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INTERIOR BOUNDARIES FOR DEGENERATE ELLIPTIC EQUATIONS OF SECOND ORDER SOME THEORY AND NUMERICAL OBSERVATIONS

G. Chobanov, N. Kutev

Dedicated to Acad. Petar Popivanov on the occasion of his 65th birthday

Abstract. For boundary value problems for degenerate-elliptic equations of second order in $\Omega \subset \mathbb{R}^n$ there are cases when a closed surface Γ exists, dividing Ω into two subdomains in such a manner that two new correct boundary value problems can be formulated without introducing new boundary conditions. Such surfaces are called interior boundaries. Some theoretical results regarding the connections between the solutions of the original problem and the two new problems are given. Some numerical experiments using the finite elements method are carried out trying to visualize the effects of the presence of such interior boundary when $n = 2$. Also some more precise study of the solutions in the case $n = 2$ is presented.

1. Introduction. After the paper by Fichera [4], boundary value problems for linear second order partial differential equations

$$(1) \qquad Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_ix_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u - f(x) = 0 \text{ in } \Omega$$

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with non negative characteristic form

$$\sum_{i,j=1}^{n} a^{ij}\xi_i\xi_j \geq 0 \quad \text{for} \quad x \in \overline{\Omega}, \xi \in \mathbb{R}^n \setminus \{0\} \tag{2}$$

are well understood in the sense that boundary conditions should not be imposed on the whole boundary but only on the non characteristic boundary $\Sigma_3 = \{x \in \partial\Omega : a^{ij}\nu_i\nu_j > 0\}$ where ν is the interior unit normal to $\partial\Omega$ and part of the characteristic boundary, i.e. $\{x \in \partial\Omega : a^{ij}\nu_i\nu_j = 0\}$. This last part is determined by means of the following function

$$\beta_{\partial\Omega,\nu}(x) = \sum_{k=1}^{n}\left(b^k(x) + \sum_{j=1}^{n} a^{kj}_{x_j}(x)\right)\nu_k \quad \text{on} \quad \partial\Omega \tag{3}$$

the rest of the boundary being subdivided as follows $\Sigma_2 = \{x \in \partial\Omega\backslash\Sigma_3 : \beta(x) > 0\}$, $\Sigma_1 = \{x \in \partial\Omega\backslash\Sigma_3 : \beta(x) < 0\}$ and $\Sigma_0 = \{x \in \partial\Omega\backslash\Sigma_3 : \beta(x) = 0\}$. Boundary conditions must be imposed on $\Sigma_2 \cup \Sigma_3$ only.

Let us note however that a function $\beta_{\Gamma,\nu}(x)$ of the form (3) can be defined for any smooth two-sided surface Γ with chosen unit normal $\nu(x)$. The equation could have characteristic surfaces Γ inside the domain Ω. So the question arises what happens if inside Ω there is a smooth characteristic surface Γ that isolates a subdomain Ω_1 and furthermore we have $\beta_{\Gamma,\nu}(x) = 0$, i.e.

$$a^{ij}(x)\nu_i(x)\nu_j(x) = 0 \quad \text{and} \quad \beta_{\Gamma,\nu}(x) = 0 \quad \text{on} \quad \Gamma \tag{4}$$

The behaviour of the solutions in situations of this type were recently studied in some detail by the authors in [1]. In order to briefly state the results some additional hypotheses and definitions are in order.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded region with a piecewise smooth boundary and $\Gamma \subset \Omega$ is smooth closed surface which divides Ω in two subdomains Ω_1 and Ω_2 such that $\Omega = \Gamma \cup \Omega_1 \cup \Omega_2$, $\Gamma \subset \partial\Omega_1$ and $\Gamma \subset \partial\Omega_2$. Suppose for simplicity and definiteness that $\partial\Omega \subset \partial\Omega_1$. Moreover let

$$a^{ij}, b^i, c, f, \psi, \Gamma \in C^\infty \tag{5}$$

and

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j = 0, \quad \xi \in \mathbb{R}^n \setminus \{0\} \tag{6}$$

only for $x \in \Gamma$ and $\xi = \nu(x)$.

