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NEW HARDY-TYPE INEQUALITIES WITH SINGULAR WEIGHTS

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Dedicated to Professor Petar Popivanov on the occasion of his 65th birthday

Abstract. We prove a new Hardy–type inequality with weights that are possibly singular at internal point and on the boundary of the domain. As an illustration some applications and examples are given.

1. Introduction. With $p \geq 2$, $n \geq 2$ consider a function $F \in C^1(R^n) \cap W^{1,p}(R^n)$ and sets $G_s = \{x \in R^n : |F(x)| = s\}$, $G_{\delta,M} = \{x \in R^n : \delta < |F(x)| < M\}$, $\delta \geq 0, M \leq \infty$. Suppose that there exist functions f, $\psi \in W^{1,p}(G_{0,M})$ and the following conditions are satisfied:

$$F \Delta_p \psi = -f \leq 0 \tag{1}$$

$$\nabla F \nabla \psi \geq 0 \tag{2}$$

Define a set of functions $U_F = \{u \in C^1(G_{0,\infty}) \text{ and } u|_{G_\delta} = o(\delta^{1/p'}) \text{ for } \delta \to 0\}$ and with

$$w = |F|^{-p} \frac{\nabla F \nabla \psi}{|\nabla \psi|^2} |\nabla \psi|^p, \quad h = \frac{F}{|F|} \left(\frac{\nabla F \nabla \psi}{|\nabla \psi|^2} \right)^{-1/p'} \frac{\nabla \psi}{|\nabla \psi|}$$

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and $u \in U_F$ consider the functions

$$L(t) = \int_{G_{0,t}} |h\nabla u|^p \, dx, \quad R(t) = \int_{G_{0,t}} w|u|^p dx, \quad N(t) = \int_{G_{0,t}} f|F|^{-p}|u|^p dx.$$

The aim of the paper is to prove a new Hardy inequality with singular weights and to give some applications.

Theorem 1. *Under the conditions (1) and (2), for every function $u \in U_F$ the following inequalities hold*

$$\text{(3)} \qquad \begin{aligned} a)\ L(t) &\geq N(t), \\ b)\ L(t) &\geq \left(\frac{1}{p'}\right)^p R(t). \end{aligned}$$

The form of the Hardy inequalities (3) depends on two functions F, ψ, satisfying (1) and (2). Also the domain where inequalities (3) take place is defined by the union of the level surfaces of function F.

Starting with the work of [1] the 1-dimensional inequality is proved

$$\text{(4)} \qquad \int_0^\infty |u'(x)|^p x^\alpha dx \geq \left(\frac{p-1-\alpha}{p}\right)^p \int_0^\infty x^{-p+\alpha}|u(x)|^p dx$$

where $1 < p < \infty$, $\alpha < p-1$, $u(x)$ is absolutely continuous on $[0, \infty)$, $u(0) = 0$.

There is a number of generalizations of (4) for n-dimensional case, see the reviews in [2, 3]. Mainly two types of Hardy inequalities are studied.

First type concerns the optimal properties of the domain $\Omega \subset R^n$, $n \geq 2$ where inequality with kernels singular on the boundary $\partial\Omega$ holds

$$\text{(5)} \qquad \int_\Omega |\nabla u(x)|^p d(x)^\alpha dx \geq C \int_\Omega d(x)^{-p+\alpha}|u(x)|^p dx$$

with $d(x) = dist(x, \partial\Omega)$, $p \geq 2$, $\alpha < p-1$, see [4, 5, 6, 7, 3, 8, 9, 10, 11] etc.

Second type concerns inequalities with a kernel, singular in internal point of Ω, i.e.

$$\text{(6)} \qquad \int_\Omega |\nabla u(x)|^2 dx \geq C \int_\Omega \frac{|u(x)|^2}{|x|^2} dx$$

where $u \in C_0^\infty(\Omega)$, $\Omega \subseteq R^n$, $0 \in \Omega$, $n \geq 3$, see [12, 13, 14, 15, 8, 16, 17] etc.

Let us note that the possibility to use two functions F and ψ in the inequalities (3) serves many new Hardy–type inequalities.

