} \det[D^2u] dx.$$

Now, since u is a minimizer we have

$$0 = \frac{d}{dt} \frac{\int_\Omega (-u-tv)|Du+tDv|^{(p-2)n} \det[D^2u + tD^2v] dx}{\left(\int_\Omega (-u-tv)^{q+1} dx \right)^{\frac{1+n(p-1)}{q+1}}} \Bigg|_{t=0}$$

$$= J \left(\int_\Omega (-u)^{q+1} dx \right)^{\frac{-1-n(p-1)}{q+1}}$$

$$+ \int_\Omega (-u)|Du|^{(p-2)n} \det[D^2u] dx \frac{d}{dt} \left(\int_\Omega (-u-tv)^{q+1} dx \right)^{\frac{-1-n(p-1)}{q+1}} \Bigg|_{t=0}$$

$$= \big(1+n(p-1)\big)\left[\int_\Omega (-v)|Du|^{(p-2)n}\det[D^2u]dx\left(\int_\Omega(-u)^{q+1}dx\right)^{\frac{-1-n(p-1)}{q+1}}\right.$$

$$\left.-\int_\Omega(-u)|Du|^{(p-2)n}\det[D^2u]dx\left(\int_\Omega(-u)^{q+1}dx\right)^{\frac{-1-n(p-1)}{q+1}-1}\int_\Omega(-u)^q(-v)dx\right].$$

Since $\int_\Omega(-u)^{q+1}dx = 1$, it follows that

$$\int_\Omega(-v)|Du|^{(p-2)n}\det[D^2u]dx = \int_\Omega(-u)|Du|^{(p-2)n}\det[D^2u]dx\int_\Omega(-u)^q(-v)dx.$$

Recalling the expression of $S_{p,q}(\Omega)$ given at the beginning of this proof we find

$$\int_\Omega(-v)(p-1)|Du|^{(p-2)n}\det[D^2u]dx = S_{p,q}(\Omega)\int_\Omega(-u)^q(-v)dx, \quad \forall v\in\Phi(\Omega),$$

from which we get

$$(p-1)|Du|^{(p-2)n}\det[D^2u] = S_{p,q}(\Omega)(-u)^q.$$

The proposition follows from the latter equation and (2). □

Proposition 3. *Let $S_{p,q}(\Omega)$ be defined as in (4) for a convex domain Ω, and let $S_{p,q}(\Omega^*_{n-1})$ be defined as in (4) for Ω^*_{n-1}. If $S_{p,q}(\Omega)$ has a minimizer $u\in\Phi(\Omega)$ then we have*

$$S_{p,q}(\Omega)\geq S_{p,q}(\Omega^*_{n-1}). \tag{12}$$

Proof. If $u\in\Phi(\Omega)$ is a minimizer for $S_{p,q}(\Omega)$ then, by using (3), we find

$$S_{p,q}(\Omega)=\frac{1}{n}\int_\Omega|Du|^{(p-2)n}T^{(n-1)}_{ij}(D^2u)u_iu_jdx, \quad \int_\Omega(-u)^{q+1}dx=1.$$

Let u^*_{n-1} be the $(n-1)$-symmetrand of u. If $\zeta=u^*_{n-1}$, by using the latter result and (11) with $\alpha=(p-1)n+1$ we find

$$S_{p,q}(\Omega)\geq\frac{1}{n}\int_{\Omega^*_{n-1}}|D\zeta|^{(p-2)n}T^{(n-1)}_{ij}(D^2\zeta)\zeta_i\zeta_jdx.$$

Since by (8) we have

$$1=\int_\Omega(-u)^{q+1}dx\leq\int_{\Omega^*_{n-1}}(-\zeta)^{q+1}dx,$$

we find

$$S_{p,q}(\Omega) \geq \frac{\frac{1}{n}\int_{\Omega^*_{n-1}} |D\zeta|^{(p-2)n} T^{(n-1)}_{ij}(D^2\zeta)\zeta_i\zeta_j dx}{\left(\int_{\Omega^*_{n-1}} (-\zeta)^{q+1} dx\right)^{\frac{1+(p-1)n}{q+1}}} \geq S_{p,q}(\Omega^*_{n-1}),$$