Also the notion of generalized mean curvature introduced by J. Serrin [9], which connects the geometric curvatures of C^2 smooth surfaces with coefficients of partial differential operators studied in their neighbourhood is needed.

If $\lambda^\tau(x)$, $\kappa_\tau(x)$ $(\tau = 1, \dots, n-1)$ are the principal directions and the principal curvatures of Γ at some point $x \in \Gamma$ and $\nu(x)$ is the interior unit normal to Γ (with respect to Ω_1) then

$$\mathcal{H}_\Gamma(x) = \sum_{\tau=1}^{n-1} \lambda^\tau A \lambda^\tau \kappa_\tau + \nu A \nu H$$

is the generalized mean curvature of Γ at the point x. Here $A = \{a^{ij}(x)\}$ and $H = (\kappa_1 + \dots + \kappa_{n-1})/(n-1)$ is the ordinary mean curvature of Γ.

Since here Γ is characteristic we have $\nu A \nu = 0$ on Γ end hence $\mathcal{H}_\Gamma(x) = \lambda^\tau A \lambda^\tau \kappa_\tau$.

Under the above assumptions applying appropriate change of variables an equation on Γ only

$$(7) \qquad - \sum_{\sigma,\tau=1}^{n-1} A^{\tau\sigma}(x) u_{\lambda_\tau \lambda_\sigma} + \sum_{\tau=1}^{n-1} B^\tau(x) u_{\lambda_\tau} + c(x) u = f(x)$$

is obtained. The hypotheses that the original equation degenerates on Γ in the normal direction only and the sufficient regularity of the coefficients imply that this is an elliptic equation on Γ that has unique classical solution $u_0(x)$ on the smooth manifold Γ.

The results in [1] now can be summarized as follows

Let $\mathcal{H}_\Gamma = 0$ for every $x \in \Gamma$. Then there exist viscosity solutions $u_1 \in C(\overline{\Omega}_1)$, $u_2 \in C(\overline{\Omega}_2)$ of the problems in Ω_1 and Ω_2) which satisfy the boundary data $u_0(x)$ on Γ, i.e. $u_1(x) = u_2(x) = u_0(x)$ on Γ. Moreover, the viscosity solution $U(x) = u_1(x)$ in Ω_1, $U(x) = u_2(x)$ in $\overline{\Omega}_2$ of the problem in Ω is Hölder continuous on Γ with exponent $\lambda \in (0, 1)$ depending on $\|a^{ij}\|_{C^2(\overline{\Omega})}$, $\|b^i\|_{C^1(\overline{\Omega})}$, c_0 and Γ.

For the definitions of viscosity solutions of the equation and the Dirichlet problem the reader is refered to [3] (see also Def. 2.1 and Def. 2.2 in [1]).

2. Visualization. Motivated by the above result in the present section we propose a model equation in dimension 2, adapted for numerical computations, in order to visualize the effects of the presence of interior boundary. Although the results are only qualitative (no convergence or approximation estimates are sought or given), they seem to give some insight into the problem. The calculations are carried out by straight application (without justifications) of the finite elements method to the model equation. The plots obtained however not only are in accordance of the previously mentioned theoretical results, but also suggest that some of the conditions imposed in the theoretical study may be redundant.