In what follows, in section 2 we will prove Theorem 1, together with the sharpness results in Theorem 2. In section 3 are shown and commented applications and two examples for some particular choices of F and ψ.

2. Main result. We start with the proof of Theorem 1.

Proof. Applying the Hölder inequality we get

$$\frac{1}{p}\int_{G_{\delta,t}} w^{1/p'} h\nabla|u|^p dx = \int_{G_{\delta,t}} w^{1/p'}|u|^{p-2}uh\nabla u dx \le \left(\int_{G_{\delta,t}} w|u|^p dx\right)^{1/p'}\left(\int_{G_{\delta,t}} |h\nabla u|^p dx\right)^{1/p},$$

and hence

$$\int_{G_{\delta,t}} |h\nabla u|^p dx \ge \left(\frac{1}{p}\right)^p \frac{\left|\int_{G_{\delta,t}} w^{1/p'} h\nabla|u|^p dx\right|^p}{\left(\int_{G_{\delta,t}} w|u|^p dx\right)^{p-1}}. \tag{7}$$

Using the definition of h and w and integrating by parts for the numerator of (7) we obtain

$$\begin{aligned}\int_{G_{\delta,t}} w^{1/p'} h\nabla|u|^p dx &= t^{1-p}\int_{G_t} \frac{\nabla F\nabla\psi}{|\nabla F|}|\nabla\psi|^{p-2}|u|^p d\sigma\\ &+(p-1)\int_{G_{\delta,t}} |F|^{-p}\nabla F\nabla\psi|\nabla\psi|^{p-2}|u|^p\\ &-\int_{G_{\delta,t}} |F|^{-p}F\Delta_p\psi|u|^p - \delta^{1-p}\int_{G_\delta}\frac{\nabla F\nabla\psi}{|\nabla F|}|\nabla\psi|^{p-2}|u|^p d\sigma.\end{aligned}$$

Recall that $u \in U_F$, so the integral over G_δ tends to 0 for $\delta \to 0$, then after the limit we get

$$\int_{G_{0,t}} w^{1/p'} h\nabla|u|^p dx \ge t\frac{d}{dt}R(t) + (p-1)R(t) + N(t).$$

Note that $L(t) < \infty$ and from (7) we obtain

$$L(t) \ge \left(\frac{1}{p}\right)^p \frac{\left(t\frac{d}{dt}R(t) + (p-1)R(t) + N(t)\right)^p}{R^{p-1}(t)}.$$

Since $t\dfrac{d}{dt}R(t) \ge 0$, $N(t) \ge 0$ and $R(t) \ge 0$ we get (3) b).

To prove (3) a) we use the Jensen inequality

$$\frac{ca^p}{c^{p'}b^{p-1}} \ge pca - (p-1)c^{p'}b, \quad c > 0. \tag{8}$$

From (8) with $a = \frac{1}{p}\left(t\frac{d}{dt}R(t) + (p-1)R(t) + N(t)\right)$, $b = R(t)$ and $c = 1$ it follows

$$L(t) \geq \left(t\frac{d}{dt}R(t) + N(t)\right).$$

and we get (3) a). □

The following sharpness result holds.

Theorem 2. *Suppose that F and ψ satisfy* (1) *and* (2)*. Then for $u_\varepsilon = |F|^{\frac{1+\varepsilon}{p'}}$, $\varepsilon > 0$ it follows that $R(t) < \infty$ and the inequality* (3) b) *is ε-sharp, i.e.:*

$$L(t) = \left(\frac{1+\varepsilon}{p'}\right)^p R(t). \tag{9}$$

P r o o f. The kernels of $L(t)$ and $R(t)$ for the case are correspondingly:

$$\begin{aligned}
|h\nabla u_\varepsilon|^p &= \left|\frac{F}{|F|}\left(\frac{\nabla F\nabla\psi}{|\nabla\psi|^2}\right)^{-1/p'}\frac{\nabla\psi}{|\nabla\psi|}\nabla u_\varepsilon\right|^p \\
&= \left(\frac{1+\varepsilon}{p'}\right)^p \left|(\nabla F\nabla\psi)^{1-\frac{1}{p'}}|F|^{\frac{1+\varepsilon}{p'}-1}|\nabla\psi|^{\frac{2}{p'}-1}\right|^p \\
&= \left(\frac{1+\varepsilon}{p'}\right)^p (\nabla F\nabla\psi)\,|F|^{\frac{1+\varepsilon}{p'}-1}|\nabla\psi|^{p-2}, \\
w|u_\varepsilon|^p &= |F|^{-p}\frac{\nabla F\nabla\psi}{|\nabla\psi|^2}|\nabla\psi|^p|u_\varepsilon|^p \\
&= |F|^{-p}\nabla F\nabla\psi|\nabla\psi|^{p-2}|F|^{\frac{1+\varepsilon}{p'}p}
\end{aligned}$$

and (9) holds. □

3. Applications.

3.1. Inequality with distance to $\partial\Omega$. With the appropriate choice of F and ψ such that $N > 0$ we can use (3) a) and to obtain a generalization of the result of [16].

Consider $\Omega \subset R^n$ and let K be a smooth surface with $\operatorname{codim} K = k$, $1 \leq k < n$. Let $d(x) = \operatorname{dist}(x, K)$, denote $\lambda = \frac{p-k}{p-1}$ and let the condition (C) from [16] holds, i.e.

$$\Delta_p d^\lambda \leq 0 \quad \text{in } \Omega\backslash K, \quad \lambda \neq 0 \tag{10}$$

An equivalent form of (10) is

$$-\lambda d\Delta d \geq \lambda(\lambda-1)(p-1) \quad \text{on} \quad \Omega\backslash K \tag{11}$$

Let $F = \psi = \psi(d)$, then condition (2) is true. As for the condition (1) using (11) and assuming that $\dfrac{\psi'}{\lambda} > 0$ we have

$$-\Delta_p \psi \geq \frac{|\psi'|^{p-2}}{d}(p-1)[(\lambda-1)\psi' - d\psi''].\tag{12}$$

For example, let us choose ψ such that

$$d\psi' = \lambda\psi V(\ln d) \quad \text{with} \quad V > 0 \tag{13}$$

We have to determine V such that condition (1), i.e. $F\Delta_p\psi \leq 0$ holds and to find the kernel of N, i.e. $N_0 = -|F|^{-p}F\Delta_p\psi$.

From (12) and (13) we get

$$d\psi'' = \psi'(\lambda V - 1) + \frac{\lambda\psi}{d}V' = V\frac{\lambda\psi}{d}(\lambda V - 1) + \frac{\lambda\psi}{d}V',$$

so

$$d(\lambda-1)\psi' - d^2\psi'' = \lambda(\lambda-1)\psi V - \lambda\psi(\lambda V - 1)V - \lambda\psi V' = \lambda\psi[\lambda(V - V^2) - V'].$$

Then

$$\begin{aligned} N_0 &= -\psi|\psi|^{-p}\Delta_p\psi \geq (p-1)\frac{\psi^{1-p}|d\psi'|^{p-2}}{d^p}[(\lambda-1)d\psi' - d^2\psi''] \\ &= (p-1)\frac{|\lambda|^{p-2}V^{p-2}}{d^p}[\lambda^2(V - V^2) - \lambda V'] \\ &= (p-1)\frac{|\lambda|^p V^{p-2}}{d^p}\left[-\frac{1}{\lambda}V' + V - V^2\right] \end{aligned}$$

and

$$N_0 \geq (p-1)\frac{|\lambda|^p V^p}{d^p}\left[-\frac{V'}{\lambda V^2} + \frac{1}{V} - 1\right].$$

Denote $G(t) = -\dfrac{V'(t)}{\lambda V(t)^2} + \dfrac{1}{V(t)} - 1$ and we need to have $G > 0$. Since

$$\left(\frac{1}{V}\right)' = -\lambda\frac{1}{V} + \lambda(1 + G).\tag{14}$$

then a solution $\frac{1}{V}$ of (14) is

$$\left(\frac{1}{V}\right) = 1 + \lambda e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} G(s) ds. \tag{15}$$

where for $\lambda > 0, t_0 = -\infty$ and for $\lambda < 0, t_0 \geq t$.