that is, (12). The proposition is proved. □

Lemma 4. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain. Suppose $f : (0,\infty) \to (0,\infty)$ is a non decreasing smooth function, and $f(s) > 0$ for $s > 0$. Let $v \in \Phi(\Omega)$ be a super-solution of*

$$\det[Q(Dv)D^2v] = f(-v) \quad in \quad \Omega.$$

If $v^*_{n-1}(x)$ is the $(n-1)$-symmetrand of v, and if $v^*(r) := v^*_{n-1}(x)$ for $r = |x|$, then

$$\left(\frac{dv^*(r)}{dr}\right)^{(p-1)n} \leq n\int_0^r t^{n-1} f(-v^*(t))dt, \tag{13}$$

with equality if and only if v is a solution and Ω is a ball.

Proof. Note that v^* is defined in $B_R = \Omega^*_{n-1}$, the ball centered in the origin and radius $R := \eta_1(\Omega)$. We also recall that

$$\Omega_t = \{x \in \Omega : v(x) < t\}, \quad \Sigma_t = \{x \in \Omega : v(x) = t\}.$$

Since v is a super-solution, using (3) we find

$$\frac{1}{n}\left(|Dv|^{(p-2)n} T^{(n-1)}_{ij}(D^2v)v_i\right)_j \leq f(-v).$$

Integration over Ω_t, for almost all values of t, leads to

$$\frac{1}{n}\int_{\Sigma_t} |Dv|^{(p-2)n-1} T^{(n-1)}_{ij}(D^2v)v_iv_j d\sigma \leq \int_{\Omega_t} f(-v)dx.$$

On using (10), this inequality becomes

$$\int_{\Sigma_t} |Dv|^{(p-1)n}\mathcal{H}_{n-1} d\sigma \leq n\int_{\Omega_t} f(-v)dx.$$

An application of Hölder's inequality then gives

$$\begin{aligned}
&\int_{\Sigma_t} \mathcal{H}_{n-1} d\sigma \\
(14) \qquad &\leq \left(\int_{\Sigma_t} |Dv|^{(p-1)n} \mathcal{H}_{n-1} d\sigma\right)^{\frac{1}{(p-1)n+1}} \left(\int_{\Sigma_t} |Dv|^{-1} \mathcal{H}_{n-1} d\sigma\right)^{\frac{(p-1)n}{(p-1)n+1}} \\
&\leq \left(n \int_{\Omega_t} f(-v) dx\right)^{\frac{1}{(p-1)n+1}} \left(\int_{\Sigma_t} |Dv|^{-1} \mathcal{H}_{n-1} d\sigma\right)^{\frac{(p-1)n}{(p-1)n+1}}.
\end{aligned}$$

Since $\mathcal{H}_{n-1}$ is the total curvature, we have

$$\int_{\Sigma_t} \mathcal{H}_{n-1} d\sigma = n\omega_n. \tag{15}$$

On the other hand, by (9) we get

$$\int_{\Sigma_t} |Dv|^{-1} \mathcal{H}_{n-1} d\sigma = \frac{1}{n-1} \frac{d}{dt} \int_{\Sigma_t} \mathcal{H}_{n-2} d\sigma. \tag{16}$$

From now on we will write $V_1(t)$ for $V_1(\Omega_t)$. By definition of $V_1(t)$ we have

$$\int_{\Sigma_t} \mathcal{H}_{n-2} d\sigma = n(n-1)V_1(t).$$

Therefore, we can rewrite (16) as

$$\int_{\Sigma_t} |Dv|^{-1} \mathcal{H}_{n-1} d\sigma = n \frac{d}{dt} V_1(t).$$

Insertion of (15) and the latter equation into (14) yields

$$n\omega_n \leq \left(n \int_{\Omega_t} f(-v) dx\right)^{\frac{1}{(p-1)n+1}} \left(n \frac{d}{dt} V_1(t)\right)^{\frac{(p-1)n}{(p-1)n+1}}.$$

After some simplification we get

$$\omega_n^{\frac{(p-1)n+1}{(p-1)n}} \leq \left(\int_{\Omega_t} f(-v) dx\right)^{\frac{1}{(p-1)n}} \frac{d}{dt} V_1(t). \tag{17}$$