Finite elements Methods are applied to equations in divergence form so lets consider a general second equation of the form

$$Lu \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a^{ij}(x)u_{x_i x_j}) + \sum_{i=1}^{m} a^i(x)u_{x_i} + c(x)u = f(x)$$

After some elementary calculations we get that for a smooth surface Γ with an unit normal vector ν the corresponding Fichera function (see (3)) in this case is

$$\beta_{\Gamma,\nu}(x) = \sum_{k=1}^{n} a_k \nu_k.$$

Now the following model operator is defined:

$$\begin{aligned} Lu = {} & \frac{\partial}{\partial x}\left((x^2+y^2-1)^2\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left((x^2+y^2-1)^2\frac{\partial u}{\partial y}\right) \\ & + \frac{\partial}{\partial x}\left(y^2\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial x}\left(xy\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(xy\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(x^2\frac{\partial u}{\partial y}\right) \\ & - y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} \end{aligned}$$

in the square $Q = \{(x,y)| -2 < x < 2, -2 < y < 2\}$ or in the disk B with radius 2. It is easily seen that the unite circle is an interior boundary. In the above definition the first line is elliptic degenerating only on the unit circle, the second is parabolic first order along the rays from the origin. the third line should give the desired values of the function $\beta_{\Gamma,\nu}(x)$ on the unit circle. If we now make polar change of coordinates the equation becomes

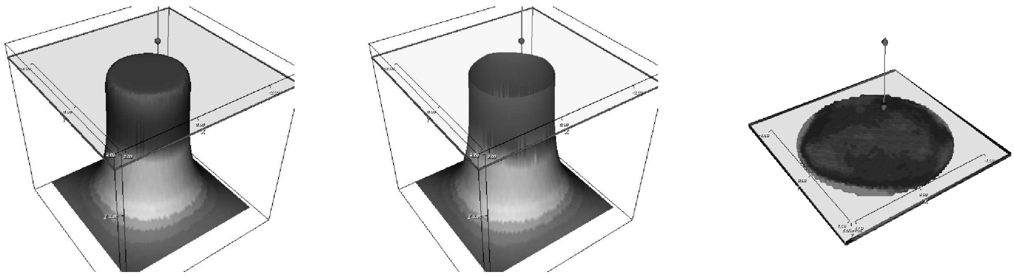
$$(\rho^2-1)^2\Delta u + 4(\rho^2-1)\rho\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial u}{\partial \phi}$$

On the unique circle $\rho = 1$ the equation on becomes

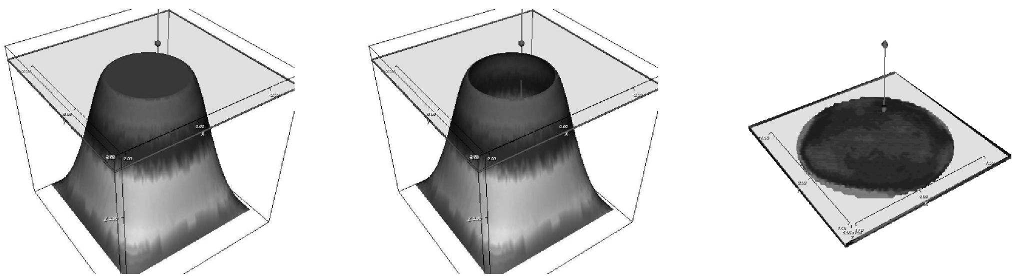
$$\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial u}{\partial \phi} + cu - f(\phi) = 0 \qquad (8)$$

(corresponding to (7)) with some 2π-periodic function f of ϕ and we must look for periodic solutions in place of the function u_0 mentioned above. We consider the simplest cases $f = 1$ and $f = x$ ($f(\phi) = \cos\phi$ in polar coordinates) and periodic solutions can be easily found with elementary means. Let D be the unit disk. We remind that the theory is valid for functions $c(x) \geq c_0 > 0$, and the bigger the constant c_0, the more regular is the solution.