From (15) we get

$$N_0 \geq (p-1) \frac{|\lambda|^p}{d^p} \frac{G(t)}{[1 + \lambda e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} G(s) ds]^p}.$$

With a change of function $G(t) = \frac{1}{p-1} H(e^{\lambda t})$ we obtain

$$N_0 \geq \frac{|\mu|^p}{d^p} \frac{H(s)}{\left[\frac{1}{p'} + \frac{1}{ps} \int_{s_0}^{s} H(\sigma) d\sigma\right]^p} = \frac{N_1}{d^p}. \tag{16}$$

where $\mu = \frac{k-p}{p}$, $s = e^{\lambda t} = d^{\lambda}$ and: for $\lambda > 0$, $H(0) = 1, s_0 = 0$ and H is increasing on the interval $(0, \delta), \delta > 0$; for $\lambda < 0$, $H(\infty) = 1$.

At this point, using Theorem 1, (3) a) we obtain the Hardy inequality

$$\int_{\Omega} |\nabla u(x)|^p dx \geq \int_{\Omega} N_0 |u(x)|^p dx \geq \int_{\Omega} \frac{N_1}{d^p} |u(x)|^p dx \tag{17}$$

Example 1. Now let us show that with a certain choice of H we can obtain the result of [16], Theorem A, equation (1.8).

Let $\lambda > 0$ and replacing $H(s) = 1 + Q(s)$ from (16) we obtain for $N_1 = d^p N_0$

$$N_1 = |\mu|^p \frac{1 + Q(t)}{\left[1 + \frac{1}{ps} \int_0^s Q(\sigma) d\sigma\right]^p}.$$

We can find $Q(s)$ such that

$$N_1 \geq |\mu|^p \frac{1}{p} \left(1 + \frac{p'}{2} \frac{1}{\ln^2(s/D)}\right), \quad D > D_0 = \max_{\Omega \setminus K} d. \tag{18}$$

Denote

$$z = \left[1 + \frac{1}{ps} \int_0^s Q(\sigma) d\sigma\right]^{1-p}$$

and to obtain (18) it is enough to find z such that

$$\frac{sz'}{1-p} + z - \frac{1}{p'} z^{p'} - \frac{1}{p}\left(1 + \frac{p'}{2}\frac{1}{\ln^2 \dfrac{s}{D}}\right) \geq 0, \tag{19}$$

$$z(0) = 1, \quad z > 0, \quad z \text{ is increasing.}$$

We are asking for z in the form

$$z = 1 + \frac{1}{\ln \dfrac{s}{D}} + \frac{b}{\ln^2 \dfrac{s}{D}}, \text{ then } z' = -\frac{1}{s \ln^2 \frac{s}{D}}\left(1 + \frac{2b}{\ln \dfrac{s}{D}}\right)$$

and for every $D > D_0$ we can find b such that $z' \geq 0$, i.e.

$$b \leq -\frac{1}{2} \ln \frac{s}{D}, \quad \text{for} \quad s < D. \tag{20}$$

Expanding the term $z^{p'}$ in (19) in a Taylor series up to the third term and simplifying, the inequality in (19) becomes

$$\frac{1}{1-p}\left(b - \frac{p-2}{6(p-1)}\right)\frac{1}{\ln^3(\frac{s}{D})} + o\left(\ln^{-3}\left(\frac{s}{D}\right)\right) \geq 0, \quad \text{for } b > \frac{p-2}{6(p-1)}.$$

So for

$$\frac{p-2}{6(p-1)} < b \leq -\frac{1}{2} \ln \frac{s}{D}, \quad \text{for} \quad s < D.$$

the inequality (19) holds and (17) becomes the result of [16]. In a similar way but using the Taylor expansion of $z^{p'}$ up to the m^{th} term we can obtain the result in [17].

Note that the equation (13) is very essential. It can be used also for Hardy inequality base on (3) b).