If $m_0 = \min_\Omega v(x)$ and $\mu(t) = |\Omega_t|$, we have

$$\int_{\Omega_t} f(-v) dx = \int_{m_0}^t f(-\tau)\mu'(\tau) d\tau = f(-t)\mu(t) + \int_{m_0}^t f'(-\tau)\mu(\tau) d\tau.$$

Using (6) we find

$$\mu(t) \le \omega_n^{1-n}\big(V_1(t)\big)^n.$$

Therefore,

$$\int_{\Omega_t} f(-v)dx \le \omega_n^{1-n}\left[f(-t)(V_1(t))^n + \int_{m_0}^t f'(-\tau)(V_1(\tau))^n d\tau\right]$$

$$= n\omega_n^{1-n}\int_{m_0}^t f(-\tau)(V_1(\tau))^{n-1}(V_1(\tau))'d\tau.$$

Putting $V_1(\tau) = \rho$, since $v^\bigstar(\rho)$ is essentially the inverse of $V_1(\tau)$ we find

$$\int_{\Omega_t} f(-v)dx \le n\omega_n^{1-n}\int_0^{V_1(t)} f(-v^\bigstar(\rho))\rho^{n-1}d\rho. \tag{18}$$

Inserting (18) into (17) we find

$$\omega_n^{\frac{(p-1)n+1}{(p-1)n}} \le \left(n\omega_n^{1-n}\int_0^{V_1(t)} f(-v^\bigstar(\rho))\rho^{n-1}d\rho\right)^{\frac{1}{(p-1)n}} \frac{d}{dt}V_1(t).$$

Now we put $V_1(t) = s$. Since

$$\frac{d}{dt}V_1(t) = \left(\frac{dv^\bigstar(s)}{ds}\right)^{-1},$$

after some simplification we get

$$\left(\frac{dv^\bigstar(s)}{ds}\right)^{(p-1)n} \le n\omega_n^{-pn}\int_0^s f(-v^\bigstar(\rho))\rho^{n-1}d\rho.$$

With the change $s = \omega_n r$, we have $v^\bigstar(s) = v^*(r)$, and

$$\frac{dv^*(r)}{dr} = \frac{dv^\bigstar(s)}{ds}\omega_n.$$

With this new variable we find

$$\left(\frac{dv^*(r)}{dr}\right)^{(p-1)n} \le n\omega_n^{-n}\int_0^{\omega_n r} \rho^{n-1} f(-v^\bigstar(\rho))d\rho.$$

Putting $\rho = \omega_n t$, after simplification we get the desired inequality (13).

If Ω is a ball and z is a solution of the given equation, all the inequalities used in the proof of the lemma are equalities. Therefore, the inequality of the lemma holds with equality sign. More easily, in this case the equality follows directly from the equation which, for radial functions $z = z(r)$, reads as

$$\frac{1}{n} r^{-n+1} \left(\left(\frac{dz(r)}{dr} \right)^{(p-1)n} \right)' = f(-z). \tag{19}$$

Finally, if equality holds for all $r \in (0, \eta_1(\Omega))$ then all the inequalities involved in the proof must be equalities. Furthermore, by equation (19) we see that $z'(r) > 0$ for $r > 0$. Hence, Ω must be a ball. The lemma is proved. $\square$

Lemma 4 in case of $f(t) = t^q$, yields

$$\left(\frac{dv^*(r)}{dr} \right)^{(p-1)n} \le n \int_0^r t^{n-1} (-v^*(t))^q dt.$$

By using a method similar to the one used in [16], we prove a lemma which we shall use later on.

Lemma 5. *Suppose $w, u \in \Phi(\Omega)$ satisfy*

$$\det[Q(Dw)D^2 w] \ge (-w)^q, \quad \det[Q(Du)D^2 u] \le (-u)^q \quad in \quad \Omega.$$

Then, $w \le u$ in Ω.

Proof. We observe first that if $w, v \in \Phi(\Omega)$ satisfy

$$\det[Q(Dw)D^2 w] > \det[Q(Dv)D^2 v],$$

then $w(x) \le v(x)$ in Ω. Indeed, by contradiction, let $x_0 \in \Omega$ such that $w(x_0) > v(x_0)$. We may assume x_0 is a point of maximum for $w(x) - v(x)$. Then, at this point we have $Q(Dw) = Q(Dv)$, and $D^2 w \le D^2 v$. As a consequence, since w and v are convex, we have $\det[D^2 w] \le \det[D^2 v]$, and $\det[Q(Dw)D^2 w] \le \det[Q(Dv)D^2 v]$, a contradiction.