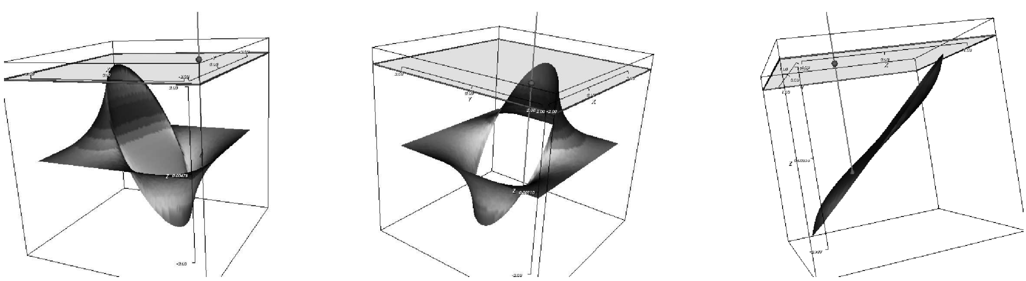
The plots below are scaled in the direction of the z-axis.



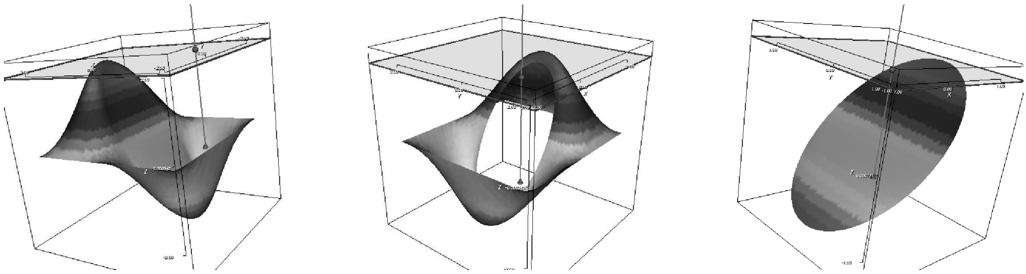
$Lu + 2u = 1$ in Q, $Q \setminus D$ and D



$Lu + 25u = 1$ in Q, $Q \setminus D$ and D



$Lu + 2u = x$ in Q, $Q \setminus D$ and D



$Lu + 25u = x$ in Q, $Q \setminus D$ and D

3. The two dimensional case. In the two-dimensional case it is possible to visualize the abstract results. It is also possible to obtain more sharp ones, for example with respect to the behaviour of the solutions in a neighbourhood of Γ. One can study the question whether the global viscosity solution is Lipshitz continuous or its gradient blows up on Γ. Below a rather simple case is considered in order to demonstrate the main ideas.

Suppose now $n = 2$ and Ω is a simply connected region in $\mathbb{R}^2$ including the unit circle $\Gamma = \{x \in \mathbb{R}^2; |x| = 1\}$. Let L be a two-dimensional operator corresponding to (1) which degenerates on Γ only, i.e. the two dimensional equivalent of (4) holds. Let $\Omega_2 = B = \{x \in \mathbb{R}^2; |x| < 1\}$ and $\Omega_1 = \Omega \setminus \overline{B}$. According to the results of our previous paper [1] the following boundary value problems

$$Lu = f \quad \text{in} \quad \Omega_2 = B \tag{9}$$

$$Lu = f \quad \text{in} \quad \Omega_1, u = \psi \quad \text{on} \quad \partial\Omega \tag{10}$$

are uniquely solvable in the sense of viscosity solutions.

Proposition 1. *Suppose* (2), (5) *and* (4) *hold. Then the boundary value problem* (9), *resp.* (10) *has a unique viscosity solution* $u_1 \in C(\overline{\Omega}_1)$, *resp* $u_2 \in C(\overline{\Omega}_2)$. *Moreover* $u_1(x) = u_2(x)$ *on* Γ *and the function* $U(x) = u_1(x)$ *in* Ω_1 *and* $U(x) = u_2(x)$ *in* $\overline{\Omega}_2$ *is the unique viscosity solution of* (1) $Lu = f$ *in* Ω *satisfying* $u = \psi$ *on* $\partial\Omega$.