Example 2. By means of (13) with $V \equiv 1$, i.e. $\psi = d^{\lambda}$, if (10) holds with $k = 1$, so that $\lambda = 1$, we can get inequality (5). Indeed, if $F = d^{\gamma}$, $0 < \gamma < 1$, then (1) and (2) hold and by (3) b) we get

$$\int_\Omega |\nabla d(x) \nabla u(x)|^p d(x)^\alpha dx \geq \left|\frac{p-1-\alpha}{p}\right|^p \int_\Omega d(x)^{-p+\alpha} |u(x)|^p dx \tag{21}$$

where $\Omega = \{x : 0 < d^{\gamma} < t\}$ and $\alpha = (1-\gamma)(p-1)$. Note that Ω can be a strip and function u in (21) should be 0 only on part of the boundary of Ω, i.e. for $\{x : d(x) = 0\}$ but not on $\{x : d(x) = t\}$.

Since $|\nabla d| = 1$, the inequality (5) follows by (21). Moreover from Theorem 2 the inequality (21) is ε-sharp.

3.2. Inequality with double singularity in the kernels. Let $\Omega \subset R^n$, $n \geq 2$ be a bounded domain, function $\psi(x) > 0$ in Ω, $\Delta_p \psi \leq 0$ and

$$\text{There exists a function } \lambda \in C^{0,1}(\Omega), \lambda(x) > 0, \text{ such that } \Omega \subset \{\psi(x) < \lambda(x)\} \text{ and } \nabla\psi\nabla\lambda \leq 0. \tag{22}$$

With $s = \dfrac{\psi}{\lambda} \in (0,1)$ define the function

$$g(s) = \begin{cases} \dfrac{1-s^m}{m} \text{ for } m \neq 0 \\ \ln\dfrac{1}{s} \text{ for } m = 0 \end{cases}, \text{ where } m \text{ will be chosen later.}$$

With ψ and $F = -\dfrac{1}{B}\psi^A g^B, B < 0$ and $m = -\dfrac{A}{B}$ the conditions (1), (2) are satisfied, indeed

$$\begin{aligned} \nabla\psi\nabla F &= -\frac{1}{B}\psi^{A-1}g^{B-1}|\nabla\psi|^2[Ag + mBg - B] = \psi^{A-1}g^{B-1}|\nabla\psi|^2 > 0 \\ -F\Delta_p\psi &= -\Delta_p\psi\frac{1}{B}\psi^A g^B \geq 0. \end{aligned}$$

Applying Theorem 1, (3) b) we get

$$\int_\Omega (\psi^{A-1}g^{B-1})^{1-p}|\nabla u|^p dx \geq \left(\frac{|B|}{p'}\right)^p \int_\Omega \psi^{A(1-p)-1}g^{B(1-p)-1}|\nabla\psi|^p|u|^p dx. \tag{23}$$

Due to the Theorem 2, the inequality (23) is sharp.

Example 3. Let $\psi = \left(\dfrac{p-1}{p-n}\right)|x|^{\frac{p-n}{p-1}}$, $p \neq n$, then $|\nabla\psi|^{p-2}\nabla\psi = |x|^{-n}x$.

Define $F = -\dfrac{1}{B_0}|\psi|^{A_0}g^{B_0}$, with $A_0 = \dfrac{\alpha p - n}{p-n}$, $B_0 = \dfrac{p\beta - 1}{p-1}$, $p\beta \neq 1$. Note that condition $B_0 < 0$ is not necessary since $\Delta_p\psi = 0$, so (1), (2) hold and inequality (23) becomes

$$\int_\Omega |x|^{p(1-\alpha)}g^{p(1-\beta)}|\nabla u|^p dx \geq \left|\frac{(p-n)B_0}{p}\right|^p \int_\Omega |x|^{-\alpha p}|g|^{-\beta p}|u|^p dx. \tag{24}$$

In the particular case $\alpha = \beta = 1$, $\lambda = 1$, so $\Omega = B_1(0)$ the inequality (24) becomes

$$\int_\Omega |\nabla u|^p dx \geq \left|\frac{p-n}{p}\right|^p \int_\Omega |x|^{-p}|g|^{-p}|u|^p dx. \tag{25}$$

Note that $|g|^{-p} \geq 1$, so the inequality (25) improves the inequality (6).

Remark. It is interesting to analyze whether the condition $u \in U_F$ can be replaced with weaker condition (26)

$$\int_{G_{0,M}} |h\nabla u|^p dx < \infty \quad \text{and} \quad u|_{F=0} = 0. \tag{26}$$

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