By (2) and the first assumption of the lemma we have

$$(p-1)|Dw|^{(p-2)n} \det[D^2 w] \ge (-w)^q.$$

Since $w \in \Phi(\Omega)$, there is a positive constant $C(n,p)$ such that

$$C(n,p)\big(\mathrm{div}|Dw|^{p-2} Dw\big)^n \ge (p-1)|Dw|^{(p-2)m} \det[D^2 w] \ge (-w)^q > 0.$$

By Hopf's lemma for p-subharmonic functions it follows that $w(x) \leq -c_1 d(x)$ in a neighborhood of $\partial\Omega$, where $d(x)$ denotes the distance from x to $\partial\Omega$ and c_1 is a suitable positive constant. Diminishing the constant c_1 we can assume the previous inequality holds in $\overline{\Omega}$. For $x \in \Omega$, let $x_b \in \partial\Omega$ such that $d(x) = |x - x_b|$. Then, since $u \in C^{0,1}(\overline{\Omega})$, we find that

$$-u(x) = |u(x)| \leq \mathrm{Lip}(u, \overline{\Omega})|x - x_b| = \mathrm{Lip}(u, \overline{\Omega})d(x).$$

Therefore $u(x) \geq -c_2 d(x)$ for $x \in \overline{\Omega}$ and some positive constant c_2. Consequently we get

$$w(x) \leq \frac{c_1}{c_2} u(x). \tag{20}$$

Let us now suppose, by way of contradiction, that $w \leq u$ in $\overline{\Omega}$ does not hold. Consider the set

$$S := \{\lambda \in [0,1] : w(x) \leq \lambda u(x) \quad \forall x \in \overline{\Omega}\}.$$

Let $\Lambda := \sup S$. By using (20) we see that $0 < \Lambda < 1$, and we have $w \leq \Lambda u$ in Ω. Since $0 \leq q < (p-1)n$, we choose $\epsilon > 0$, sufficiently small, that $\Lambda^q > (\Lambda + \epsilon)^{(p-1)n}$. The following chain of inequalities holds in Ω.

$$\begin{aligned} \det[Q(Dw)D^2 w)] &\geq (-w)^q \\ &\geq \big(-\Lambda u\big)^q \\ &> (\Lambda + \epsilon)^{(p-1)n}(-u)^q \\ &\geq (\Lambda + \epsilon)^{(p-1)n} \det[Q(Du)D^2 u] \\ &= \det[Q(Du_\epsilon)D^2 u_\epsilon], \quad u_\epsilon = (\Lambda + \epsilon)u. \end{aligned}$$

By our observation above we have $w(x) \leq (\Lambda + \epsilon)u(x)$ in $\overline{\Omega}$. Since this inequality contradicts the choice of Λ, the lemma is proved. □

Remark. By Proposition 2 and Lemma 5, problem (2) has at most a minimizer. Furthermore, if B is a ball then the minimizer v for $S_{p,q}(B)$ is radially symmetric. Indeed, v satisfies (5) with $\Omega = B$. Since the operator $\det[Q(Dv)D^2 v]$ is invariant for rotations, if v were not symmetric then, by a suitable rotation, we would find a different solution $\tilde{v}$, contradicting the uniqueness for problem (5).

Theorem 6. *Let $v(x) \in \Phi(\Omega)$ be a super-solution of problem (1) in Ω, and let $v^*_{n-1}(x)$ be its $(n-1)$-symmetrand. With $R := \eta_1(\Omega)$, let $z(r)$ be a sub-solution of problem (1) in the ball with radius R. If $v^*(r) = v^*_{n-1}(x)$ for $|x| = r$, we have*

$$v^*(r) \geq z(r), \quad 0 \leq r \leq R.$$

Proof. Let w be a radial solution of

$$\det[Q(Dw)D^2w] = (-v^*)^q \text{ in } \Omega^*_{n-1} \text{ and } w = 0 \text{ on } \partial\Omega^*_{n-1}. \tag{21}$$