Let us make a polar change of variables

$$z_1 = \sqrt{x_1^2 + x_2^2}, \quad z_2 = \arg(x_1 + ix_2)$$

in the domain $\{x \in \mathbb{R}^2; 1-\delta < |x| < 1+\delta\} \subset \Omega$. (Some additional hypotheses on δ will be made later). If Pw is the operator in the new variables, then

$$Pw = -\sum_{i,j=1}^{2} A^{ij}(z) w_{z_i z_j} + \sum_{i=1}^{2} B^i(z) w_{z_i} + C(z)w - F(z) = 0$$

in G, where $G = \{z \in \mathbb{R}^2; 1-\delta < z_1 < 1+\delta,\ 0 \le z_2 < 2\pi\}$.

Note that Γ is transformed in the line $l = \{z \in \mathbb{R}^2; z_1 = 1,\ 0 \le z_2 < 2\pi\}$ with zero generalized curvature $\mathcal{H}_l$. Hence from Th. 5.1 in [1], it follows that

$$w_1(1, z_2) = w_2(1, z_2) = w_0(z),$$

where w_1, w_2 are the images of u_1, u_2 in $G_1 = G \cap \{z_1 > 1\}$, $G_2 = G \cap \{z_1 < 1\}$ and w_0 is the unique 2π-periodic solution of the equation (analogue of (8))

$$-A^{22}(1, z_2)(w_0)_{z_2 z_2} + B^1(1, z_2)(w_0)_{z_2} + C(1, z_2)w_0 - F(1, z_2) = 0$$

(see Ch 2, § 2.9 in [11]). Moreover the viscosity solution $w(x)$ is Hölder continuous with exponent α close to 0 (see Th. 5.1 in [1]).

Theorem 1. *Suppose* (1), (2), (5) *and* (6). *If*

$$B^1_{z_1}(1, z_2) + C(1, z_2) > 0 \tag{11}$$

for every $0 \le z_2 < 2\pi$. *Then* $w_1(x)$ *is Lipshitz continuous in a neighbourhood of* l *in* G_1 *and therefore* $u_1(x)$ *is Lipshitz continuous in a neighbourhood of* Γ *in* Ω_1.

Remark 1. From (6) and (5) the viscosity solution $u \in C^\infty(\overline{\Omega}\backslash\Gamma)$ (see [5], [6], [7], [8]), while the regularity in a neighbourhood of Γ remains an open question.

P r o o f. Consider in G_1 the barrier function $h(z) = N(z_1 - 1) + w_0(z_2)$ Suppose δ is sufficiently small so that

$$\frac{B^1(z)}{z-1} + C(z) \ge k > 0 \tag{12}$$

in G_1. The existence of such δ follows from (11). Indeed, the Fichera function on l is

$$\beta_l(z_2) = B^1(1, z_2) + A^{11}_{z_1}(1, z_2) + A^{12}_{z_2}(1, z_2) = 0$$

for $0 \le z_2 < 2\pi$. The equality $A^{11}(1, z_2) = 0$ holds since l is a characteristic. Now $A^{11}(z_1, z_2) \ge 0$ implies that the coefficient A^{11} has a minimum for $z_1 = 1$, whence $A^{11}_{z_1}(1, z_2) = 0$. The inequality (2) implies

$$(A^{12}(1, z_2))^2 \le A^{11}(1, z_2)A^{22}(1, z_2) = 0$$

hence $A^{12}_{z_2}(1, z_2) = 0$. Now it follows that $B^1(1, z_2) = 0$ and (11) implies (12) for δ sufficiently small. Let N be so large that

$$N\delta + w_0(z_2) > w_1(1 + \delta, z_2)$$

for $0 \le z_2 < 2\pi$. Simple computations give

$$\begin{aligned} Ph = N[B^1(z) + C(z)(z_1 - 1)] + F(1, z_2) - F(z) \\ \ge N(z_1 - 1)k - (z_1 - 1) \sup |\nabla F| \ge 0 \end{aligned}$$

for every $z \in G_1$ when $Nk \ge \sup |\nabla F|$.