If $w(x) = w(r)$ with $|x| = r$, w is given explicitly by

$$w(r) = -n^{\frac{1}{(p-1)n}} \int_r^R \left(\int_0^s t^{n-1}(-v^*(t))^q dt \right)^{\frac{1}{(p-1)n}} ds.$$

Therefore w satisfies

$$\left(\frac{dw(r)}{dr} \right)^{(p-1)n} = n \int_0^r t^{n-1}(-v^*(t))^q dt.$$

Comparing this equation and the inequality (13) with $f(t) = t^q$, we see that

$$\frac{dv^*(r)}{dr} \leq \frac{dw(r)}{dr}, \quad 0 < r < R.$$

Integrating on (r, R) for any $0 < r < R$, we get

$$v^*(r) \geq w(r) \quad \text{for } 0 \leq r \leq R.$$

With $v^*(x) = v^*(r)$ and $w(x) = w(r)$ for $|x| = r$, we have

$$v^*(x) \geq w(x) \quad \text{for } x \in \Omega^*_{n-1}. \tag{22}$$

Using (21) and (22), we find that

$$\det[Q(Dw)D^2w] = (-v^*)^q \leq (-w)^q \quad \text{in } \Omega^*_{n-1}.$$

Summarizing, we see that w and z satisfy

$$\begin{cases} \det[Q(Dw)D^2w] \leq (-w)^q & \text{in } \Omega^*_{n-1} \text{ and } w = 0 \text{ on } \partial\Omega^*_{n-1} \\ \det[Q(Dz)D^2z] \geq (-z)^q & \text{in } \Omega^*_{n-1} \text{ and } z = 0 \text{ on } \partial\Omega^*_{n-1}. \end{cases}$$

By Lemma 5, we have

$$w(x) \geq z(x) \text{ in } \Omega^*_{n-1}. \tag{23}$$

Thus, from (22) and (23) we conclude

$$v^*(x) \geq z(x) \text{ in } \Omega^*_{n-1}.$$

The theorem follows. $\square$

Corollary 7. *Let $g : [0, \infty) \to [0, \infty)$ be a non-decreasing smooth function such that $g(0) = 0$ and $g(t) > 0$ for $t > 0$. Under the assumptions of Theorem 6, we have*

$$\int_\Omega g(-v)dx \leq \int_{\Omega^*_{n-1}} g(-z)dx.$$

Moreover we have

$$\inf_\Omega v(x) \geq \inf_{\Omega^*_{n-1}} z(x).$$

Furthermore, equality holds in each of these inequalities if and only if v and z are solutions of the corresponding equations and Ω is a ball.

Proof. Let $\mu(t) = |\{x \in \Omega : v(x) < t\}|$, $\mu^*(t) = |\{x \in \Omega^*_{n-1} : v^*_{n-1}(x) < t\}|$. By (7) we have

$$\mu(t) \leq \mu^*(t) \quad \forall t \in (m_0, 0), \quad m_0 = \min_\Omega v.$$

We note that $m_0 = v^*_{n-1}(0)$, and

$$\begin{aligned}\int_\Omega g(-v)dx &= \int_{m_0}^0 g(-t)d\mu(t) = \int_{m_0}^0 g'(-t)\mu(t)dt \\ &\leq \int_{m_0}^0 g'(-t)\mu^*(t)dt = \int_{m_0}^0 g(-t)d\mu^*(t) \\ &= \int_{\Omega^*_{n-1}} g(-v^*_{n-1})dx.\end{aligned}$$

The first statement follows since we have $-v^*_{n-1}(x) \leq -z(x)$ (see Theorem 6). The second statement is true because

$$\inf_\Omega v(x) = v^*_{n-1}(0) \geq z(0) = \inf_{\Omega^*_{n-1}} z(x).$$

Finally, when equality holds in each of these inequalities we must have $v^*_{n-1}(x) = z(x)\ \forall x \in \Omega^*_{n-1}$. Hence, we must have equality in the inequality of Lemma 4, but this implies Ω is a ball. The corollary is proved. $\square$

For $u \in \Phi(\Omega)$, we define the Hessian integral

$$I(\Omega, u) := \int_\Omega (-u) \det[Q(Du)D^2u]dx.$$

Proposition 8. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain. Let $v \in \Phi(\Omega)$ be a super-solution of*

$$\det[Q(Dv)D^2v] = (-v)^q \quad in \quad \Omega,$$

and let $v^*_{n-1}(x)$ be its $(n-1)$-symmetrand. If $z \in \Phi(\Omega^*_{n-1})$ is a sub-solution of the above equation in the ball Ω^*_{n-1}, we have

$$I(\Omega, v) \le I(\Omega^*_{n-1}, z).$$

P r o o f.

$$\begin{aligned} I(\Omega, v) &= \int_\Omega (-v) \det[Q(Dv)D^2v]dx \\ &\le \int_\Omega (-v)(-v)^q dx \quad \text{since } v \text{ is a super-solution} \\ &\le \int_{\Omega^*_{n-1}} (-z)(-z)^q dx \quad \text{by Corollary 7} \\ &\le \int_{\Omega^*_{n-1}} (-z) \det[Q(Dz)D^2z]dx \quad \text{since } z \text{ is a sub-solution} \\ &= I(\Omega^*_{n-1}, z). \end{aligned}$$

The proposition is proved. $\square$

4. Eigenvalues.

In this Section we consider the case $q = (p-1)n$. Let

$$\lambda(\Omega) = \inf\left\{ \int_\Omega (-v) \det[Q(Dv)D^2v]dx : \quad v \in \Phi(\Omega), \quad \int_\Omega (-v)^{(p-1)n+1} dx = 1 \right\}. \tag{24}$$

Of course, we have

$$\lambda(\Omega) = \inf \left\{ \frac{\displaystyle\int_\Omega (-v)\ \det[Q(Dv)D^2v]dx}{\displaystyle\int_\Omega (-v)^{(p-1)n+1}dx} : \quad v \in \Phi(\Omega) \right\}.$$

Proceeding as in the proof of Proposition 2, we find that if $u \in \Phi(\Omega)$ is a minimizer of (24) then it solves the eigenvalue problem

$$\text{(25)}\quad \det[Q(Du)D^2u] = \lambda(\Omega)(-u)^{(p-1)n},\ \ \lambda(\Omega) > 0,\ \ u < 0 \text{ in } \Omega,\ \ u = 0 \text{ on } \partial\Omega.$$

Recall that $p \geq 2$. Of course, $\lambda(\Omega)$ depends also on p, but in the sequel of this section p will be fixed. We have

Proposition 9. *Given a convex domain Ω, let Ω^*_{n-1} be the ball with radius $R := \eta_1(\Omega)$. If $\lambda(\Omega)$ has a minimizer $u \in \Phi(\Omega)$ then*

$$\lambda(\Omega) \geq \lambda(\Omega^*_{n-1}).$$

Proof. If $u \in \Phi(\Omega)$ is a minimizer for $\lambda(\Omega)$, using (3) we find

$$\lambda(\Omega) = \frac{1}{n}\int_\Omega |Du|^{(p-2)n} T^{(n-1)}_{ij}(D^2u)u_iu_jdx, \quad \int_\Omega (-u)^{(p-1)n+1}dx = 1.$$

Putting $\zeta = u^*_{n-1}$, by using (11) with $\alpha = (p-1)n+1$ we find

$$\lambda(\Omega) \geq \frac{1}{n}\int_{\Omega^*_{n-1}} |D\zeta|^{(p-2)n} T^{(n-1)}_{ij}(D^2\zeta)\zeta_i\zeta_jdx.$$

On the other hand, by (8) we have

$$1 = \int_\Omega (-u)^{(p-1)n+1}dx \leq \int_{\Omega^*_{n-1}} (-\zeta)^{(p-1)n+1}dx.$$

Therefore, using again (3) we find

$$\lambda(\Omega) \geq \frac{\displaystyle\frac{1}{n}\int_{\Omega^*_{n-1}} |D\zeta|^{(p-2)n} T^{(n-1)}_{ij}(D^2\zeta)\zeta_i\zeta_jdx}{\displaystyle\int_{\Omega^*_{n-1}} (-\zeta)^{(p-1)n+1}dx}$$

$$= \frac{\displaystyle\int_{\Omega^*_{n-1}} (-\zeta)\ \det[Q(D\zeta)D^2\zeta]dx}{\displaystyle\int_{\Omega^*_{n-1}} (-\zeta)^{(p-1)n+1}dx} \geq \lambda(\Omega^*_{n-1}).$$