Since $h(1, z_2) = 0$, $h(1+\delta, z_2) > w_1(1+\delta, z_2)$ and $h(z)$ is a 2π-periodic function of z_2, it follows from the comparison principle, Th 3.2 and Lemma 4.2 in [1] , that $w_1(z) \le h(z)$ in G_1. Hence $w_1(z) - w_0(z) \le Nk|z_1 - 1|$ in G_1.

Similar argument involving the barrier function $h_1(z) = -N(z_1 - 1) + w_0(z_2)$ gives an estimate from below, so now $|w_1(z) - w_0(z)| \le Nk|z_1 - 1|$ in G_1. This proves Theorem 1.

Remark 2. Similar argument holds also in G_2 using obvious modifications of the barrier functions.

Remark 3. The condition (11) is probably close also to the necessary one for it can be proved using barrier functions of the form $w_0(z_1) + (z_2 - 1)^\alpha$, $0 < \alpha < 1$ and α close to 1, that if

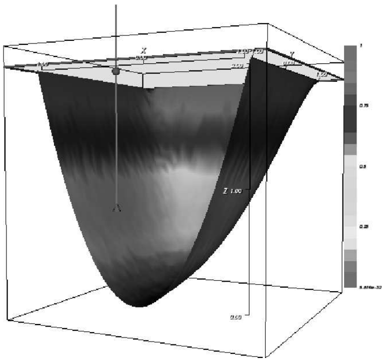
$$B^1_{z_1}(1, z_2) + C(1, z_2) < 0$$

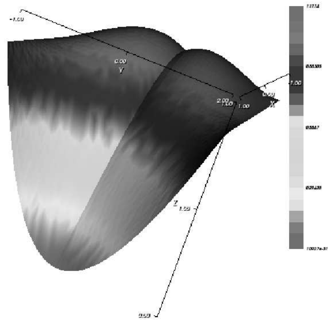
then for special choice of the right-hand side f we have gradient blow up in transversal to Γ direction.

4. Comments. Another numerical example is produced by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial y} y^2 \frac{\partial u}{\partial y} - cu$$

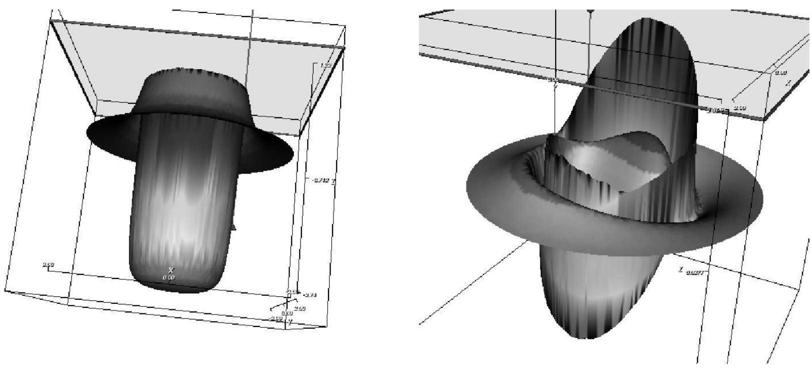
on $Q = \{(x, y)| -1 < x < 1, -1 < y < 1\}$ with the line $y = 0$ as interior boundary. Hence the solution must satisfy on $y = 0$ the boundary value problem $u'' - cu - f(x, 0) = 0$ for $-1 < x < 1$ with the appropriate values for $u(-1)$ and $u(1)$. The next two plots illustrate this situation.





In the visualization above was used the integrated environment FreeFem++-cs (`http://www.ann.jussieu.fr/~lehyaric/ffcs/index.htm`) providing an intuitive graphical interface to FreeFem++ (`http://www.freefem.org/ff++/`) on a machine running Fedora 14.

The condition $c(x) \geq c_0 > 0$ is essential in all the theoretical considerations above. On the other hand direct application of the finite element method gives some results when this is not the case. Some of the plots are given here. These probably should be further studied.



$$Lu - u = 1 \quad \text{and} \quad Lu - 10u = x \quad \text{in} \quad B$$

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