The proposition is proved. □

Let us consider a fixed eigenfunction u of problem (25), and let B_{R_0} be the ball centered at the origin and such that $\lambda(B_{R_0}) = \lambda(\Omega)$. Let v be the (radial) eigenfunction corresponding to $\lambda(B_{R_0})$ normalized either such that

$$\inf_{B_{R_0}} v(x) = \inf_{\Omega} u(x), \tag{26}$$

or such that

$$\int_{B_{R_0}} (-v(x))^\beta dx = \int_{\Omega^*_{n-1}} (-u^*(x))^\beta dx, \quad 0 < \beta < \infty. \tag{27}$$

Theorem 10. *Let u be a fixed eigenfunction of problem* (25), *and let $u^* = u^*_{n-1}$ be its $(n-1)$-symmetrand. Let B_{R_0} be a ball with radius R_0, centered at the origin, and such that $\lambda(B_{R_0}) = \lambda(\Omega) =: \lambda$. Let v be an eigenfunction of problem* (25) *with $\Omega = B_{R_0}$. Let $u^*(x) = u^*(r)$ and $v(x) = v(r)$ for $|x| = r$. If v is normalized as in* (26) *then*

$$u^*(r) \le v(r), \quad 0 < r < R_0.$$

If v is normalized as in (27) *then*

$$\int_0^r t^{n-1}(-u^*(t))^\beta dt \le \int_0^r t^{n-1}(-v(t))^\beta dt, \quad 0 < r < R_0.$$

P r o o f. Since $\lambda(B_{R_0}) = \lambda(\Omega)$, by Proposition 9 we have $R_0 \le R$, where R is the radius of Ω^*_{n-1}. If $R_0 = R$ then $\Omega = B_{R_0}$, and there is nothing to prove. Thus, assume $R_0 < R$.

Let v be normalized as in (26). Since $u^*(0) = v(0)$ and $u^*(R_0) < v(R)$, if $u^*(r) \le v(r)$ does not hold, there exists a point r_0 such that $u^*(r_0) = v(r_0)$ and either $u^*(r) \ge v(r)$ or $u^*(r) \le v(r)$ for $0 < r < r_0$ with the inequalities being strict at some point. By Lemma 4 with $f(t) = \lambda t^{(p-1)n}$, we have

$$\left(\frac{du^*(r)}{dr}\right)^{(p-1)n} \le n\lambda \int_0^r t^{n-1}(-u^*(t))^{(p-1)n} dt, \tag{28}$$

and

$$\left(\frac{dv(r)}{dr}\right)^{(p-1)n} = n\lambda \int_0^r t^{n-1}(-v(t))^{(p-1)n} dt. \tag{29}$$

In case of $u^*(r) \geq v(r)$ on $(0, r_0)$, by (28) and (29) we get

$$\frac{du^*(r)}{dr} \leq \frac{dv(r)}{dr}, \quad 0 < r < r_0,$$

with the inequality being strict at some point. Integration on $(0, r_0)$ yields $u^*(r_0) < v(r_0)$, a contradiction. In case of $u^*(r) \leq v(r)$ on $(0, r_0)$, we proceed as follows. Define

$$w(r) = \begin{cases} u^*(r) & r \in (0, r_0], \\ \\ v(r) & r \in (r_0, R_0]. \end{cases}$$

By (28) and (29) we get

$$(30) \qquad \big(w'(r)\big)^{(p-1)n} \leq n\lambda \int_0^r t^{n-1}(-w(t))^{(p-1)n} dt, \quad 0 < r < R_0.$$

Furthermore, we have $w(r) < 0$ on $(0, R_0)$, $w(R_0) = 0$, and clearly $w(r)$ is not equal to $cv(r)$ for any constant c. Therefore,

$$(31) \qquad \lambda < \frac{\int_{B_{R_0}} (-w) \det[Q(Dw)D^2 w] dx}{\int_{B_{R_0}} (-w)^{(p-1)n+1} dx}.$$

Since w is a radial function, with $w(r) = w(x)$ for $r = |x|$, we have (see [14], page 99)

$$\det[D^2 w] = r^{-n+1} \left(\frac{1}{n}(w')^n\right)'.$$

Hence, since $\det[Q(Dw)D^2 w] = (p-1)|Dw|^{(p-2)n} \det[D^2 w]$, we find

$$\det[Q(Dw)D^2 w] = r^{-n+1} \left(\frac{1}{n}(w')^{(p-1)n}\right)'.$$

Therefore, using the latter equation and inequality (30) we get

$$\begin{aligned}\int_{B_{R_0}}(-w)\det[Q(Dw)D^2w]dx &= n\omega_n\int_0^{R_0}(-w)\left(\frac{1}{n}(w')^{(p-1)n}\right)'dr\\ &= \omega_n\int_0^{R_0}w'(w')^{(p-1)n}dr\\ &\le \lambda n\omega_n\int_0^{R_0}w'(r)dr\int_0^r t^{n-1}(-w(t))^{(p-1)n}dt\\ &= \lambda n\omega_n\int_0^{R_0}r^{n-1}(-w(r))^{(p-1)n+1}dr\\ &= \lambda\int_{B_{R_0}}(-w)^{(p-1)n+1}dx.\end{aligned}$$

Insertion of this inequality into (31) yields $\lambda < \lambda$, a contradiction. Hence, we must have $u^*(r) \le v(r)$ on $[0, R_0]$, as claimed.

Let v be normalized as in (27). Since $R_0 < R$ and $-u^* > 0$ on $[R_0, R)$, we have

$$\int_0^{R_0} r^{n-1}(-u^*(r))^{\beta}dr < \int_0^{R_0} r^{n-1}(-v(r))^{\beta}dr. \tag{32}$$

Since $v(R_0) = 0$ and $u^*(R_0) < 0$, it follows that there is (at least) one point $r_0 \in (0, R_0)$ such that $u^*(r_0) = v(r_0)$. We claim that there is only one point r_0 such that $u^*(r_0) = v(r_0)$. By contradiction, assume $u^*(r_1) = v(r_1)$, $u^*(r_0) = v(r_0)$, $u^*(r) < v(r)$ on $(0, r_1)$ and $u^*(r) \ge v(r)$ on (r_1, r_0). Putting

$$w(r) = \begin{cases} u^*(r) & r \in (0, r_1], \\ v(r) & r \in (r_1, R_0]. \end{cases}$$

we see that w satisfies inequality (30). Arguing as in the previous case we get a contradiction.

Now let $u^*(r_1) = v(r_1)$, $u^*(r_0) = v(r_0)$, $u^*(r) \ge v(r)$ on $(0, r_1)$ and $u^*(r) < v(r)$ on (r_1, r_0). Putting

$$w(r) = \begin{cases} v(r) & r \in (0, r_1], \\ u^*(r) & r \in (r_1, r_0], \\ v(r) & r \in (r_0, R_0], \end{cases}$$

w satisfies again inequality (30), and arguing again as in the previous case we still get a contradiction. Hence, we must have,

$$u^*(r) > v(r), \quad 0 < r < r_0,$$

and

$$u^*(r) < v(r), \quad r_0 < r < R_0.$$

Let us write inequality (32) as

$$\int_{r_0}^{R_0} t^{n-1}\big[(-u^*(t))^\beta - (-v(t))^\beta\big]\,dt < \int_0^{r_0} t^{n-1}\big[(-v(t))^\beta - (-u^*(t))^\beta\big]\,dt.$$

Since $-u^*(t) > -v(t)$ for $r_0 < t < R_0$, it follows that, for any $r \in [r_0, R_0]$,

$$\int_{r_0}^{r} t^{n-1}\big[(-u^*(t))^\beta - (-v(t))^\beta\big]\,dt \le \int_0^{r_0} t^{n-1}\big[(-v(t))^\beta - (-u^*(t))^\beta\big]\,dt,$$

that is,

$$\int_0^r t^{n-1}(-u^*(t))^\beta dt \le \int_0^r t^{n-1}(-v(t))^\beta dt, \quad r \in [0, R_0].$$

The proof of the theorem is completed. □

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